A Chebyshev Based Spectral Method for Solving Boundary Layer Flow of a Fractional-Order Oldroyd-B Fluid

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ABSTRACT

We focused on developing an accurate numerical scheme for the flow of a fractional-order Oldroyd–B fluid model with the non-isothermal property. In many cases, the direct application of the Chebyshev tau method using the operational matrix of the Chebyshev polynomials usually leads to an accurate solution. However, in some cases, dealing with non-linearity and coupling can be tedious. In this study, we present a numerical method based on Chebyshev polynomials of the first kind and interpolation using Gauss–Lobatto quadrature. The coefficients of the series expansion of the pseudospectral method are obtained through integration of the Chebyshev polynomials orthogonality condition. The numerical results show that the scheme is accurate and reliable. The effects of the fractional-order objective stress rate of the Oldroyd–B fluid on the velocity and shear stress are also presented. The error bound theorems presented in this study support the findings of the numerical computations.

1. INTRODUCTION

The Oldroyd–B fluid is a rate type model for viscoelastic fluids that have been studied extensively since its formulation by Oldroyd [1]. This model is part of a large class of rate type viscoelastic fluids models, which include, but are not limited to the upper convected Maxwell model, Jeffrey model and Burgers’ model. The Oldroyd-B model gives a good representation of the rheological response of viscoelastic fluids in shear flow. The Cauchy stress tensor for the classical Oldroyd–B constitutive model has the form

\[ S_{ij} = -p\delta_{ij} + \tau_{ij}, \]

\[ \left[ 1 + \lambda_0 D^\gamma \right] \tau_{ij} = \mu \left[ 1 + \lambda_1 D^\gamma \right] e_{ij}, \]

where, \( \tau_{ij} \) is the extra stress tensor, \( e_{ij} \) is the stress tensor. The material parameters \( \mu, \lambda_0 \) and \( \lambda_1 \) are the viscosity, stress relaxation time and retardation time respectively and \( D \) is the upper convected time derivative. The constitutive model of the Oldroyd–B is a generalization of the Maxwell fluid model [2]. Unlike the Maxwell model, the Oldroyd–B fluid uses the objective stress rate that takes into account frame indifference of the deformation rate. The deformation rate is represented by the upper convected derivative in Eq. (2). Because of the memory retention characteristics of viscoelastic fluids, recent studies have suggested using non–local time derivatives [3]. Non–local objective stress rate exhibits complex dynamical and viscoelastic behaviours. Taking advantage of the inherent non–locality and memory retention characteristics of fractional derivatives, we generalize the Oldroyd–B constitutive relation by replacing the local time derivatives in Eq. (2) with fractional time derivative [4]:

\[ \left[ 1 + \lambda^\alpha \partial_t^\alpha \right] \nabla_{\alpha} \tau_{ij} = \mu \left[ 1 + \lambda_1^\beta \nabla_{\beta} \right] e_{ij}, \]

where, \( D_{\alpha} = \frac{\partial^\alpha}{\partial t^\alpha} + \mathbf{U} \cdot \nabla \tau - \tau \nabla \mathbf{U} - (\nabla \mathbf{U})^T \). In the constitutive model Eq. (3), \( 0<\alpha, \beta \leq 1 \) and \( \mu, \lambda, \lambda_1 > 0 \). If \( \alpha=\beta=1 \), it should be noted that the model is physically unrealistic. This case corresponds to an increasing relaxation function [5]. Hence, the constraint on the order of the derivatives must satisfy \( 0<\alpha, \beta \leq 1 \). It is evident that if \( \alpha=\beta=1 \), the constitutive relation is the classical Oldroyd–B fluid. Setting \( \lambda_1 = 0 \) in Eq. (3) corresponds to the fractional Maxwell fluid, if \( \lambda=0 \), the model is equivalent to the fractional second-grade fluid, if \( 0<\lambda_1 \), we obtain the fractional Jeffrey fluid [6] and when \( \lambda=\lambda_1 = 0 \), the model corresponds to the classical Newtonian fluid. If we consider an incompressible fluid with constant pressure and using established notation, the momentum conservation equation for an unsteady flow of a magnetohydrodynamic (MHD) fluid is:

\[ \rho \frac{\partial \mathbf{U}}{\partial t} + \mathbf{U} \cdot \nabla \mathbf{U} = \nabla \tau_{ij} - \sigma B_0^2 \mathbf{U}, \]

where, \( \rho \) is the fluid density, \( \sigma \) is the electrical conductivity of the fluid and \( B_0 \) is the applied magnetic field. To write the conservation equation and the shear stress relation in non–dimensional form, we assume a reference length \( L \) and velocity \( U_0 \), so that we define \( t = \frac{iL}{U_0}, \mathbf{y} = \frac{\mathbf{y}}{L} \) for all \( \mathbf{y} \) in
the domain of the flow, \( \mathbf{U}(y, t) = \mathbf{U}_0 \tilde{\mathbf{U}}(\tilde{y}, \tilde{t}) \), \( \tau_{ij}(y, t) = \mu \mathbf{U}_0 \tilde{\mathbf{U}}^{-1} \tilde{\tau}_{ij}(\tilde{y}, \tilde{t}) \), and \( \lambda_r = \tilde{\lambda}_r \mathbf{U}_0 \tilde{\mathbf{U}}^{-1} \). Therefore, we write Eq. (3) and Eq. (5) in the dimensionless form (dropping the tilde for convenience):

\[
Re \left[ \frac{\partial \mathbf{U}}{\partial t} + \mathbf{U} \cdot \nabla \mathbf{U} \right] = \nabla \tau_{ij} - Ha^2 \mathbf{U}
\tag{6}
\]

\[
\left[ 1 + We^a D^a \right] \tau_{ij} = \left[ 1 + \lambda_r \frac{\partial}{\partial y} \right] e_{ij}.
\tag{7}
\]

Here, \( Re = \rho \mathbf{U} \mathbf{L} \mu r^{-1} \) is the Reynolds number, \( Ha = B_0 \mathbf{L} (\sigma / \mu)^{3/2} \) is the Hartmann number, \( We = \mathbf{U}_0 \mathbf{L}^{-1} \) is the Weissenberg number, \( \lambda_r \) is a dimensionless retardation parameter. Some special cases of interest include the Jeffrey fluid (0 < \( \lambda_r < We \)), Maxwell fluid (\( \lambda_r = 0 \)), second-order fluid (\( We = 0 \)), Newtonian fluid (\( We = \lambda_r = 0 \)) and hydrodynamic fluids (\( Ha = 0 \)).

Exact solutions to fractional partial differential equations are, in general, challenging to find. Even when found, the solution may contain complicated integrals and special functions that have to be approximated numerically. Several studies have proposed various methods, both numerical and analytic, for solving fractional differential equations [7-12]. However, recent studies have identified spectral methods as efficient for approximating solutions of fractional partial differential equations. Spectral methods are favoured because of their spectral rate of convergence and accuracy, especially for differential equations with sufficiently smooth solutions. Doha et al. [13] and Atabakzadeh et al. [14] presented the operational matrix of the shifted Chebyshev polynomial and used the polynomial to approximate multi-order fractional ordinary differential equations. Liu et al. [15] presented numerical solutions to multiterm variable-order fractional ordinary differential equations using the Chebyshev polynomial of the second kind as the basis function. Operational matrices of the shifted Chebyshev wavelets, generalized Laguerre polynomials, and Bernstein polynomials are presented in the studies of Benattia and Belghaba [16], Bhrayw and Alghamdi [17], Baseri et al. [18] respectively. These matrices are used to approximate the solutions of fractional differential equations. The numerical results that were reported in the studies as mentioned earlier are typical of spectral method, and they exhibit the spectral convergence property of spectral methods. From literature, many authors have used the spectral tau method to solve fractional differential equations with different orthogonal polynomials as basis functions. In this study, we propose a spectral method that uses the shifted Chebyshev polynomials of the first kind as basis functions and interpolates using Gauss–Lobatto quadrature. We obtain the coefficients of the series expansion in the spectral method by integrating the orthogonal condition of the Chebyshev polynomials using Chebyshev–Gauss–Lobatto quadrature. We then apply the resulting fractional differentiation matrix to solve fractional partial differential equations that describe the unsteady boundary layer flow of a generalized MHD Oldroyd–B fluid and a non–isothermal flow of an Oldroyd–B fluid (see Section 4). We also investigate the effects of derivatives of arbitrary order in Eq. (6) and Eq. (7) on fluid velocity and shear stress.

2. FRACTIONAL ORDER DIFFERENTIATION MATRIX

The most used definitions of fractional operators are the Riemann-Liouville and Caputo fractional operators. In the Riemann-Liouville case, we have the definition

\[
Re \mathbf{D}_{\alpha} H(y) = \frac{1}{\Gamma(n - \alpha)} \int_0^y (y - \zeta)^{n-\alpha-1} \mathbf{H}(\zeta) d\zeta
\tag{8}
\]

and the Caputo case is defined as

\[
\mathbf{D}_{\alpha} H(y) = \frac{1}{\Gamma(n - \alpha)} \int_0^y (y - \zeta)^{n-\alpha-1} \mathbf{H}(\zeta) d\zeta
\tag{9}
\]

In both cases, \( n-1 < \alpha < n \), and \( n \in \mathbb{N} \) and \( \mathbf{H}(y) \) is a continuously bounded function with \( n \) derivatives in \([0, L]\), for \( y > 0 \). For an in-depth exploration of these definitions, see Podlubny [19], Atangana [20]. Using the Caputo operator, the following holds

\[
\mathbf{D}_{\alpha} K = 0, \quad K \text{ is a constant},
\tag{10}
\]

\[
\mathbf{D}_{\alpha} y_j = \begin{cases} 0 & \text{for } j \in \mathbb{N}_0 \text{and } j < [\alpha] \\ \frac{\Gamma(j + 1)}{\Gamma(j + 1 - \alpha)} y^{j-\alpha} & \text{for } j \in \mathbb{N}_0, j \geq [\alpha] 
\end{cases}
\tag{11}
\]

Considering that most physical problems are defined in \([0, L]\), \( L \) being a truncation of the semi-infinite domain, we define the series form of the shifted Chebyshev polynomials \( T_{n,n} \) of degree \( n > 0 \) as ([21, 22])

\[
T_{n,n} = n \sum_{j=0}^{n} \frac{(-1)^{n-j}(n + j - 1)! 2^{2j} y_j}{(n - j)! (2j)! L^j} y_j,
\tag{12}
\]

which satisfies the orthogonality condition

\[
\int_0^L T_{n,n}(y) T_{m,n}(y) w_L(y) dy = \delta_{mn} h_n.
\tag{13}
\]

Here, the weight function of the shifted Chebyshev polynomials is given by \( w_L(y) = 1/\sqrt{Ly - y^2} \) and \( h_n=\pi n/2 \), with \( c_0=2 \) and \( c_0=1 \) for \( n \geq 1 \).

At this point, we discuss some lemmas and theorems that are used in developing the approximation of the fractional-order derivative of a square-integrable function that is expanded by a shifted Chebyshev series and integrated using the Chebyshev–Gauss–Lobatto quadrature.

**Remark 2.1:** For the shifted Chebyshev–Gauss–Lobatto quadrature, the Christoffel numbers are the same as those of the Chebyshev–Gauss–Lobatto quadrature. The shifted Gauss–Lobatto nodes are defined as ([21])

\[
y_j = \frac{L}{2} \cos \left( \frac{\pi j}{N} \right) + \frac{L}{2}
\tag{14}
\]

and the associated Christoffel weight number \( w_{L,j} = \pi c_i N \), \( 0 \leq j \leq N \), where \( c_0<c_N=2 \) and \( c_j=1 \) for \( j=1,2,...,N-1 \).
Lemma 2.2: Assume that $u(y)$ is an integrable function defined on the domain $[0, L]$, then the function can be expanded in terms of shifted Chebyshev polynomials as a $N+1$ truncated series:

$$u_N(y) = \sum_{n=0}^{N} u_n T_{L,n}(y),$$

(15)

where, the coefficients $u_n$ satisfy the orthogonality condition, which is given in discrete form as

$$u_n = \frac{1}{h_n} \sum_{j=0}^{N} \frac{\pi}{c_j} u(y_j) T_{L,n}(y_j), \quad n = 0, \ldots, N.$$

where

(16)

Lemma 2.3: Let $T_{L,n}(y)$ be a $n$–th order shifted Chebyshev

$$q_{j,k} = \begin{cases} 0 & \text{if } k > n \end{cases} k \sum_{r=0}^{n} (-1)^{k-r}(k+r-1)!2^{2r}L^{-a} \Gamma(j-r+n+\frac{1}{2}) \Gamma(j-a+r+1)$$

(19)

Proposition 2.5: If $u(y,t)$ is a function defined on the rectangular domain $[0, L] \times [0, T]$, then it can be approximated at the shifted Gauss–Lobatto nodes in terms of $N+1$ shifted Chebyshev polynomials so that we have

$$vec(\tilde{u}) = [I_t \otimes I_y] vec(\bar{u}).$$

(23)

where, the superscript indicates the derivative with respect to the temporal variable and the subscript is the derivative with respect to the spatial variable and $D^n$ is from Theorem 2.4.

3. ERROR ESTIMATION FOR THE APPROXIMATION

Assume that $u(y)$ is a square-integrable function and $w_L(y)$ is a Lebesgue integrable function defined in the interval $\mathbb{I} = [0, L]$. Then, we can define a $L_{w_L}^2$ space in which $u(y)$ is measurable and the norm $\|u(y)\|_{w_L}$ is defined as

$$\|u(y)\|_{w_L} = \left( \int_{\mathbb{I}} |u(y)|^2 w_L(y) dy \right)^{1/2} < \infty,$$

(24)

such that the norm is induced by the inner product.
\[(u(y), \hat{u}(y)) = \int_0^L u(y)\hat{u}(y)w_L(y)dy. \quad (25)\]

If \(u(y)\) is the exact solution and \(u_N(y)\) is the approximated solution based on the Chebyshev polynomials given in Eq. (15) with the coefficients \(u_0\) obtained by the normalized inner product

\[u_n = \frac{(u_N(y), T_{L_n}(y))}{\|T_{L_n}(y)\|_{w_L}^2}\]  

(26)

We can define an error bound for the approximation in the \(L^2_{w_L}\) norm.

**Theorem 3.1** (Error estimation for a single variable approximation): Given the shifted interpolation nodes defined in Eq. (14) and let \(P_N u(y)\) be the approximation through these nodes given in Eq. (15), where \(P_N\) is the space of all Chebyshev polynomials of degree less than or equal to \(N\). Assume that \(d^{N+1}u/dy^{N+1}\) exist and is continuous on the interval \(I\), then the error bound is defined as

\[\|u(y) - P_N u(y)\| \leq \frac{\max_{0 \leq y \leq 1} |d^{N+1}u(y)|}{\Gamma(N + 2)} \frac{L^{2N+2} \sqrt{\pi}}{\Gamma(2N + 3)} \]  

(27)

**Proof.** Consider the generalized Taylor's approximation of \(u(y)\) in which the error bound is known as

\[\|R_N(y)\| \leq \frac{|y|^{N+1}}{\Gamma(N + 2)} \max_{0 \leq y \leq 1} \frac{|d^{N+1}u(y)|}{dy^{N+1}} \]  

(28)

then for any \(y\) in the collocation points

\[\|u(y) - P_N u(y)\|_{w_L}^2 \leq \|R_N(y)\|_{w_L}^2 \leq \frac{1}{\Gamma(N + 2)} \left( \max_{0 \leq y \leq 1} \frac{|d^{N+1}u(y)|}{dy^{N+1}} \right)^2 \int_0^L y^{2N+2} dy \]  

(29)

\[= \frac{1}{\Gamma(N + 2)} \left( \max_{0 \leq y \leq 1} \frac{|d^{N+1}u(y)|}{dy^{N+1}} \right)^2 \frac{(2N + 5)!}{(2N + 3)!} \]  

(30)

This leads to the desired result.

**Theorem 3.2:** Let \(u: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}\) be a continuously differentiable function such that at least \((N + 1)\)th partial derivative with respect to \(y\), \((N + 1)\)th partial derivative with respect to \(t\) and \((N + 1)\)th mixed derivative with respect to \(y\) and \(t\) exist, then based on the mean value theorem, the following remainder formula holds \((25)\)

\[\left| R_{N_y, N_t}(y, t) \right| \leq \frac{K_1 y^{N_y+1} + K_2 \Gamma(N_y + 2)}{\Gamma(N_y + 2)} \]  

(31)

where, \(0 \leq y \leq L\), \(0 \leq t \leq T\) and \(K_1\), \(K_2\), \(K_3\) are constants, defined as

\[K_1 = \sup \left\{ \left| \frac{\partial^{N_y+1}u(y, t)}{\partial y^{N_y+1}} \right| : y, t \in \mathbb{R} \times \mathbb{R} \right\}, \]  

(32)

\[K_2 = \sup \left\{ \left| \frac{\partial^{N_y+N_t+1}u(y, t)}{\partial y^{N_y+1} \partial t^{N_t+1}} \right| : y, t \in \mathbb{R} \times \mathbb{R} \right\}, \]  

(33)

respectively.

**Theorem 3.3** (Error bound for functions of \(1+1\) variables approximation): We define the error bound for the approximation of a function of two variables as

\[\|R_{N_y, N_t}(y, t)\|_{w_L} \leq \frac{K_1 \Gamma(2N_y + 5)}{(\Gamma(N_y + 2))^2} \]  

(34)

**Proof.** Given that \(P_{N_y, N_t} u(y, t)\) is the space of all Chebyshev polynomials which approximate \(u(y, t).\) In a similar sense as in Theorem 3.1, for any points \(y, t\) in the collocation points and using the relation in Eq. (31), we define

\[\left( u(y, t) - P_{N_y, N_t} u(y, t) \right)^2 \]  

(35)

\[\leq \int_{\mathbb{R}^2} \left( \frac{K_1 y^{N_y+1} + K_2 \Gamma(N_y + 2)}{\Gamma(N_y + 2)} \right)^2 \frac{1}{\sqrt{t^2 - t^2 \sqrt{L^2 - y^2}}} \frac{1}{\sqrt{t^2 - t^2 \sqrt{L^2 - y^2}}} dy dt \]  

(36)

\[= \frac{K_1^2}{(\Gamma(N_y + 2))^2} \left( \int_{\mathbb{R}^2} \frac{y^{2N_y+2}}{\sqrt{t^2 - t^2 \sqrt{L^2 - y^2}}} dy dt \right) \]  

(37)

The integrals in Eq. (37) are evaluated as

\[\int_{\mathbb{R}^2} \frac{1}{\sqrt{t^2 - t^2 \sqrt{L^2 - y^2}}} \frac{y^{2N_y+2}}{\sqrt{t^2 - t^2 \sqrt{L^2 - y^2}}} dy dt = \pi \sqrt{\pi L^{2N_y+2}} \frac{\Gamma(2N_y + 5)}{\Gamma(2N_y + 3)} \]  

(38)

\[\int_{\mathbb{R}^2} \frac{1}{\sqrt{t^2 - t^2 \sqrt{L^2 - y^2}}} dy dt = \pi \sqrt{\pi L^{2N_t+2}} \frac{\Gamma(2N_t + 5)}{\Gamma(2N_t + 3)} \]  

(39)
\[
\int \frac{t^{2N_t+2}y^{2N_y+2}}{\sqrt{1-t^2}\sqrt{1-y^2}} dy dt = \frac{\pi t^{2N_t+2}y^{2N_y+2}}{\Gamma(2N_t + \frac{5}{2}) \Gamma(2N_y + \frac{5}{2}) \Gamma(2N_t + 3) \Gamma(2N_y + 3)}
\]

Substituting Eq. (38) to Eq. (40) into Eq. (37) completes the proof. If we consider \( u_{N_y,N_t}(y,t) \), the \( L_{\omega L,W_T} \) orthogonal projection of \( u(y,t) \) onto \( P_{N_y,N_t} \), then

\[
\left| u(y,t) - u_{N_y,N_t}(y,t) \right| = \left| u(y,t) - P_{N_y,N_t}u(y,t) \right| + |u(y,t) - u_{N_y,N_t}(y,t)|
\]

and boundary conditions with a sine oscillation at the wall and initial conditions

\[
\begin{align*}
\omega &= 0, & \alpha &= 0.25, & \beta &= 0.75, & t &= 10, & Re &= 0.1, & Ha &= 0.5. & \\
\text{We consider the cases: } We &= 1 < \lambda &= 1.5, & \text{Jeffrey fluid } (\lambda &= 1.2 < We < 1.5), & \text{Maxwell fluid } (We &= 1.5, \lambda &= 0), & \text{second-grade fluid } (We &= 0, \lambda &= 1.2) & & \text{and Newtonian fluid.}
\end{align*}
\]

Figure 2 and Figure 3 illustrate the behavior of the velocity and shear stress for different values of the fractional orders \( \alpha \) and \( \beta \). Figure 2 shows that as the order \( \alpha \) increases, the shear stress decreases. While the net effect of increasing \( \alpha \) is that the shear stress increases. Figure 3, it can be seen that as \( \beta \) increases, the shear stress decreases while the amplitude becomes smaller. We remark that the degree of \( \alpha \) is related to the relaxation time. Hence, the lack of reverse flow or small amplitude of oscillation for small values of \( \alpha \) can be associated with the short memory of the fluid and slow response to shear stress.
Table 1. Residual and condition number for the problem in Section 4.1

<table>
<thead>
<tr>
<th>(a,β)</th>
<th>(Nt,Ny)</th>
<th>Res α</th>
<th>CN α</th>
<th>Res τ</th>
<th>CN τ</th>
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<tr>
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<td>(5,5)</td>
<td>4.1482e-16</td>
<td>203.5</td>
<td>1.3210e-15</td>
<td>17.35</td>
</tr>
<tr>
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<td>2.8511e-14</td>
<td>6.0310e+04</td>
<td>7.2893e-15</td>
<td>18.99</td>
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<tr>
<td>(15,15)</td>
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<td>6.1347e+11</td>
<td>7.1746e-07</td>
<td>8.6119e+04</td>
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</tr>
<tr>
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<td>1.1055e-15</td>
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</tr>
<tr>
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</tr>
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<td>477.8</td>
<td>2.0091e-15</td>
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<td>7.8238e-15</td>
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<td>2.6527e+10</td>
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<td>4.1076e+06</td>
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</tr>
</tbody>
</table>

Figure 1. Velocity and shear stress of special cases of the fractional Oldroyd-B fluid with \( \omega=70+n\pi/2 \)

Figure 2. Variation of the tangential shear stress profiles with \( \omega=10+\pi/2 \) for (a) different \( \alpha \) with \( \beta=1 \) and (b) different \( \beta \) with \( \alpha=0.25 \)

Figure 3. Velocity profiles with \( \omega=10+\pi/2 \) for: (a) different \( \alpha \) with \( \beta=1 \) and (b) different \( \beta \) with \( \alpha=0.25 \)
4.2 The non-isothermal flow of an Oldroyd–B fluid

Here we consider the unsteady non–isothermal flow of a generalized Oldroyd–B fluid. The thermal property of the fluid (γ), the specific heat (c) and the thermal conductivity (κ) are constant and isotropic, except the viscosity which is assumed to be temperature-dependent with reference viscosity, µ₀. We use [27]

\[
Re \left[ \frac{\partial u}{\partial t} + We^a \frac{\partial u_1}{\partial t} \right] + \frac{\partial u}{\partial y} \left( 1 + \frac{1}{\theta_T} \right) + \lambda_T^a \frac{\partial^2 u_1}{\partial y^2} + 1 + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial y^2} = 0
\]

\[
Re \left[ \frac{\partial \theta}{\partial t} + We^a \frac{\partial \theta_1}{\partial t} \right] - \frac{1}{Pr} \frac{\partial^2 \theta}{\partial y^2} + We^a \frac{\partial \theta}{\partial y} - Ec \left[ \frac{1}{1 + \theta_T} \frac{\partial u}{\partial y} + \lambda_T^a \frac{\partial^2 \theta}{\partial t^2} \frac{\partial u}{\partial y} \right] = 0
\]

\[
1 + We^a \frac{\partial u}{\partial t} \left[ \frac{\partial u}{\partial y} + \lambda_T^a \frac{\partial^2 \theta_1}{\partial y^2} \right].
\]

\[
e^a \left[ \theta(y, 0) = \frac{\partial \theta}{\partial y} \theta(y, 0) = 0 \forall y > 0 \right.
\]

\[
\theta(0, t) = 1, \theta(y, t) = 0 \text{ as } y \to \infty, t > 0.
\]

We seek a solution in the form of Eq. (22) for each dependent variable and apply the quasi–linearization method to linearize the system of equations (see Bellman and Kalaba [30], Motsa et al. [31]):

**Table 2.** Residuals for the dependent variables in the problem in Section 4.2

<table>
<thead>
<tr>
<th>(a,b)</th>
<th>(N,N₁)</th>
<th>Re₈₉₂</th>
<th>Re₈₅₂</th>
<th>Re₅₂₉</th>
</tr>
</thead>
<tbody>
<tr>
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**Figure 4.** Velocity and temperature profiles for non-isothermal flow of the special cases of the fractional Oldroyd-B fluid

\[
\mu(T) = \frac{\mu_0}{1 + \gamma(T - T_0)}
\]
Table 2 shows the maximum residual error for $0 \leq t \leq 10$ for different values of $\alpha$ and $\beta$. In Figure 4, the velocity and temperature profiles are illustrated for the special cases of the fractional Oldroyd–B fluid with $\alpha=0.25$, $\beta=0.5$, $Re=1.1$, $Pr=5$, $Ec=0.2$, $\theta=0.33$ and $\omega=10+\pi/2$. We consider the cases: $W_e=0.2<\lambda r=1.5$, Jeffrey fluid ($\lambda r=1.2<W_e=1.5$), Maxwell fluid ($W_e=1.5$, $\lambda r=0$), second-grade fluid ($W_e=0$, $\lambda r=1.2$) and Newtonian fluid. It can be seen that, as in Section 4.1, the second-order fluid has the smallest amplitude while the Maxwell fluid has the highest oscillation amplitude. However, for the temperature profiles, the Maxwell fluid reaches its thermal boundary layer region quicker than the other fluids, and the second-grade fluid has the most extensive thermal boundary layer. Figure 4b also show increased temperature close to the wall. The study of Ishaq et al. [28] attributes this behaviour to the accumulation of energy in the fluid particles in the vicinity of the wall because the viscous effect is profoundly felt in this region. The effects of different orders of derivatives are shown in Figure 5. Unlike the problem in Section 4.1, where velocity profiles are either strictly increasing or decreasing functions of $\alpha$ and $\beta$, in this flow, the velocity profiles intersect. We ascribe this behaviour to the coupling effect and non–isothermic nature of the fluid.

**REFERENCES**


NOMENCLATURE

- $B_0$: the applied magnetic field, $N \cdot s / C \cdot m$
- $c$: specific heat capacity, $J \cdot K / kg$
- $E_c$: Eckert number
- $\epsilon_0$: stress tensor, $Nim^2$
- $H_a$: Hartmann number
- $k$: thermal conductivity, $W / m \cdot K$
- $Pr$: reference Prandtl number
- $Re$: Reynolds number
- $U_0$: reference velocity, $m/s$
- $We$: Weissenberg number

Greek symbols

- $\alpha, \beta$: fractional orders
- $\theta$: temperature ratio
- $\lambda$: relaxation time, $s$
- $\lambda_0$: retardation time, $s / dimensionless retardation time$
- $\mu$: viscosity, $kg/m/s$
- $\mu_0$: reference viscosity, $kg/m/s$
- $\rho$: fluid density, $kg/m^3$
- $\sigma$: electrical conductivity, $S/m$
- $\tau$: shear stress, $N/m^2$
- $\omega$: frequency of oscillation, $s^{-1}$
APPENDIX

The vectors $R_{1r}$ and $R_{2r}$ in Eq. (56) and Eq. (57) are defined as

$$R_{1r} = a_{or} \frac{\partial^2 u_r}{\partial y^2} + a_{1r} \frac{\partial u_r}{\partial y} + a_{2r} \frac{\partial^\beta u_r}{\partial t^\beta \partial y^2} + a_{3r} \frac{\partial^\beta u_r}{\partial t^\beta \partial y} + a_{4r} \frac{\partial u_r}{\partial y} + a_{5r} \frac{\partial^\beta u_r}{\partial y}$$

where,

$$Nu_r = \frac{\partial u_r}{\partial y} \left( \frac{1}{1 + \theta_e \theta_r} \right) - \lambda_r^\beta \frac{\partial^\beta u_r}{\partial t^\beta \partial y} \frac{\partial}{\partial y} \left( \frac{1}{1 + \theta_e \theta_r} \right) - \frac{1}{1 + \theta_e \theta_r} \frac{\partial^2 u_r}{\partial y^2} - \frac{1}{1 + \theta_e \theta_r} \frac{\partial^\beta u_r}{\partial t^\beta \partial y^2}$$

$$a_{or} = -(1 + \theta_e \theta_r)^{-1}, \quad a_{1r} = \theta_e (1 + \theta_e \theta_r)^{-2} \frac{\partial \theta_r}{\partial y}, \quad a_{2r} = -(1 + \theta_e \theta_r)^{-1}, \quad a_{3r} = \theta_e \lambda_r^\beta (1 + \theta_e \theta_r)^{-2} \frac{\partial \theta_r}{\partial y}$$

$$a_{4r} = \theta_e (1 + \theta_e \theta_r)^{-2} \left( \frac{\partial u_r}{\partial y} + \lambda_r^\beta \frac{\partial^\beta u_r}{\partial t^\beta \partial y} \right),$$

$$a_{5r} = -2 \theta_e^2 (1 + \theta_e \theta_r)^{-2} \left( \frac{\partial u_r}{\partial y} + \lambda_r^\beta \frac{\partial^\beta u_r}{\partial t^\beta \partial y} \right) + \theta_e (1 + \theta_e \theta_r)^{-2} \left( \frac{\partial^2 u_r}{\partial y^2} + \frac{\partial^\beta u_r}{\partial t^\beta \partial y^2} \right)$$

and

$$R_{2r} = b_{or} \frac{\partial u_r}{\partial y} + b_{1r} \frac{\partial^\beta u_r}{\partial t^\beta \partial y} + b_{2r} \theta_r - N\theta r.$$

Here,

$$N\theta_r = -Ec \left[ \frac{1}{1 + \theta_e \theta_r} \left( \frac{\partial u_r}{\partial y} \right)^2 + \lambda_r^\beta \frac{\partial^\beta u_r}{\partial t^\beta \partial y} \frac{\partial u_r}{\partial y} + \frac{\partial u_r}{\partial y} \frac{\partial^\beta u_r}{\partial t^\beta \partial y} \frac{\partial u_r}{\partial y} \right],$$

$$b_{or} = -2Ec (1 + \theta_e \theta_r)^{-1} \frac{\partial u_r}{\partial y} - \lambda_r^\beta (1 + \theta_e \theta_r)^{-1} \frac{\partial^\beta u_r}{\partial t^\beta \partial y} \frac{\partial u_r}{\partial y},$$

$$b_{1r} = -\lambda_r^\beta (1 + \theta_e \theta_r)^{-1} \frac{\partial u_r}{\partial y},$$

$$b_{2r} = \theta_e (1 + \theta_e \theta_r)^{-1} \left( Ec \left( \frac{\partial u_r}{\partial y} \right)^2 + \lambda_r^\beta \frac{\partial^\beta u_r}{\partial t^\beta \partial y} \frac{\partial u_r}{\partial y} \right).$$