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Effective Technique for Converting Ill-Posed Volterra Equation to Integro-Differential Equation and Solving It

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ABSTRACT

The different types of integral equations are very important in practice life. Volterra integral equations of the first kind are not a lower interest them, then the study of the values of the solutions and methods for solving these equations with continuous kernels is a must be a step. However, it is well known that these equations are ill- posed problems. Therefore, in this paper, we will provide a new technique for finding solutions to these problems, by using conversion these integral equations of the first kind to integro-differential equations of the second kind using Taylor series. In this article, we apply this technique with some numerical methods such as modified Simpson method and finite difference method. Finally, we will present four numerical examples that demonstrate the performance and efficiency of our technique.

1. INTRODUCTION

Integral equation of the first kind is considered ill-posed problem because it does not satisfy the following three properties:

(1) Existence of a solution.
(2) Uniqueness of a solution.
(3) Continuous dependence of the solution $\phi(t)$ on the data $f(t)$. This property means that small errors in the data $f(t)$ should cause small errors in the solution $\phi(t)$.

Any problem that satisfies the three previous properties is called well-posed problem. For any ill-posed problem, a very small change on the data $f(t)$ can give a large change in the solution $\phi(t)$. Methods for obtaining a stable approximate solution of an ill-posed problem are called regularization methods. The classic theory of regularization is well-developed for linear ill-posed problems. For example, Tikhonov regularization which was established independently by Phillips [1] and Tikhonov [2]. The method of regularization consists of replacing ill-posed problem by well-posed problem [3-6], For Volterra integral equations, we may differentiate these equations with respect to t to obtain Volterra integral equations of the second kind which are known to be a well-posed problem. This converting is applied [3, 4, 7] and in our last article [8].

This paper is concerned with the first kind linear Volterra integral equations:

$$\int_a^t k(t,x)\phi(x)dx = f(t), \quad a \le x \le t \le b \tag{1}$$

where, $\phi(t)$ is unknown, $k(t,x)$ is a kernel, and the function $f(t)$ are given real-valued functions, are known and continuous in $a \le x \le t \le b$. It will often be useful to write Eq. (1) in the form:

$$A(\varphi) = f,$$

where, the operator A is defined by:

$$A\varphi(t) : A(t) = \int_a^t k(t,x)\varphi(x)dx, \quad t \in [a,b]$$

where, $A(\phi)$: X→Y is a linear and continuous operator, X and Y are the Hilbert spaces, $f \in Y$ is the given element. We assume that the equation $A(\phi)=f$, has the unique solution $\bar{\varphi}^0 \in X$.

Tikhonov regularization consists in approximation of the desired solution by the minimize of the functional:

$$\mathbf{F}_N(\varphi) = \left\| T(\varphi) - f^{\delta} \right\|_Y^2 + \alpha \left\| \varphi - \varphi_0 \right\|_X^2 .$$

In the functional f^δ denotes perturbation of f, $\alpha>0$, $\|.\|_Y$ and $\|.\|_X$ denote the norm on the Hilbert spaces Y and X, respectively.

Theorem 01: Assume there exists a minimum norm solution of $A(\phi)=f$, denoted by $\bar{\varphi}^0 \in X$. Let $\{f_k\}_{k\in N}$ be a sequence where $f_k \to f^\delta$ and let ϕ_k be a minimizer of F_N where f^δ is replaced by f_k. Then there exists a convergent subsequence of $\{\phi_k\}_{k\in N}$ and the limit of every convergent subsequence is a minimizer of F_N [7].

For more information on these topics, see for example in researches [7, 9].

In this paper, we will apply a new technique for conversion a linear Volterra integral equation of first kind to a linear

Volterra integro-differential equation of second kind by using Taylor series.

2. MAIN RESULT (THE REGULARIZATION METHOD)

We will transfer the integral Eq. (1) to an integro-differential equation of the second kind defined in [0,1]. It is important to note that the solution for an ill-posed problem may not exist, and if it does exist it may not be unique. We will use in this section the Taylor series and Leibnitz rule.

Let $A(t)$ be a function with derivatives of all orders with respect to t in an interval [0,1] than for $0<t-\varepsilon<t<t+\varepsilon<1$, with $\varepsilon\to 0$. The Taylor series is given by:

$$A(t+\varepsilon)=A(t)+\frac{\varepsilon}{1!}.\frac{\partial A(t)}{\partial t}+\frac{\varepsilon^2}{2!}.\frac{\partial^2 A(t)}{\partial t^2}+\dots$$
$$\dots+\frac{\varepsilon^n}{n!}.\frac{\partial^n A(t)}{\partial t^n}+O(\varepsilon^n) \tag{2}$$

where, $O(\varepsilon^n)$ is an unknown error term of approximation.

In last our work [8], we transformed the Volterra integral Eq. (1) to an equivalent integral equation of the second kind defined in the interval [0,1] by using Taylor series of the first order, and we found this equivalent equation:

$$\phi_\varepsilon(t)+\int_0^t K(t,x)\phi_\varepsilon(x)dx=f_\varepsilon(t) \tag{3}$$

where, $K(t,x)=\frac{k(t,x)+\varepsilon\frac{\partial k(t,x)}{\partial t}}{\varepsilon k(t,t)}$, $f_\varepsilon(t)=\frac{f(t+\varepsilon)}{\varepsilon k(t,t)}$ and $\phi_\varepsilon=\phi$ if $\varepsilon\to 0$. For more information, see researches [9].

Now, by using the Taylor series of the second order Eq. (2) and Leibnitz rule for equation of first kind Eq. (1), we find:

$$A(t+\varepsilon)=A(t)+\varepsilon\int_0^t\frac{\partial k(t,x)}{\partial t}\varphi(x)dx+\frac{\varepsilon^2}{2!}\frac{\partial k(t,x)}{\partial t}\varphi(t)+$$
$$+\frac{\varepsilon^2}{2!}k(t,t)\varphi'(t)+\frac{\varepsilon^2}{2!}\int_0^t\frac{\partial^2 k(t,x)}{\partial t^2}\varphi(x)dx+\varepsilon k(t,t)\varphi(t)+O(\varepsilon^2),$$

Then, the equivalent equation of Volterra equation of the first kind Eq. (1) is given by:

$$\frac{\varepsilon^2}{2!}k(t,t)\varphi'(t)=-\int_0^t\left(k(t,x)+\varepsilon\frac{\partial k(t,x)}{\partial t}+\frac{\varepsilon^2}{2!}\frac{\partial^2 k(t,x)}{\partial t^2}\right)\varphi(x)dx$$
$$-\left(\varepsilon k(t,t)+\frac{\varepsilon^2}{2!}\frac{\partial k(t,t)}{\partial t}\right)\varphi(t)+f(t+\varepsilon)+O(\varepsilon^2),$$

after simplify, we obtain:

$$\varphi'(t)=\frac{-2}{\varepsilon^2 k(t,t)}\int_0^t\left(k(t,x)+\varepsilon\frac{\partial k(t,x)}{\partial t}+\frac{\varepsilon^2}{2!}\frac{\partial^2 k(t,x)}{\partial t^2}\right)\varphi(x)dx$$
$$-\frac{1}{k(t,t)}\left(\frac{2}{\varepsilon}+\frac{\partial k(t,t)}{\partial t}\right)\varphi(t)+\frac{2}{\varepsilon^2 k(t,t)}f(t+\varepsilon),$$

where, $k(t,t)\neq 0$, $\frac{\partial k(t,x)}{\partial t}\neq 0$ for $t\in[0,1]$ and $\varepsilon\to 0$, we obtain the Volterra integro-differential equation of the second kind of the following form:

$$\phi'(t)=K_1(t)\phi_\varepsilon(t)+\int_0^t K(t,x)\phi_\varepsilon(x)dx+F_\varepsilon(t) \tag{4}$$

where,

$$K(t,x)=\frac{-2}{\varepsilon^2 k(t,t)}\left(k(t,x)+\varepsilon\frac{\partial k(t,x)}{\partial t}+\frac{\varepsilon^2}{2!}\frac{\partial^2 k(t,x)}{\partial t^2}\right),$$
$$K_1(t)=\frac{-1}{k(t,t)}\left(\frac{2}{\varepsilon}+\frac{\partial k(t,t)}{\partial t}\right),$$
$$F_\varepsilon(t)=\frac{2}{\varepsilon^2 k(t,t)}f(t+\varepsilon),$$

and, $\phi_\varepsilon(t)=\phi(t)$ if $\varepsilon\to 0$. Substituting $t=0$ into Eq. (3) gives the initial condition $\phi_\varepsilon(0)=\phi_0$.

Now, we apply modified Simpson method [10] for Eq. (4) and take $\phi_\varepsilon(t)=\phi(t)$. Consider let:

$$t_0=0<t_1<\dots<t_{2j-1}<t_{2j}<\dots<t_{2n}=1,$$

be an equidistant subdivision of a step $h=t_{2j+1}-t_{2j}$ for $j=0,1,2,\dots,n$. Our objective then, it's to approximate the solutions of the equivalent Eq. (4) to the nodes of even indices (at the point t_{2j}), then the modified Simpson have the form:

$$\int_{t_{2j}}^{t_{2j+2}} f(t)dt=\frac{h}{3}\left[f(t_{2j})+4f(t_{2j+1})+f(t_{2j+2})\right], \tag{5}$$

with the error of integration is:

$$E(h)=-2\frac{(h/2)^5}{90}\left(f(\zeta)\right)^{(4)}$$

Now, by the numerical integration formulas of modified Simpson method Eq. (5) and finite difference formulation for the integro-differential Eq. (4), we obtain the following iteration formula:

$$\phi'(t_{2j})=\frac{h}{3}\sum_{i=0}^{j-1}\begin{bmatrix}K(t_{2j},x_{2i})\phi(t_{2i})\\+4K(t_{2j},x_{2i+1})\phi(t_{2i+1})+K(t_{2j},x_{2i+2})\phi(t_{2i+2})\end{bmatrix}$$
$$+K_1(t_{2j})\phi(t_{2j})+F(t_{2j}) \tag{6}$$

We approximate φ'_{2j} and φ_{2i+1} by $\frac{\varphi_{2j+2}-\varphi_{2j}}{2h}$ and $\frac{\varphi_{2i}+\varphi_{2i+2}}{2}$, respectively. The Eq. (6) becomes:

$$\phi_{2j+2}=\frac{2h^2}{3}\left(\sum_{i=0}^{j-1}(K_{2j,2i}+2K_{2j,2i+1})\phi_{2i}+\sum_{i=0}^{j-1}(2K_{2j,2i+1}+K_{2j,2i+2})\phi_{2i+2}\right)$$
$$+(2h.K_1+1)\phi_{2j}+2h.F_{2j}.$$

By recurrence, we can to calculate the approximation solutions ϕ of the Eq. (4) in all points t_{2j} for $j=0, 1, \dots, n$. Cleary from Eq. (3) the initial value of e is $\phi(0)=\phi_0=f_\varepsilon(0)$.

3. NUMERICAL EXAMPLES

In what follows, we will apply our technique of regularization for linear ill-posed Volterra equations, and we will present four illustrative numerical examples where we transform these equations to second kind Volterra integro-differential equations. Examples will be used to highlight the reliability of the regularization method.

Example 01: We consider the linear Volterra integral equation of the first kind [8]:

$$\int_0^t \left(t-x+10^2\right)\phi(x)dx=\frac{t^2}{2}+10^2 t. \tag{7}$$

Can be transformed toit to Volterra integral equation of the second kind given by:

$$\phi_\varepsilon(t)+\frac{1}{\varepsilon 10^2}\int_0^t \left(t-x+10^2+\varepsilon\right)\phi_\varepsilon(x)dx=\frac{\left(t+\varepsilon\right)^2}{2\varepsilon.10^2}+\frac{t+\varepsilon}{\varepsilon}. \tag{8}$$

And Eq. **(**7) can be transformed to Volterra integro-differential equation of the second kind given by:

$$\begin{aligned}&\phi'_\varepsilon(t)+2\left(\frac{1}{\varepsilon}+\frac{1}{10^2}\right)\phi_\varepsilon(t)\\&=\frac{-2}{\varepsilon^2 10^2}\int_0^t \left(t-x+10^2+\varepsilon\right)\phi_\varepsilon(x)dx+\left(\frac{t+\varepsilon}{10\varepsilon}\right)^2+\frac{2}{\varepsilon^2}\left(t+\varepsilon\right),\end{aligned} \tag{9}$$

for $t\in[0,1]$ with the initial condition $\phi_\varepsilon(0)=\frac{\varepsilon}{2}10^{-2}+1$, et la solution exact $\phi(t)=1$.

Table 1. Comparison of the exact, the approximate solutions and the absolute errors for **Example 01** of Eq. (9) obtained by Taylor approximation (ε=0.1) and modified Simpson method (n=40)

t_{2j}	ES	AS of Eq. (9)	Err of Eq. (9)
0	1.0000	1.0500	5.0000E-02
0.1	1.0000	0.8510	1.4900E-01
0.2	1.0000	0.9458	5.4155E-02
0.3	1.0000	0.9886	1.1400E-02
0.4	1.0000	0.9989	1.1472E-03
0.5	1.0000	1.0003	3.3281E-04
0.6	1.0000	1.0003	3.4682E-04
0.7	1.0000	1.0003	2.7918E-04
0.8	1.0000	1.0003	2.5477E-04
0.9	1.0000	1.0002	2.4940E-04
1	1.0000	1.0002	2.4855E-04

Note: AS: The approximate solutions; Err: The absolute errors

Example 02: We consider the linear Volterra integral equation of the first kind:

$$\int_0^t e^{x+t}\phi(x)dx=te^t. \tag{10}$$

Can be transformed to it to Volterra integral equation of the second kind given by:

$$\phi_\varepsilon(t)+\frac{1+\varepsilon}{\varepsilon}\int_0^t e^{x-t}\phi_\varepsilon(x)dx=\frac{t+\varepsilon}{\varepsilon}e^{\varepsilon-t}. \tag{11}$$

And Eq. (10) can be transformed to Volterra integro-differential equation of the second kind given by:

$$\begin{aligned}&\phi'_\varepsilon(t)+2\left(\frac{1}{\varepsilon}+1\right)\phi_\varepsilon(t)\\&+\frac{2+2\varepsilon+\varepsilon^2}{\varepsilon^2}\int_0^t e^{-t+x}\phi_\varepsilon(x)dx=\frac{2\left(t+\varepsilon\right)}{\varepsilon^2}e^{\varepsilon-t}\end{aligned} \tag{12}$$

for $t\in[0,1]$ with the initial condition $\phi_\varepsilon(0)=\frac{1+\varepsilon}{\varepsilon}e^{\varepsilon}$ and $\varepsilon\to0$, et la solution exact given by$\phi(t)=e^{-t}$.

Table 2. Comparison of the exact, the approximate solutions and the absolute errors for **Example 02** of Eq. (12**)** obtained by Taylor approximation (ε=0.1) and modified Simpson method (n=40)

t_{2j}	ES	AS of Eq. (12)	Err of Eq. (12)
0	1.0000	1.1052	1.0517E-01
0.1	0.9048	0.7876	1.1725E-01
0.2	0.8187	0.7831	3.5616E-02
0.3	0.7480	0.7343	6.5622E-03
0.4	0.6703	0.6695	7.7296E-04
0.5	0.6065	0.6065	1.5206E-05
0.6	0.5488	0.5489	6.6996E-05
0.7	0.4966	0.4966	5.5256E-05
0.8	0.4493	0.4494	4.6996E-05
0.9	0.4066	0.4066	4.1838E-05
1	0.3679	0.3679	3.7750E-05

Note: AS: The approximate solutions; Err: The absolute errors

Example 03: We consider the linear Volterra integral equation of the first kind [11]:

$$\int_0^t e^{t-x}\phi(x)dx=\sin(t). \tag{13}$$

Can be transformed toit to Volterra integral equation of the second kind given by:

$$\phi_\varepsilon(t)+\int_0^t\left(\frac{1}{\varepsilon}+1\right)e^{t-x}\phi_\varepsilon(x)dx=\frac{1}{\varepsilon}\sin(t+\varepsilon). \tag{14}$$

And Eq. (13) can be transformed to Volterra integro-differential equation of the second kind given by:

$$\begin{aligned}&\phi'_\varepsilon(t)+2\left(\frac{1}{\varepsilon}+1\right)\phi_\varepsilon(t)\\&+\frac{2+2\varepsilon+\varepsilon^2}{\varepsilon^2}\int_0^t e^{t-x}\phi_\varepsilon(x)dx=\frac{2}{\varepsilon}\sin(t+\varepsilon)\end{aligned} \tag{15}$$

For $t\in[0,1]$ with the initial condition $\phi_\varepsilon(0)=\frac{1}{\varepsilon}sin(\varepsilon)$ and$\varepsilon\to0$, et la solution exact given by:

$$\phi(t)=cos(x)-sin(x).$$

Table 3. Comparison of the exact, the approximate solutions and the absolute errors for **Example 03** of Eq. (15) obtained by Taylor approximation (ε=0.1) and modified Simpson method (n=40)

t_{2j}	ES	AS of Eq. (15)	Err of Eq. (15)
0	1.0000	0.9983	1.6658E-03
0.1	0.8952	0.7509	1.4423E-01
0.2	0.7841	0.7688	1.2578E-02
0.3	0.6598	0.7023	4.2451E-02
0.4	0.5316	0.5896	5.7936E-02
0.5	0.3982	0.4586	6.0422E-02
0.6	0.2607	0.3196	5.8875E-02
0.7	0.1206	0.1765	5.5850E-02
0.8	-0.0206	0.0314	5.2042E-02
0.9	-0.1617	-0.1141	4.7659E-02
1	-0.3012	-0.2584	4.2786E-02

Note: AS: The approximate solutions; Err: The absolute errors

Example 04: We consider the linear Volterra integral equation of the first kind:

$$\int_0^t (3t-3x+1)\phi(x)dx = \frac{5}{2}t^2(1+t). \tag{16}$$

Can be transformed to it to the Volterra integral equation of the second kind given by:

$$\begin{aligned}&\phi_\varepsilon(t)+\frac{1}{\varepsilon}\int_0^t (3t-3x+1+3\varepsilon)\phi_\varepsilon(x)dx\\ &=\frac{5}{2\varepsilon}(t+\varepsilon)^2(1+t+\varepsilon).\end{aligned} \tag{17}$$

and Eq. (16) can be transformed to Volterra integro-differential equation of the second kind given by:

$$\begin{aligned}&\phi_\varepsilon^{'}(t)+\left(\frac{2}{\varepsilon}+3\right)\phi_\varepsilon(t)\\ &+\frac{2}{\varepsilon^2}\int_0^t (3t-3x+1+3\varepsilon)\phi_\varepsilon(x)dx\\ &=\frac{5}{\varepsilon^2}(t+\varepsilon)^2(1+t+\varepsilon)\end{aligned} \tag{18}$$

for $t\in[0,1]$ with the initial condition $\phi_\varepsilon(0)=\frac{5}{2}\varepsilon(1+\varepsilon)$ and $\varepsilon\to 0$, et la solution exact given by $\phi(t)=5t$.

Table 4. Comparison of the exact, the approximate solutions and the absolute errors for **Example 04** of Eq. (18) obtained by Taylor approximation (ε=0.1) and modified Simpson method (n=40)

t_{2j}	ES	AS of Eq. (18)	Err of Eq. (18)
0	0.0000	0.2750	2.7500E-01
0.1	0.5000	0.4376	6.2419E-02
0.2	1.0000	0.9529	4.7088E-02
0.3	1.5000	1.4940	6.0162E-03
0.4	2.0000	2.0036	3.5568E-03
0.5	2.5000	2.5017	1.6775E-03
0.6	3.0000	3.0001	1.0373E-04
0.7	3.5000	3.4999	1.3103E-04
0.8	4.0000	4.0000	2.9592E-05
0.9	4.5000	4.5000	1.9307E-05
1	5.0000	5.0000	1.8227E-05

Note: AS: The approximate solutions; Err: The absolute errors

After reading all the Tables 1-4 of the examples, we observed that the absolute error is very small, this indicates the effectiveness of our method in converting Volterra equation of first kind.

In this work we used the modified Simpson method, and can be used other numerical methods such as the trapezoidal method, the Newton-Kantorovich method [12-14].

4. CONCLUSIONS

In this paper we have solved linear Volterra integral equations of the first kind by converting them into Volterra integro-differential equation of the second kind and then applying the modified Simpson method and the finite difference method. We tested this technique by using four different examples. It is observed that all absolute errors have approached zero which was shown that numerical results were acceptable for all types of Volterra integral equation of the first kind.

REFERENCES

[1] Phillips, D.L. (1962). A technique for the numerical solution of certain integral equations of the first kind. Journal of the ACM (JACM), 9(1): 84-97.

[2] Tikhonov, A.N. (1963). On the solution of incorrectly posed problem and the method of regularization. Soviet Math, (4): 1035-1038.

[3] Abdul-Majid, W. (2011). Linear and nonlinear integral equations: Methods and applications. Higher Education Press Springer-Verlag Berlin Heidelberg, London, New York.

[4] Hochstadt, H. (1973). Integral Equations. John Wiley & Sons, Inc. Canada.

[5] Vasin, V.V. (2006). Some tendencies in the Tikhonov regularization of ill-posed problems. Journal of Inverse and Ill-posed Problems, 14(8): 813-840.https://doi.org/10.1515/156939406779768328

[6] He, J.H. (2000). Variational iteration method for autonomous ordinary differential systems. Applied Mathematics and Computation, 114(2-3): 115-123.https://doi.org/10.1016/S0096-3003(99)00104-6

[7] Lenzen, F., Scherzer, O. (2004). Tikhonov type regularization methods: History and recent progress. ECCOMAS.

[8] Guechi, S., Guechi, M. (2021). Taylor approximation for solving linear and nonlinear Ill-Posed Volterra Equations via an iteration method. General Letters in Mathematics, 11(2): 18-25. http://dx.doi.org/10.31559/glm2021.11.2.1

[9] Engl, H.W., Hanke, M., Neubauer, A. (1996). Regularization of inverse problems. In Mathematics and Its Applications (MAIA, volume 375). Kluwer Academic Publishers Group, Dordrecht.

[10] Nadir, M., Rahmoune, A. (2007). Modified method for solving linear Volterra integral equations of second kind using Simpson's rule. IJMM, 1(1): 141-146.

[11] Polyanin, A.D, Manzhirov, A.V. (2008). Handbook of integral equations. CRC Press, FI, USA.

[12] Nadir, M., Guechi S. (2016). The combination of modified Simpson and Newton-Kantrovich methods for solving nonlinear integral equations. Advanced Studies in Contemporary Mathema.

[13] Nadir, M, Guechi S. (2016). Solutions of integral

equations in the Urysohn form via some numerical methods. The International Arab Conference on Mathematics and computations (IACMC-2016), Zarqa, Jordan.

[14] Nadir, M, Guechi S. (2016). Integral equations and their relationship to differential equations with initial conditions. General Letters in Mathematics, 1(1): 23-31.