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An Investigation of Generalized Fuzzy Integral Ro-Transform

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ABSTRACT

Due of its numerous and significant applications in a variety of industries, fuzzy differential equations have been applied in various fields during the past few decades. Fuzzy integral transforms are the simplest and most widely used mathematical techniques for solving differential, partial, and integral equations, and we presented this paper in order to keep up with the field's rapid development and progress in the area of fuzzy differential equations. Our research will be restricted to the solution of fuzzy differential equations of the first order. Under certain conditions, fuzzy differential equations may be represented as a dynamical system model. In this paper, generalized differentiability of the fuzzy Ro-transform (FRT) and various R-transform features are proven. A generic formula for the nth-order fuzzy derivative is then produced using highly generalized H-differentiability principles, starting with a formula for the third-order fuzzy derivative. The effectiveness of this fuzzy Ro-transform is then demonstrated using a real-world example (the liquid tank system) to highlight the initial value problem. Due of its numerous and significant applications in a variety of industries, fuzzy differential equations have been applied in various fields during the past few decades. Fuzzy integral transforms are the simplest and most widely used mathematical techniques for solving differential, partial, and integral equations, and we presented this paper in order to keep up with the field's rapid development and progress in the area of fuzzy differential equations. Our research will be restricted to the solution of fuzzy differential equations of the first order. Under certain conditions, fuzzy differential equations may be represented as a dynamical system model. In this paper, generalized differentiability of the fuzzy Ro-transform (FRT) and various R-transform features are proven. A generic formula for the nth-order fuzzy derivative is then produced using highly generalized H-differentiability principles, starting with a formula for the third-order fuzzy derivative. The effectiveness of this fuzzy Ro-transform is then demonstrated using a real-world example (the liquid tank system) to highlight the initial value problem.

1. INTRODUCTION

Fuzzy integral transformations, which are a crucial component of the fuzzy analysis theory, have been instrumental in solving fuzzy differential and integral equations over the past few decades. A branch of mathematics called the fuzzy transformation has been used to model physical and chemical processes as well as engineering fields. Differential and integral equations with adequate boundary conditions can be used to model a wide variety of intriguing physical issues. Fourier, Laplace, Sumudu, and others investigated this section. Indeed, Dubois and Prade [1], Puri and Ralescu [2], and Chang and Zadeh [3] were the next to create fuzzy derivative notions. Hasan and Alkiffai wrote a book about fuzzy complex integral transforms that they termed the Ro- transform [4]. The formulas for the third-order fuzzy derivative and the generalized differentiability of the fuzzy Ro-transform (FRT) are found in this paper, and the realistic equation is solved using these formulas. is acquired utilizing concepts of highly generalized H-differentiability arranged as follows in this work. Section 2 provides basic principles. Some fuzzy Ro-transform features are discussed in Section 3. Fuzzy Ro-transform for the third order derivative is found in Section 4. The general formula of the fuzzy Ro-transform was introduced in Section 5. Finally in order in Section 6, a generalization of fuzzy Ro-transform technique (n^{th}-order fuzzy derivative) is addressed with a realistic example contains a liquid tank equation.

2. BASIC CONCEPTS

Definition 2.1 [5]

In parametric form, a fuzzy number is a pair of functions that fulfill the following conditions:

1. $\underset{\sim}{\omega}(\vartheta)$ is a left continuous function in (0,1] bounded non-decreasing and right continuous at 0.
2. $\bar{\omega}(\vartheta)$ is a left continuous function in (0,1] bounded non-increasing, and right continuous at 0.
3. $\underset{\sim}{\omega}(\vartheta) \leq \bar{\omega}(\vartheta), 0 \leq \vartheta \leq 1$.

For $\omega = \underset{\sim}{\omega}(\vartheta), \bar{\omega}(\vartheta)$ and $\nu = \underset{\sim}{\nu}(\vartheta), \bar{\nu}(\vartheta)$, φ>0 we define the add *ition* $\omega \oplus \nu$ subtraction $\omega \ominus \nu$ and scalar multiplication by φ>0 as follows:

(a) *Addition*: $\omega \oplus \nu = \underline{\omega}(\vartheta) + \underline{\nu}(\vartheta), \bar{\omega}(\vartheta) + \bar{\nu}(\vartheta)$.

(b) *Subtraction*: $\omega \ominus \nu = \underline{\omega}(\vartheta) - \bar{\nu}(\vartheta), \bar{\omega}(\vartheta) - \underline{\nu}(\vartheta)$.

(c) *Scalar* multiplication: $\varphi \odot \omega = \begin{Bmatrix} (\varphi\underline{\omega}, \varphi\bar{\omega}) & \varphi \geq 0 \\ (\varphi\bar{\omega}, \varphi\underline{\omega}) & \varphi < 0 \end{Bmatrix}$.

Definition 2.2 [6]

Assume that $\omega, \nu \in E$. If there exists $\rho \in E$ such that $\omega + \nu = \rho$ then ρ is called the Hukuhara difference of ω and ν and it is identified by $\omega \ominus \nu$.

Definition 2.3 [7]

Let $\varphi(\varpi): (a,b) \to E$; be continuous fuzzy-valued function, φ is strongly generalized difference at $\varpi_0 \in (a,b)$. If there was an element $\varphi'(\varpi_0) \in E$ then:

1. For all $\hbar > 0$ small enough $\exists \varphi(\varpi_0 + \hbar) \ominus \varphi(\varpi_0), \exists \varphi(\varpi_0) \ominus \varphi(\varpi_0 - \hbar)$ and the limit is:
$$\varphi'(\varpi_0) = \lim_{\hbar \to 0^+} \frac{\varphi(\varpi_0 + \hbar) \ominus \varphi(\varpi_0)}{\hbar} = \lim_{\hbar \to 0^+} \frac{\varphi(\varpi_0) \ominus \varphi(\varpi_0 - \hbar)}{\hbar}?$$
Or
2. For all $\hbar > 0$ small enough $\exists \varphi(\varpi_0) \ominus \varphi(\varpi_0 + \hbar), \exists \varphi(\varpi_0 - \hbar) \ominus \varphi(\varpi_0)$ and the limit is:
$$\varphi'(\varpi_0) = \lim_{\hbar \to 0^+} \frac{\varphi(\varpi_0) \ominus \varphi(\varpi_0 + \hbar)}{-\hbar} = \lim_{\hbar \to 0^+} \frac{\varphi(\varpi_0 - \hbar) \ominus \varphi(\varpi_0)}{-\hbar}$$
Or
3. For all $\hbar > 0$ small enough $\exists \varphi(\varpi_0 + \hbar) \ominus \varphi(\varpi_0), \exists \varphi(\varpi_0 - \hbar) \ominus \varphi(\varpi_0)$ and the limit is:
$$\varphi'(\varpi_0) = \lim_{\hbar \to 0^+} \frac{\varphi(\varpi_0 + \hbar) \ominus \varphi(\varpi_0)}{\hbar} = \lim_{\hbar \to 0^+} \frac{\varphi(\varpi_0 - \hbar) \ominus \varphi(\varpi_0)}{-\hbar}.$$
Or
For all $\hbar > 0$ small enough $\exists \varphi(\varpi_0) \ominus \varphi(\varpi_0 + \hbar), \exists \varphi(\varpi_0 - \hbar) \ominus \varphi(\varpi_0)$ and the limit is:
$$\varphi'(\varpi_0) = \lim_{\hbar \to 0^+} \frac{\varphi(\varpi_0) \ominus \varphi(\varpi_0 + \hbar)}{-h} = \lim_{\hbar \to 0^+} \frac{\varphi(\varpi_0 - \hbar) \ominus \varphi(\varpi_0)}{\hbar}.$$

Theorem 2.4 [5]

Let $\varphi: R \to E$ be a function and indicate $\varphi(\varpi) = \left(\underline{\varphi}(\varpi;\vartheta), \bar{\varphi}(\varpi;\vartheta)\right)$ foreach $\vartheta \in [0,1]$. Then:

1- If φ is the 1st form then $\underline{\varphi}(\varpi;\vartheta)$ and $\bar{\varphi}(\varpi;\vartheta)$ are differentiable functions and $\varphi'(\varpi) = \underline{\varphi}(\varpi;\vartheta), \bar{\varphi}(\varpi;\vartheta)$.

2- If φ is the 2nd form, then $\underline{\varphi}(\varpi;\vartheta)$ and $\bar{\varphi}(\varpi;\vartheta)$ are differentiable functions and $\varphi'(\varpi) = \bar{\varphi}(\varpi;\vartheta), \underline{\varphi}(\varpi;\vartheta)$.

Theorem 2.5 [8]

Let $\varphi(\varpi;\vartheta): R \to \mathrm{E}$ and it is represented by $[\underline{\varphi}(\varpi;\vartheta), \bar{\varphi}(\varpi;\vartheta)]$. For any fixed $\vartheta \in 0{,}1$ assume that $\underline{\varphi}(\varpi;\vartheta)$ and $\bar{\varphi}(\varpi;\vartheta)$ are Riemann-integrable functions on [a, b] for every $b \geq a$, there are two positive functions $\underline{M}_\vartheta$ and $\overline{M}_\vartheta$ such that $\int_a^b |\underline{f}(\varpi;\vartheta)|\, d\varpi \leq \underline{M}_\vartheta$ and $\int_a^b |\bar{\varphi}(\varpi;\vartheta)|\, d\varpi \leq \overline{M_\vartheta}$. Then, $f(\varpi)$ is an improper fuzzy Riemann-integrable function on $[a, \infty)$. Furthermore, we have:

$$\int_a^\infty \varphi(\varpi)\, d\varpi = \left[\int_a^\infty \underline{\varphi}(\varpi;\vartheta)\, d\varpi, \int_a^\infty \bar{\varphi}(\varpi;\vartheta)\, d\varpi\right]$$

Definition 2.6 [4]

Let $\varphi(\varpi)$ be a continuous fuzzy-valued function. Suppose that $\upsilon^2 e^{-\left(i\sqrt[\Omega]{\upsilon}\right)\varpi} \odot \varphi(\varpi) d\varpi$ is an improper fuzzy Riemann-integrable on [0, ∞), then $\upsilon^2 \int_0^\infty e^{-\left(i\sqrt[\Omega]{\upsilon}\right)\varpi} \varphi(\varpi) d\varpi$ is called fuzzy Ro-transform and it is denoted as:

$$\tilde{\Re}(\upsilon) = R[\varphi(\varpi)] = \upsilon^2 \int_0^\infty e^{-\left(i\sqrt[\Omega]{\upsilon}\right)\varpi} \varphi(\varpi) d\varpi, \qquad \mathrm{n} \geq 1.$$

From Theorem 2:

$$\upsilon^2 \int_0^\infty e^{-\left(i\sqrt[\Omega]{\upsilon}\right)\varpi} \varphi(\varpi) d\varpi = \left(\upsilon^2 \int_0^\infty e^{-\left(i\sqrt[\Omega]{\upsilon}\right)\varpi} \underline{\varphi}(\varpi;\vartheta) d\varpi\right), \left(\upsilon^2 \int_0^\infty e^{-\left(i\sqrt[\Omega]{\upsilon}\right)\varpi} \bar{\varphi}(\varpi;\vartheta) d\varpi\right).$$

By the classical definition of Ro-transform:

$$\gamma\left[\underline{\varphi}(\varpi;\vartheta)\right] = \upsilon^2 \int_0^\infty e^{-\left(i\sqrt[\Omega]{\upsilon}\right)\varpi} \underline{\varphi}(\varpi;\vartheta) d\varpi\ , \quad \gamma\left[\bar{\varphi}(\varpi;\vartheta)\right] = \upsilon^2 \int_0^\infty e^{-\left(i\sqrt[\Omega]{\upsilon}\right)\varpi} \bar{\varphi}(\varpi;\vartheta) d\varpi.$$

So:

$$R\left[\varphi(\varpi;\vartheta)\right] = \gamma\left[\underline{\varphi}(\varpi;\vartheta)\right], \gamma\left[\bar{\varphi}(\varpi;\vartheta)\right].$$

Theorem 2.7 [9]

Let $\varphi(\varpi), \varphi'(\varpi), \ldots, \varphi^{n-1}(\varpi)$ be piecewisecontinuous fuzzy valuedfunctions on [0, ∞).

Let $\varphi^{(i_1)}(\varpi), \varphi^{(i_2)}(\varpi), \ldots, \varphi^{(i_\eta)}(\varpi)$ are the 2nd form differentiable functions for $0 \leq i_1 \leq i_2 \leq \cdots \leq i_\eta \leq n-1$ and $\varphi^{(\rho)}$ is the 1st form differentiable function for $\rho \neq i_j$, $j = 1, 2, \ldots, \eta$, then:

1. If η is an even number, then:

$$L\left(\varphi^{(n)}(\varpi)\right) = \zeta^n L(\varphi(\varpi)) \ominus \zeta^{n-1} \varphi(0) \odot \sum_{\kappa=1}^{n-1} \zeta^{n-(\kappa+1)} \varphi^{(\kappa)}(0). \tag{1}$$

such that:

$$\odot = \begin{cases} \ominus, \text{ if the number of the functions in the } 2^{nd} \text{ form} \\ \text{differetiable functions among } i_1, \ldots, i_\kappa \text{ is an even number.} \\ -, \text{ if the number of the functions in the } 2^{nd} \text{ form} \\ \text{differetiable functions among } i_1, \ldots, i_\kappa \text{ is an odd number.} \end{cases}$$

2. If η is an odd number, then:

$$L\left(\varphi^{(n)}(\varpi)\right)=-\zeta^{n-1}\varphi(0)\ominus\left(-\zeta^{n}\right)$$
$$L(\varphi(\varpi))\odot\sum_{\kappa=1}^{n-1}\zeta^{n-(\kappa+1)}\varphi^{(\kappa)}(0). \tag{2}$$

such that

$$\odot=\begin{cases}\ominus, \text{ if the number of the function in the } 2^{nd} \text{ form} \\ \text{among } i_1,...,i_\kappa \text{ is an odd number.} \\ -, \text{ if the number of the function in the } 2^{nd} \text{ form} \\ \text{among } i_1,...,i_\kappa \text{ is an even number.}\end{cases}$$

Theorem 2.8 [4]

Suppose that $\varphi'(\varpi)$ is continuous fuzzy-valued function and $f(\varpi)$ the primitiveof $\varphi'(\varpi)$on [0, ∞), then:

1. If φ is the 1st form differentiable function.

$$R\left[\varphi'(\varpi)\right]=\left(i\sqrt[\Omega]{\upsilon}\right)R\left[\varphi(\varpi)\right]\ominus\upsilon^2\varphi(0).$$

2. If φ is the 2nd form differentiable function.

$$R\left[\varphi'(\varpi)\right]=-\upsilon^2\varphi(0)\ominus\left(-i\sqrt[\Omega]{\upsilon}\right)R\left[\varphi(\varpi)\right].$$

Theorem 2.9 [4]

Let, $\varphi(\varpi)$, $\varphi'(\varpi)$ are continuous fuzzy-valued functions on [0, ∞), fuzzy derivative of fuzzy Ro-transform from second order will be as following:

1. If φ, φ' are the 1st form differentiable functions then:

$$R\left[\varphi''(\varpi)\right]=\left(i\sqrt[\Omega]{\upsilon}\right)^2R\left[\varphi(\varpi)\right]\ominus\upsilon^2\left(i\sqrt[\Omega]{\upsilon}\right)\varphi(0)\ominus\upsilon^2\varphi'(0).$$

2. If φ is the 1st form differentiable function and φ' is the 2nd form differentiable function then:

$$R\left[\varphi''(\varpi)\right]=-\upsilon^2\left(i\sqrt[\Omega]{\upsilon}\right)\varphi(0)\ominus\left(-i\sqrt[\Omega]{\upsilon}\right)^2R\left[\varphi(\varpi)\right]\ominus\upsilon^2\varphi'(0).$$

3. If φ is the 2nd form differentiable function and φ' is the 1st form differentiable function then:

$$R\left[\varphi''(\varpi)\right]=-\upsilon^2\left(i\sqrt[\Omega]{\upsilon}\right)\varphi(0)\ominus\left(-i\sqrt[\Omega]{\upsilon}\right)^2R\left[\varphi(\varpi)\right]-\upsilon^2\varphi'(0).$$

4. If φ,φ' are the 2nd form differentiable functions then:

$$R\left[\varphi''(\varpi)\right]=\left(i\sqrt[\Omega]{\upsilon}\right)^2R\left[\varphi(\varpi)\right]\ominus\upsilon^2\left(i\sqrt[\Omega]{\upsilon}\right)\varphi(0)-\upsilon^2\varphi'(0).$$

3. SOME PROPERTIES OF FUZZY RO-TRANSFORM

Theorem 3.1

Let $\varphi(\varpi)$ be fuzzy function, υ^2 be positive real function and $i\sqrt[\Omega]{\upsilon}$ be positive complex function, then the derivatives of $\varphi(\varpi)$ for nth –order will be:

1. $R\{\varpi\varphi(\varpi)\}=-\dfrac{v^2}{i\sqrt[\Omega]{\upsilon}}\left(\dfrac{R(\upsilon)}{\upsilon^2}\right)'.$

2. $R\left\{\varpi^2\varphi(\varpi)\right\}=(-1)^2\dfrac{v^2}{i\sqrt[\Omega]{\upsilon}}\left(\dfrac{1}{i\sqrt[\Omega]{\upsilon}}\left(\dfrac{R(\upsilon)}{\upsilon^2}\right)'\right)'.$

3. $R\left\{\varpi^n\varphi(\varpi)\right\}=(-1)^n\dfrac{v^2}{i\sqrt[\Omega]{\upsilon}}\left(\dfrac{1}{i\sqrt[\Omega]{\upsilon}}\left(\dfrac{1}{i\sqrt[\Omega]{\upsilon}}\left(\cdots\left(\dfrac{1}{i\sqrt[\Omega]{\upsilon}}\left(\dfrac{R(\upsilon)}{\upsilon^2}\right)'\right)'\right)'\right)'\cdots\right)'.$

Proof:

1. Since:

$$R\{\varphi(\varpi),\upsilon\}=\upsilon^2\int_0^\infty\underline{\varphi}(\varpi;\vartheta)e^{-\left(i\sqrt[\Omega]{\upsilon}\right)\varpi}d\varpi,\upsilon^2\int_0^\infty\overline{\varphi}(\varpi;\vartheta)e^{-\left(i\sqrt[\Omega]{\upsilon}\right)\varpi}d\varpi$$
$$\frac{R\{\varphi(\varpi),\upsilon\}}{\upsilon^2}=\int_0^\infty\underline{\varphi}(\varpi;\vartheta)e^{-\left(i\sqrt[\Omega]{\upsilon}\right)\varpi}d\varpi,\int_0^\infty\overline{\varphi}(\varpi;\vartheta)e^{-\left(i\sqrt[\Omega]{\upsilon}\right)\varpi}d\varpi \tag{3}$$

Derivative Eq. (3) with respect to υ, to get:

$$\left(\frac{R\{\varphi(\varpi),\upsilon\}}{\upsilon^2}\right)'=\frac{d}{d\upsilon}\left[\int_0^\infty\underline{\varphi}(\varpi;\vartheta)e^{-\left(i\sqrt[\Omega]{\upsilon}\right)\varpi}d\varpi,\int_0^\infty\overline{\varphi}(\varpi;\vartheta)e^{-\left(i\sqrt[\Omega]{\upsilon}\right)\varpi}d\varpi\right]$$
$$\left(\frac{R\{\varphi(\varpi),\upsilon\}}{\upsilon^2}\right)'=-\left(i\sqrt[\Omega]{\upsilon}\right)\int_0^\infty\varpi\underline{\varphi}(\varpi;\vartheta)e^{-\left(i\sqrt[\Omega]{\upsilon}\right)\varpi}d\varpi,$$
$$-\left(\sqrt[\Omega]{\upsilon}\right)\int_0^\infty\varpi\overline{\varphi}(\varpi;\vartheta)e^{-\left(i\sqrt[\Omega]{\upsilon}\right)\varpi}d\varpi$$

From Eq. (3):

$$\left(\frac{R\{\varphi(\varpi),\upsilon\}}{\upsilon^2}\right)'=-\left(i\sqrt[\Omega]{\upsilon}\right)\frac{\gamma\{\varpi\underline{\varphi}(\varpi;\vartheta),\upsilon\}}{\upsilon^2},\ -\left(i\sqrt[\Omega]{\upsilon}\right)\frac{\gamma\{\varpi\overline{\varphi}(\varpi;\vartheta),\upsilon\}}{\upsilon^2}.$$
$$\left(\frac{R\{\varphi(\varpi),\upsilon\}}{\upsilon^2}\right)'=-\left(i\sqrt[\Omega]{\upsilon}\right)\frac{R\{\varpi\varphi(\varpi),\upsilon\}}{\upsilon^2}$$

Then: $R\{\varpi\varphi(\varpi),\upsilon\}=-\dfrac{\upsilon^2}{\left(i\sqrt[\Omega]{\upsilon}\right)}\left(\dfrac{R\{\varphi(\varpi),\upsilon\}}{\upsilon^2}\right)'.$

2. From the first part, we have:

$$R\{\varpi\varphi(\varpi),\upsilon\}=-\frac{\upsilon^2}{\left(i\sqrt[\Omega]{\upsilon}\right)}\left(\frac{R\{\varphi(\varpi),\upsilon\}}{\upsilon^2}\right)'.$$

Taking the derivative for both sides of the above equation:

$$-\left(i\sqrt[\Omega]{\upsilon}\right)\frac{1}{\upsilon^2}\int_0^\infty \varpi^2\underline{\varphi}(\varpi;\vartheta)e^{-\left(i\sqrt[\Omega]{\upsilon}\right)\varpi}d\varpi,-\left(i\sqrt[\Omega]{\upsilon}\right)\int_0^\infty \varpi^2\overline{\varphi}(\varpi;\vartheta)e^{-\left(i\sqrt[\Omega]{\upsilon}\right)\varpi}d\varpi.$$

$$=\left(-\frac{\upsilon^2}{\left(i\sqrt[\Omega]{\upsilon}\right)}\left(\frac{R\{\varphi(\varpi),\upsilon\}}{\upsilon^2}\right)'\right)'$$

$$-\frac{i\sqrt[\Omega]{\upsilon}}{\upsilon^2}\gamma\left\{\varpi^2\underline{\varphi}(\varpi;\vartheta)\right\},-\frac{\left(i\sqrt[\Omega]{\upsilon}\right)}{\upsilon^2}\gamma\left\{\varpi^2\overline{\varphi}(\varpi;\varpi)\right\}$$

$$=\left(-\frac{\upsilon^2}{\left(i\sqrt[\Omega]{\upsilon}\right)}\left(\frac{R\{\varphi(\varpi),\upsilon\}}{\upsilon^2}\right)'\right)'.$$

Thus:

$$R\left\{\varpi^2\varphi(\varpi)\right\}=(-1)^2\frac{\upsilon^2}{\left(i\sqrt[\Omega]{\upsilon}\right)}\left(-\frac{\upsilon^2}{\left(i\sqrt[\Omega]{\upsilon}\right)}\left(\frac{R\{\varphi(\varpi),\upsilon\}}{\upsilon^2}\right)'\right)'.$$

3. Since from the second part, we have:

$$R\left\{\varpi^2\varphi(\varpi)\right\}=(-1)^2\frac{\upsilon^2}{\left(i\sqrt[\Omega]{\upsilon}\right)}\left(-\frac{\upsilon^2}{\left(i\sqrt[\Omega]{\upsilon}\right)}\left(\frac{R\{\varphi(\varpi),\upsilon\}}{\upsilon^2}\right)'\right)',$$

Derivative both sides of last equation for (n-2) times, to get:

$$R\left\{\varpi^n\varphi(\varpi)\right\}=(-1)^n\frac{v^2}{i\sqrt[\Omega]{\upsilon}}\left(\frac{1}{i\sqrt[\Omega]{\upsilon}}\left(\frac{1}{i\sqrt[\Omega]{\upsilon}}\left(\cdots\left(\frac{1}{i\sqrt[\Omega]{\upsilon}}\left(\frac{R(\upsilon)}{\upsilon^2}\right)'\right)'\right)'\right)'\cdots\right)'$$

Theorem 3.2

Let υ^2 and $i\sqrt[\Omega]{\upsilon}$ are differentiable functions, such that the function $\varphi(\varpi)$ and $(\upsilon^2)'\neq 0$ are fuzzy functions then:

$$R\left\{\varpi\varphi^{(n)}(\varpi)\right\}=-\frac{\upsilon^2}{i\sqrt[\Omega]{\upsilon}}\frac{d}{d\upsilon}\left(\frac{R\left(\varphi^{(n)}(\varpi)\right)}{\upsilon^2}\right).$$

Proof: Since:

$$R\left\{\varphi^{(n)}(\varpi),\upsilon\right\}=\upsilon^2\int_0^\infty\underline{\varphi}^{(n)}(\varpi;\vartheta)e^{-\left(i\sqrt[\Omega]{\upsilon}\right)\varpi}d\varpi,$$

$$\upsilon^2\int_0^\infty\overline{\varphi}^{(n)}(\varpi;\vartheta)e^{-\left(i\sqrt[\Omega]{\upsilon}\right)\varpi}d\varpi \qquad (4)$$

$$\frac{R\left\{\varphi^{(n)}(\varpi),\upsilon\right\}}{\upsilon^2}=\int_0^\infty\underline{\varphi}^{(n)}(\varpi;\vartheta)e^{-\left(i\sqrt[\Omega]{\upsilon}\right)\varpi}d\varpi,\int_0^\infty\overline{\varphi}^{(n)}(\varpi;\vartheta)e^{-\left(i\sqrt[\Omega]{\upsilon}\right)\varpi}d\varpi$$

Derivative Eq. (4) with respect to υ, then:

$$\frac{d}{d\upsilon}\left(\frac{R\left\{\varphi^{(n)}(\varpi),\upsilon\right\}}{\upsilon^2}\right)=\frac{d}{d\upsilon}\begin{bmatrix}\int_0^\infty\underline{\varphi}^{(n)}(\varpi;\vartheta)e^{-\left(i\sqrt[\Omega]{\upsilon}\right)\varpi}d\varpi,\\ \int_0^\infty\overline{\varphi}^{(n)}(\varpi;\vartheta)e^{-\left(i\sqrt[\Omega]{\upsilon}\right)\varpi}d\varpi\end{bmatrix},$$

From Eq. (4):

$$\frac{d}{d\upsilon}\left(\frac{R\left\{\varphi^{(n)}(\varpi),\upsilon\right\}}{\upsilon^2}\right)=-\left(i\sqrt[\Omega]{\upsilon}\right)\frac{\gamma\left\{\varpi\underline{\varphi}^{(n)}(\varpi;\vartheta),\upsilon\right\}}{\upsilon^2},-\left(i\sqrt[\Omega]{\upsilon}\right)\frac{\gamma\left\{\varpi\overline{\varphi}^{(n)}(\varpi;\vartheta),\upsilon\right\}}{\upsilon^2}.$$

$$\frac{d}{d\upsilon}\left(\frac{R\left\{\varphi^{(n)}(\varpi),\upsilon\right\}}{\upsilon^2}\right)=-\left(i\sqrt[\Omega]{\upsilon}\right)\frac{R\left\{\varpi\varphi^{(n)}(\varpi),\upsilon\right\}}{\upsilon^2},$$

$$R\left\{\varpi\varphi^{(n)}(\varpi),\upsilon\right\}=-\frac{\upsilon^2}{\left(i\sqrt[\Omega]{\upsilon}\right)}\frac{d}{d\upsilon}\left(\frac{R\left\{\varphi^{(n)}(\varpi),\upsilon\right\}}{\upsilon^2}\right).$$

$$R\left\{\varpi\varphi^{(n)}(\varpi),\upsilon\right\}=-\frac{\upsilon^2}{\left(i\sqrt[\Omega]{\upsilon}\right)}\frac{d}{d\upsilon}\left(\frac{R\left\{\varphi^{(n)}(\varpi),\upsilon\right\}}{\upsilon^2}\right).$$

4. FUZZY RO-TRANSFORM FOR THIRD ORDER DERIVATIVE

Theorem 4.1

Let that, $\varphi(\varpi)$, $\varphi'(\varpi)$ and $\varphi''(\varpi)$ are continuous fuzzy-valued function on [0, ∞), fuzzy derivative of fuzzy Ro-transform from third order will be as following:

1. If there are φ and φ', φ'' from the 1st form differentiable functions then:

$$R\left[\varphi'''(\varpi)\right]=\left(i\sqrt[\Omega]{\upsilon}\right)^3R\left[\varphi(\varpi)\right]\ominus\upsilon^2\left(i\sqrt[\Omega]{\upsilon}\right)^2\varphi(0)\ominus\upsilon^2\left(i\sqrt[\Omega]{\upsilon}\right)\varphi'(0)\ominus\upsilon^2\varphi''(0).$$

2. If there are φ', φ'' from the 1st form differentiable functions and φ is the 2nd form differentiable function then:

$$R\left[\varphi'''(\varpi)\right]=-\upsilon^2\left(i\sqrt[\Omega]{\upsilon}\right)^2\varphi(0)\ominus\left(-i\sqrt[\Omega]{\upsilon}\right)^3R\left[\varphi(\varpi)\right]-\upsilon^2\left(i\sqrt[\Omega]{\upsilon}\right)\varphi'(0)-\upsilon^2\varphi''(0).$$

3. If there are φ, φ'' from the 1st form differentiable functions and is the φ' 2nd form differentiable function then:

$$R\left[\varphi'''(\varpi)\right]=-\upsilon^2\left(i\sqrt[\Omega]{\upsilon}\right)^2\varphi(0)\ominus\left(-i\sqrt[\Omega]{\upsilon}\right)^3R\left[\varphi(\varpi)\right]\ominus\upsilon^2\left(i\sqrt[\Omega]{\upsilon}\right)\varphi'(0)\ominus\upsilon^2\varphi''(0).$$

4. If there are φ, φ' from the 1st form differentiable functions and φ'' is the 2nd function with form differentiation then:

$$R\left[\varphi'''(\varpi)\right]=-\upsilon^2\left(i\sqrt[\Omega]{\upsilon}\right)^2\varphi(0)\ominus\left(-i\sqrt[\Omega]{\upsilon}\right)^3R\left[\varphi(\varpi)\right]$$
$$-\upsilon^2\left(i\sqrt[\Omega]{\upsilon}\right)\varphi'(0)\ominus\upsilon^2\varphi''(0).$$

5. If there are φ, φ' from the 2nd form differentiable functions and φ'' is the 1st form differentiable function then:

$$R\left[\varphi'''(\varpi)\right]=\left(i\sqrt[\Omega]{\upsilon}\right)^3R\left[\varphi(\varpi)\right]\ominus\upsilon^2\left(i\sqrt[\Omega]{\upsilon}\right)^2\varphi(0)$$
$$-\upsilon^2\left(i\sqrt[\Omega]{\upsilon}\right)\varphi'(0)-\upsilon^2\varphi''(0).$$

6. If there are φ, φ'' from the 2nd form differentiable functions and φ' is the 1st form differentiable function then:

$$R\left[\varphi'''(\varpi)\right]=\left(i\sqrt[\Omega]{\upsilon}\right)^3R\left[\varphi(\varpi)\right]\ominus\upsilon^2\left(i\sqrt[\Omega]{\upsilon}\right)^2\varphi(0)$$
$$\ominus\upsilon^2\left(i\sqrt[\Omega]{\upsilon}\right)\varphi'(0)-\upsilon^2\varphi''(0).$$

7. If there are φ', φ'' from the 2nd form differentiable functions and φ is the 1st form differentiable function then:

$$R\left[\varphi'''(\varpi)\right]=\left(i\sqrt[\Omega]{\upsilon}\right)^3R\left[\varphi(\varpi)\right]\ominus\upsilon^2\left(i\sqrt[\Omega]{\upsilon}\right)^2\varphi(0)$$
$$-\upsilon^2\left(i\sqrt[\Omega]{\upsilon}\right)\varphi'(0)\ominus\upsilon^2\varphi''(0).$$

8. If there are φ, φ', φ'' from the 2nd differentiable functions form then:

$$R\left[\varphi'''(\varpi)\right]=-\upsilon^2\left(i\sqrt[\Omega]{\upsilon}\right)^2\varphi(0)\ominus\left(-i\sqrt[\Omega]{\upsilon}\right)^3R\left[\varphi(\varpi)\right]$$
$$\ominus\upsilon^2\left(i\sqrt[\Omega]{\upsilon}\right)\varphi'(0)-\upsilon^2\varphi''(0).$$

Proof: We will prove four cases:
Case 1

1. Since φ, φ', φ'' are the 1st form differentiable functions so for any arbitrary $\vartheta\in[0,1]$, from Theorem 2.4/1:

$$R\left[\varphi'''(\varpi)\right]=\gamma\left[\underline{\varphi}'''(\varpi,\vartheta)\right],\gamma\left[\overline{\varphi}'''(\varpi,\vartheta)\right] \tag{5}$$

By classical derivative of Ro-transform:

$$\begin{aligned}\gamma\left[\underline{\varphi}'''(\varpi,\vartheta)\right]&=\left(i\sqrt[\Omega]{\upsilon}\right)^3\gamma\left[\underline{\varphi}(\varpi,\vartheta)\right]-\upsilon^2\left(i\sqrt[\Omega]{\upsilon}\right)^2\underline{\varphi}(0,\vartheta)\\&-\upsilon^2\left(i\sqrt[\Omega]{\upsilon}\right)\underline{\varphi'(0,\vartheta)}-\upsilon^2\underline{\varphi''(0,\vartheta)},\\\gamma\left[\overline{\varphi}'''(\varpi,\vartheta)\right]&=\left(i\sqrt[\Omega]{\upsilon}\right)^3\gamma\left[\overline{\varphi}(\varpi,\vartheta)\right]-\upsilon^2\left(i\sqrt[\Omega]{\upsilon}\right)^2\overline{\varphi}(0,\vartheta)\\&-\upsilon^2\left(i\sqrt[\Omega]{\upsilon}\right)\overline{\varphi'(0,\vartheta)}-\upsilon^2\overline{\varphi''(0,\vartheta)}.\end{aligned} \tag{6}$$

Substitute (6) in (5) and since $\varphi'(\varpi)$, $\varphi'^{(\varpi)}$ are the 1st form differentiable function and by Theorem 2.5:

$$R\left[\varphi'''(\varpi)\right]=\left(i\sqrt[\Omega]{\upsilon}\right)^3R\left[\varphi(\varpi)\right]\ominus\upsilon^2\left(i\sqrt[\Omega]{\upsilon}\right)^2\varphi(0)$$
$$\ominus\upsilon^2\left(i\sqrt[\Omega]{\upsilon}\right)\varphi'(0)\ominus\upsilon^2\varphi''(0).$$

2. Since φ' and φ'' are the 1st form differentiable functions and φ is the 2nd function with form differentiation, from Theorem 2.4/2:

$$R\left[\varphi''(\varpi)\right]=\gamma\left[\overline{\varphi}'''(\varpi,\vartheta)\right],\gamma\left[\underline{\varphi}'''(\varpi,\vartheta)\right] \tag{7}$$

By classical derivative of Ro-transform:

$$\begin{aligned}\gamma\left[\underline{\varphi}'''(\varpi,\vartheta)\right]&=\left(i\sqrt[\Omega]{\upsilon}\right)^3\gamma\left[\underline{\varphi}(\varpi,\vartheta)\right]-\upsilon^2\left(i\sqrt[\Omega]{\upsilon}\right)^2\underline{\varphi}(0,\vartheta)\\&-\upsilon^2\left(i\sqrt[\Omega]{\upsilon}\right)\underline{\varphi'(0,\vartheta)}-\upsilon^2\underline{\varphi''(0,\vartheta)},\\\gamma\left[\overline{\varphi}'''(\varpi,\vartheta)\right]&=\left(i\sqrt[\Omega]{\upsilon}\right)^3\gamma\left[\overline{\varphi}(\varpi,\vartheta)\right]-\upsilon^2\left(i\sqrt[\Omega]{\upsilon}\right)^2\overline{\varphi}(0,\vartheta)\\&-\upsilon^2\left(i\sqrt[\Omega]{\upsilon}\right)\overline{\varphi'(0,\vartheta)}-\upsilon^2\overline{\varphi''(0,\vartheta)}.\end{aligned} \tag{8}$$

Substitute (8) in (7) and since $\varphi'(\varpi)$, $\varphi'^{(\varpi)}$ are 1st function with form differentiation and by Theorem 2.5:

$$R\left[\varphi'''(\varpi)\right]=-\upsilon^2\left(i\sqrt[\Omega]{\upsilon}\right)^2\varphi(0)\ominus\left(i\sqrt[\Omega]{\upsilon}\right)^3R\left[\varphi(\varpi)\right]$$
$$-\upsilon^2\left(i\sqrt[\Omega]{\upsilon}\right)\varphi'(0)-\upsilon^2\varphi''(0).$$

3. Since φ, φ' are the 2nd form differentiable functions and is the φ'' 1st form differentiable function, from Theorem 2.4/1:

$$R\left[\varphi'''(\varpi)\right]=\gamma\left[\underline{\varphi}'''(\varpi,\vartheta)\right],\gamma\left[\overline{\varphi}'''(\varpi,\vartheta)\right].$$

By Theorem 6/3:

$$\begin{aligned}\gamma\left[\underline{\varphi}'''(\varpi,\vartheta)\right]&=\left(i\sqrt[\Omega]{\upsilon}\right)^3\gamma\left[\underline{\varphi}(\varpi,\vartheta)\right]-\upsilon^2\left(i\sqrt[\Omega]{\upsilon}\right)^2\underline{\varphi}(0,\vartheta)\\&-\upsilon^2\left(i\sqrt[\Omega]{\upsilon}\right)\underline{\varphi'(0,\vartheta)}-\upsilon^2\underline{\varphi''(0,\vartheta)},\\\gamma\left[\overline{\varphi}'''(\varpi,\vartheta)\right]&=\left(i\sqrt[\Omega]{\upsilon}\right)^3\gamma\left[\overline{\varphi}(\varpi,\vartheta)\right]-\upsilon^2\left(i\sqrt[\Omega]{\upsilon}\right)^2\overline{\varphi}(0,\vartheta)\\&-\upsilon^2\left(i\sqrt[\Omega]{\upsilon}\right)\overline{\varphi'(0,\vartheta)}-\upsilon^2\overline{\varphi''(0,\vartheta)}.\end{aligned}$$

Then:

$$R\left[\varphi'''(\varpi)\right]=\left(i\sqrt[\Omega]{\upsilon}\right)^3R\left[\varphi(\varpi)\right]\ominus\upsilon^2\left(i\sqrt[\Omega]{\upsilon}\right)^2\varphi(0)$$
$$-\upsilon^2\left(i\sqrt[\Omega]{\upsilon}\right)\varphi'(0)-\upsilon^2\varphi''(0).$$

4. Since φ, φ', φ'' are the 2nd form differentiable functions, from Theorem 2.4/2:

$$R\left[\varphi'''(\varpi)\right]=\gamma\left[\overline{\varphi}'''(\varpi,\vartheta)\right],\gamma\left[\underline{\varphi}'''(\varpi,\vartheta)\right]$$

By classical derivative of Ro-transform:

$$\gamma\left[\underline{\varphi}'''(\varpi,\vartheta)\right]=\left(i\sqrt[\Omega]{\upsilon}\right)^{3}\gamma\left[\underline{\varphi}(\varpi,\vartheta)\right]-\upsilon^{2}\left(i\sqrt[\Omega]{\upsilon}\right)^{2}\underline{\varphi}(0,\vartheta)$$
$$-\upsilon^{2}\left(i\sqrt[\Omega]{\upsilon}\right)\underline{\varphi'(0,\vartheta)}-\upsilon^{2}\underline{\varphi''(0,\vartheta)}.$$
$$\gamma\left[\bar{\varphi}'''(\varpi,\vartheta)\right]=\left(i\sqrt[\Omega]{\upsilon}\right)^{3}\gamma\left[\bar{\varphi}(\varpi,\vartheta)\right]-\upsilon^{2}\left(i\sqrt[\Omega]{\upsilon}\right)^{2}\bar{\varphi}(0,\vartheta)$$
$$-\upsilon^{2}\left(i\sqrt[\Omega]{\upsilon}\right)\overline{\varphi'(0,\vartheta)}-\upsilon^{2}\overline{\varphi''(0,\vartheta)},$$

Then:

$$R\left[\varphi'''(\varpi)\right]=-\upsilon^{2}\left(i\sqrt[\Omega]{\upsilon}\right)^{2}\varphi(0)\ominus\left(-i\sqrt[\Omega]{\upsilon}\right)^{3}R\left[\varphi(\varpi)\right]$$
$$\ominus\upsilon^{2}\left(i\sqrt[\Omega]{\upsilon}\right)\varphi'(0)-\upsilon^{2}\varphi''(0).$$

5. FUZZY RO-TRANSFORM FOR FUZZY NTH- ORDER DERIVATIVE

Theorem 5.1

Assume that $\varphi(\varpi),\varphi'(\varpi),\ldots,\varphi^{n-1}(\varpi)$ are continuous fuzzy valued functions on [0, ∞) and $\varphi^{(n)}(\varpi)$ is piecewise continuous fuzzy-valued function on [0, ∞). $\varphi^{(i_1)}(\varpi),\varphi^{(i_2)}(\varpi),\ldots,\varphi^{(i_\varepsilon)}(\varpi)$ are the 2nd form differentiable functions for $0\le i_1\le i_2\ldots\le i_\varepsilon\le n-1$ and $\varphi^{(p)}$is the 1st form differentiable function for $p\neq i_j$, $j=1,2,\ldots,\varepsilon$, and if the ϑ-cut is fuzzy-valued function $\varphi(\varpi)$ is called by $\varphi(\varpi)=\left[\underline{\varphi}(\varpi;\vartheta),\bar{\varphi}(\varpi;\vartheta)\right]$, then:

1. If ε is an even number, then:

$$R\left(\varphi^{(n)}(\varpi)\right)=\left(i\sqrt[\Omega]{\upsilon}\right)^{n}R\left(\varphi(\varpi)\right)\ominus\upsilon^{2}\left(i\sqrt[\Omega]{\upsilon}\right)^{n-1}\varphi(0)$$
$$®\sum_{\lambda=1}^{n-1}\upsilon^{2}\left(i\sqrt[\Omega]{\upsilon}\right)^{n-(\lambda+1)}\varphi^{(\lambda)}(0).$$

such that:

$$®=\begin{cases}\ominus,\ \text{if thenumber of the } 2^{nd}\text{ form differentiable}\\ \text{functionsbetween } i_1,\ldots,i_\lambda \text{ has an even number.}\\ -,\ \text{if the number of the } 2^{nd}\text{ form differentiable}\\ \text{functionsbetween } i_1,\ldots,i_\lambda \text{ has an odd number.}\end{cases}$$

2. If ε is an odd number, then:

$$R\left(\varphi^{(n)}(\varpi)\right)=-\upsilon^{2}\left(i\sqrt[\Omega]{\upsilon}\right)^{n-1}\varphi(0)\ominus\upsilon^{2}\left(-i\sqrt[\Omega]{\upsilon}\right)^{n-1}R\left(\varphi(\varpi)\right)$$
$$®\sum_{\lambda=1}^{n-1}\upsilon^{2}\left(i\sqrt[\Omega]{\upsilon}\right)^{n-(\lambda+1)}\varphi^{(\lambda)}(0).$$

such that:

$$®=\begin{cases}\ominus,\ \text{if the number of the } 2^{nd}\text{ form differentiable}\\ \text{functions between } i_1,\ldots,i_\lambda \text{ has an odd number.}\\ -,\ \text{if the number of the } 2^{nd}\text{ form differentiable}\\ \text{functions between } i_1,\ldots,i_\lambda \text{ has an even number.}\end{cases}$$

Proof: the proof depends on the duality between fuzzy Laplace-Ro-transforms, from Theorem 7 [5]:

$$\tilde{\mathfrak{R}}(\upsilon)=R\left[\varphi(\varpi)\right]\qquad F(\zeta)=L\left[\varphi(\aleph)\right].$$
$$\tilde{\mathfrak{R}}_n(\upsilon)=R\left[\varphi^{(n)}(\varpi)\right]\text{ and }F_n(\zeta)=L\left[\varphi^{(n)}(\aleph)\right].$$

From Theorem 7 [5]:

$$\tilde{\mathfrak{R}}_n(\upsilon)=R\left[\varphi^{(n)}(\varpi)\right]=\upsilon^{2}\varphi_n\left(i\sqrt[\Omega]{\upsilon}\right).$$

1. Since ε is an even number. Then by Theorem 2.7/1, Eq. (1) becomes:

$$\tilde{\mathfrak{R}}_n(\upsilon)=\upsilon^{2}\left[\left(i\sqrt[\Omega]{\upsilon}\right)^{n}F\left(i\sqrt[\Omega]{\upsilon}\right)\ominus\left(i\sqrt[\Omega]{\upsilon}\right)^{(n-1)}\varphi(0)®\sum_{\lambda=1}^{n-1}\left(i\sqrt[\Omega]{\upsilon}\right)^{n-(\lambda+1)}\varphi^{(\lambda)}(0)\right]$$
$$=\left(i\sqrt[\Omega]{\upsilon}\right)^{n}\left[\upsilon^{2}F\left(i\sqrt[\Omega]{\upsilon}\right)\right]\ominus\upsilon^{2}\left(i\sqrt[\Omega]{\upsilon}\right)^{(n-1)}\varphi(0)$$
$$®\,\upsilon^{2}\sum_{\lambda=1}^{n-1}\left(i\sqrt[\Omega]{\upsilon}\right)^{n-(\lambda+1)}\varphi^{(\lambda)}(0)$$
$$=\left(i\sqrt[\Omega]{\upsilon}\right)^{n}R\left[\varphi(\varpi)\right]\ominus\upsilon^{2}\left(i\sqrt[\Omega]{\upsilon}\right)^{(n-1)}\varphi(0)$$
$$®\,\upsilon^{2}\sum_{\lambda=1}^{n-1}\left(i\sqrt[\Omega]{\upsilon}\right)^{n-(\lambda+1)}\varphi^{(\lambda)}(0).$$

2. Since ε is an odd number. Then by Theorem 2.7/2, Eq. (2) becomes:

$$\tilde{\mathfrak{R}}_n(\upsilon)=\upsilon^{2}\begin{bmatrix}-\left(i\sqrt[\Omega]{\upsilon}\right)^{(n-1)}\varphi(0)\ominus\left(-i\sqrt[\Omega]{\upsilon}\right)^{n}F\left(i\sqrt[\Omega]{\upsilon}\right)\\ ®\sum_{\lambda=1}^{n-1}\left(i\sqrt[\Omega]{\upsilon}\right)^{n-(\lambda+1)}\varphi^{(\lambda)}(0)\end{bmatrix}$$
$$=-\upsilon^{2}\left(i\sqrt[\Omega]{\upsilon}\right)^{(n-1)}\varphi(0)\ominus\left(-i\sqrt[\Omega]{\upsilon}\right)^{n}\left[\upsilon^{2}F\left(i\sqrt[\Omega]{\upsilon}\right)\right]$$
$$®\,\upsilon^{2}\sum_{\lambda=1}^{n-1}\left(i\sqrt[\Omega]{\upsilon}\right)^{n-(\lambda+1)}\varphi^{(\lambda)}(0)$$
$$=-\upsilon^{2}\left(i\sqrt[\Omega]{\upsilon}\right)^{(n-1)}\varphi(0)\ominus\left(-i\sqrt[\Omega]{\upsilon}\right)^{n}R\left[\varphi(\varpi)\right]$$
$$®\,\upsilon^{2}\sum_{\lambda=1}^{n-1}\left(i\sqrt[\Omega]{\upsilon}\right)^{n-(\lambda+1)}\varphi^{(\lambda)}(0).$$

6. APPLICATION

The applicability of the fuzzy Ro-transformation for solving fuzzy differential equations is demonstrated by the example below.

Example 6.1

Tank system is shown in Figure 1. Assume $I=0$ to be inflow disturbances of the tank, which generates vibration in liquid level $\wp$, where $\mathfrak{R}=1$ is the flow obstruction that can be curbed using the valve. $A=1$ is the cross section of the tank:

$$\wp'(\varpi)=-\frac{1}{A\mathfrak{R}}\wp(\varpi)+\frac{I}{A'},\quad \wp(0)=\left(\wp(0,r),\wp(r,0)\right).$$

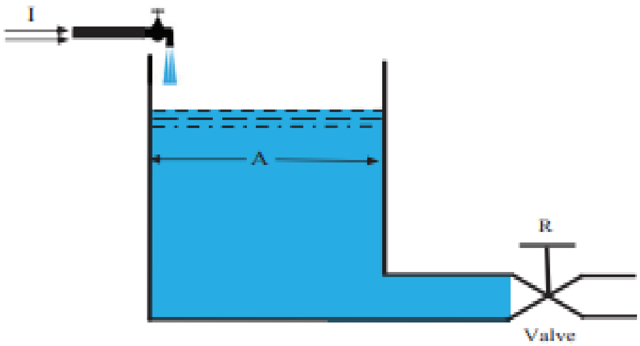

Figure 1. Liquid tank system

Case 1

Take fuzzy Ro-transform for both sides of the original equation:

$$\left(i\sqrt[\Omega]{\upsilon}\right)R\left[\wp(\varpi)\right]\ominus\upsilon^{2}\wp(0)=-R\left[\wp(\varpi)\right] \tag{9}$$

Eq. (9) becomes:

$$\begin{aligned}&\left(i\sqrt[\Omega]{\upsilon}\right)\gamma\left[\underline{\wp}(\varpi;\vartheta)\right]-\upsilon^{2}\underline{\wp}(0;\vartheta)=-\gamma\left[\underline{\wp}(\varpi;\vartheta)\right],\\&\left(i\sqrt[\Omega]{\upsilon}\right)\gamma\left[\overline{\wp}(\varpi;\vartheta)\right]-\upsilon^{2}\overline{\wp}(0;\vartheta)=-\gamma\left[\overline{\wp}(\varpi;\vartheta)\right].\end{aligned} \tag{10}$$

Solve Eq. (10), then:

$$\gamma\left[\underline{\wp}(\varpi;\vartheta)\right]=\frac{\upsilon^{2}}{\left(i\sqrt[\Omega]{\upsilon}\right)+1}\underline{\wp}(0;\varpi),$$

$$\gamma\left[\overline{\wp}(\varpi;\vartheta)\right]=\frac{\upsilon^{2}}{\left(i\sqrt[\Omega]{\upsilon}\right)+1}\overline{\wp}(0;\varpi).$$

Take the inverse fuzzy Ro-transform for above equations:

$$\underline{\wp}(\varpi;\vartheta)=e^{-\varpi}\underline{\wp}(0;\varpi),\overline{\wp}(\varpi;\vartheta)=e^{-\varpi}\overline{\wp}(0;\varpi).$$

Case 2

Use fuzzy Ro-transform for both sides of the original equation to get:

$$-\upsilon^{2}\wp(0)\ominus\left(-i\sqrt[\Omega]{\upsilon}\right)R\left[\wp(\varpi)\right]=-R\left[\wp(\varpi)\right] \tag{11}$$

Eq. (11) becomes:

$$\begin{aligned}&\left(i\sqrt[\Omega]{\upsilon}\right)\gamma\left[\overline{\wp}(\varpi;\vartheta)\right]-\upsilon^{2}\overline{\wp}(0;\vartheta)=-\gamma\left[\underline{\wp}(\varpi;\vartheta)\right],\\&\left(i\sqrt[\Omega]{\upsilon}\right)\gamma\left[\underline{\wp}(\varpi;\vartheta)\right]-\upsilon^{2}\underline{\wp}(0;\vartheta)=-\gamma\left[\overline{\wp}(\varpi;\vartheta)\right].\end{aligned} \tag{12}$$

Solve Eq. (12):

$$\gamma\left[\underline{\wp}(\varpi;\vartheta)\right]=\frac{\upsilon^{2}\left(i\sqrt[\Omega]{\upsilon}\right)}{\left(i\sqrt[\Omega]{\upsilon}\right)^{2}-1}\overline{\wp}(0;\varpi)-\frac{\upsilon^{2}}{\left(i\sqrt[\Omega]{\upsilon}\right)^{2}-1}\underline{\wp}(0;\varpi),$$

$$\gamma\left[\overline{\wp}(\varpi;\vartheta)\right]=\frac{\upsilon^{2}\left(i\sqrt[\Omega]{\upsilon}\right)}{\left(i\sqrt[\Omega]{\upsilon}\right)^{2}-1}\underline{\wp}(0;\varpi)-\frac{\upsilon^{2}}{\left(i\sqrt[\Omega]{\upsilon}\right)^{2}-1}\overline{\wp}(0;\varpi).$$

Use the inverse fuzzy Ro-transform for above equations, to get:

$$\underline{\wp}(\varpi;\vartheta)=\cosh\varpi\,\overline{\wp}(0;\varpi)-\sinh\varpi\,\underline{\wp}(0;\varpi),$$

$$\overline{\wp}(\varpi;\vartheta)=\cosh\varpi\,\underline{\wp}(0;\varpi)-\sinh\varpi\,\overline{\wp}(0;\varpi).$$

7. CONCLUSION

In this study, we establish various fuzzy Ro-transform properties, discover fuzzy Ro-transform derivatives for the third and generalized orders, and apply these formulas to resolve the liquid tank system problem.

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