Stability Results for a Class of Nonlinear Caputo Volterra-Fredholm System: Physics and Engineering Application

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ABSTRACT
This study intends to provide and prove a novel stability theorem for the non-linear Volterra-Fredholm integro-differential equation with Caputo fractional derivative using the weighted space method and fixed-point technique. Specifically, the study investigates the H-U-R stability and semi-U-H-R stability results. Eventually, the investigation discusses an example of the capability of this method.

1. INTRODUCTION
As a result of their frequent appearance in a wide range of engineering and scientific disciplines, systems of fractional differential and integral equations are currently the focus of active research [1]. A system of integral-differential equations must therefore have approximate solutions. Besides, fractional derivatives provide a powerful tool for many types of physical modeling, such as stochastic dynamical systems, electrodynamics of complex medium, plasma physics, signal processing, economics, and so on researches [2, 3].

Budak et al. [4] reported that the stability issue of differential equations solutions presented. One of the most essential topics in differential equation theory is Ulam-Hyers stability. Because of the broad scope of fractional calculus, many authors focused on the study of stability for fractional differential equations [5-8]. In the same regard, fractional integro-differential equations also drew the attention of several authors [9-16].

Chalishajar and Kumar [5] enhanced a new direction of research via studied the existence and uniqueness of the solutions as well as discussed two types of stability. In same regard, Khan et al. [7] used Perov’s fixed point theorem and generalized metric space to derive some relaxed requirements for the uniqueness of positive solutions to the aforementioned problem. Dong et al. [9] investigated the Ulam-Hyers stability and Ulam-Hyers-Rassias stability of the random fractional integro-differential equation using the fixed point theorem.

The stability theory of fractional integro-differential equations is a significant branch of fractional calculus. The Ulam-type stability of an integro-differential equation implies that we can find the exact solution to the problem near an approximate solution. Several varieties of Ulam-type stability for nonlinear fractional integro-differential equations have been studied in recent decades [5, 7, 15-17].

Recently, Sevgi and Sevli [12] examined the U-H stability and the U-H-R stability in formulations of fixed-point techniques for the nonlinear Volterra equation:

\[ \Delta'(v) = A(u, \Delta(v)) + \int_{0}^{v} \Phi(v, \varsigma, \Delta(\varsigma)) \, d\varsigma \]  

Vu and Van Hao [15] addressed the nonlinear IVP of the Volterra equations, and they used the successive approximation approach to explain the U-H and U-H-R stability of the following equations.

\[ \Delta'(v) = A(u, \Delta(v)) + \int_{a}^{v} \Phi(v, \varsigma, \Delta(\varsigma)) \, d\varsigma, \quad v \in [a, b] \]  

\[ \Delta(a) = \Delta_{0} \]  


\[ ^{\nu}D_{0+}^{\alpha, \beta, \psi} \Delta(v) = A(v, \Delta(v)) + \int_{0}^{v} \Phi(v, \varsigma, \Delta(\varsigma)) \, d\varsigma \]  

where, \(^{\nu}D_{0+}^{\alpha, \beta, \psi}\) is the \(\nu\)-Hilfer fractional derivative.

Herein, the current study is interested in the following Caputo fractional nonlinear Volterra-Fredholm integro-differential problem:

\[ \theta'(v) + ^{\alpha}D_{0+}^{\alpha, \beta, \theta} \theta(v) = g(v, \theta(v)) + \int_{0}^{v} \Psi(v, \varsigma, \theta(\varsigma)) \, d\varsigma + \int_{0}^{1} \psi(v, \varsigma, \theta(\varsigma)) \, d\varsigma, \quad v \in [0, 1] \]  

\[ \theta(0) = \eta \]  

where, \( \theta \in C^{1}[0,1], 0 < \alpha < 1, Y, \Psi, [0,1] \times [0,1] \times \mathbb{R} \rightarrow \mathbb{R} \) and \( g: [0,1] \times \mathbb{R} \rightarrow \mathbb{R} \) are continuous functions.

Motivated by the above studies, the current study will investigate another problem of stability theorem for the nonlinear Volterra-Fredholm integro-differential equation with Caputo fractional derivative using the weighted space method and fixed-point technique.

Therefore, the aim of this work is to investigate the H-U, H-U-R, and semi-U-H-R stability for the system (5) under some new standards.
2. PRELIMINARIES

In this segment, the study introduces some useful preliminaries for fractional derivatives [20, 21]. Moreover, we recall concepts of stability for Eq. (5).

Let:

\[ \rho(\theta, \omega) = \sup_{v \in [0,1]} \frac{|\theta(v) - \omega(v)|}{\xi(v)} , \theta, \omega \in C^1[0,1] \]  

(7)

The weighted metric, where function \( \xi \) is a continuous non-decreasing defined as \( \xi : [0,1] \rightarrow (0, +\infty) \) then there is \( \xi \in [0,1] \), satisfies:

\[ \int_0^\nu E_{1-\alpha, \nu} \left( - (v - \zeta)^{1-\alpha} \right) \xi(\zeta) d\zeta \leq \xi(v) \]  

(8)

Obviously, \( C^1[0,1], \rho \) is a complete metric space.

**Definition 2.1** [20, 21] Let \( f : (0, +\infty) \rightarrow \mathbb{R} \) be integrable function, the R-L fractional integral is given by:

\[ I_0^\nu f(\Lambda) = \frac{1}{\Gamma(\alpha)} \int_0^\nu (\Lambda - v)^{\alpha-1} f(v) dv, \Lambda > 0, 0 < \alpha < 1. \]  

(9)

**Definition 2.2** [20, 21] The left Caputo fractional derivative of differentiable function \( f(v) \) is given by:

\[ \overset{\cdot}{D}_a^\nu f(\Lambda) = I_{1-\alpha}^{\nu} \left( \frac{1}{(1-\alpha)} \right) \int_0^\nu (\Lambda - v)^{-\alpha} f'(v) dv, 0 < \alpha < 1. \]  

(10)

**Definition 2.3** [20, 21] The function of Mittag-Leffler is given by:

\[ E_{\alpha, \beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha+\beta)}, \beta, \alpha, z \in \mathbb{C}, \text{Re}(\beta) > 0, \text{Re}(\alpha) > 0 \]  

(11)

The Laplace transform of the Mittag-Leffler and Caputo derivative given by:

\[ \mathcal{L} \left\{ v^{\beta-1} E_{\alpha, \beta}(\pm \alpha v^\beta) \right\}(z) = \frac{z^{\alpha-\beta}}{(\alpha+\beta)}, \text{Re}(\alpha) > 0, \beta, \alpha \in \mathbb{C} \]  

(12)

\[ \mathcal{L} \left\{ v^\beta (\pm \alpha v^\beta) \right\}(z) = \frac{k! z^{\alpha-\beta}}{\Gamma(k\alpha+\beta)}, \text{Re}(\alpha) > 0, \beta, \alpha \in \mathbb{C} \]  

(13)

where:

\[ E_{\alpha, \beta}^{(k)}(y) = \frac{d^k}{dy^k} E_{\alpha, \beta}(y) = \sum_{j=0}^{\infty} \frac{(j+k)^{j+1}}{j! \Gamma(j+1+k\alpha+\beta)}, k = 0,1,2, ... \]  

(14)

and

\[ \mathcal{L} \left\{ \overset{\cdot}{D}_a^\nu f(v) \right\}(z) = \xi^\alpha \hat{f}(z) - \xi^{\alpha-1} f(0), 0 < \alpha < 1. \]  

(15)

**Definition 2.4** [20, 21] If \( \Lambda(v) \) is a given differential function, satisfying:

\[ |\Lambda'(v)+cD_0^\nu \Lambda(v) - g(v, \Lambda(v))| - \int_0^\nu Y(v, c, \Lambda(\zeta)) d\zeta - \int_0^\nu \Psi(v, c, \Lambda(\zeta)) d\zeta| \leq \theta, v \in [0,1], \theta > 0, \]  

(16)

There is \( C>0 \) and \( \Theta(v) \) is a solution of the system in Eq. (5), where:

\[ |\Lambda(v) - \Theta(v)| \leq C\theta, v \in [0,1] \]  

(17)

Then, the system in Eq. (5) has the U-H stability.

If \( \Lambda(v) \) satisfies (16) there is a solution \( \Theta(v) \) of the system (5) and \( C>0 \), such that:

\[ |\Lambda(v) - \Theta(v)| \leq C\phi(v), v \in [0,1] \]  

(18)

where, the function \( \phi \) is continuous nonnegative defined as \( \phi : [0,1] \rightarrow (0, +\infty) \), then the system (1) has the semi-U-H-R stability.

If \( \phi : [0,1] \rightarrow (0, +\infty) \) is a continuous and \( \Lambda(v) \) satisfying:

\[ |\Lambda'(v)+cD_0^\nu \Lambda(v) - g(v, \Lambda(v))| - \int_0^\nu Y(v, c, \Lambda(\zeta)) d\zeta - \int_0^\nu \Psi(v, c, \Lambda(\zeta)) d\zeta| \leq \phi(v) \]  

(19)

There is \( C>0 \) and \( \Theta(v) \) is a solution of the system (5), where:

\[ |\Lambda(v) - \Theta(v)| < -C\phi(v), v \in [0,1] \]  

(20)

Then, the system in Eq. (1) has the U-H-R stability.

3. STABILITY RESULTS

The study will investigate in this segment the stabilities of U-H-R, semi-U-H-R and U-H for the system (1) in \( C^1[0,1] \).

3.1 U-H-R stability for the system in Eq. (5)

Here, the study will investigate the equivalent integral equation of the system (5) and study the U-H-R stability for the system (5) in \( C^1[0,1], \rho \).

**Lemma 3.1** Assume that \( f : [0,1] \rightarrow \mathbb{R} \) is a continuous function, and \( 0 < \alpha < 1, \Theta(v) \in C^1[0,1] \), the unique solution of the following equation.

\[ \Theta'(v)+cD_0^\nu \Theta(v) = f(v), \Theta(0) = \eta \]  

(21)

is given by:

\[ \Theta(v) = \eta + \int_0^\nu E_{1-\alpha, \nu} \left( -(v - \zeta)^{1-\alpha} \right) f(\zeta) d\zeta. \]  

(22)

**Proof:** The Laplace transforms of both \( \Theta'(v) \) and \( \overset{\cdot}{D}_a^\nu \Theta(v) \) exist for \( \Theta(v) \in C^1[0,1] \), applying the Laplace transform on two sides of Eq. (22). Then,

\[ s\Theta(\zeta) - \eta + \xi^\alpha \Theta(\zeta) - \xi^{\alpha-1} \Theta(0) = \frac{\hat{f}(\zeta)}{\xi^\alpha} \]  

(23)

\[ \Theta(\zeta) = \frac{1}{\xi^\alpha} \eta + \frac{1}{\xi^{\alpha+1}} \hat{f}(\zeta) \]  

(24)

It can take the inverse Laplace transform on the both sides of Eq. (23), then get:}
\[ \Theta(v) = \eta + \int_0^v E_{1-a,1}(-(v-\zeta)^{1-a}) f(\zeta) d\zeta \quad (25) \]

Then, \( \Theta(v) \) satisfies Eq. (21) \( \Leftrightarrow \) \( \Theta(v) \) satisfies Eq. (23). As a result, Eq. (23) is the equivalent integral equation of Eq. (21).

**Theorem 3.2** Assume that a function \( \zeta \) is continuous non-decreasing defined as \( \zeta : [0,1] \rightarrow (0,\infty) \), and there exists \( \xi \in [0,1] \), satisfying:

\[ \int_0^\xi E_{1-a,1}(-(v-\zeta)^{1-a}) \zeta(\zeta) d\zeta \leq \xi \zeta(v) \quad (26) \]

The following hypotheses are introduced:

[D1] Assume that a continuous function \( g \) defined as \( g : [0,1] \times \mathbb{R} \rightarrow \mathbb{R} \)

such that:
\[ |g(v,h_1) - g(v,h_2)| \leq e_1 |h_1 - h_2|, v \in [0,1], h_1, h_2 \in \mathbb{R} \quad (27) \]

with \( e_1 > 0 \).

[D2] Suppose that the kernels \( Y, \Psi : [0,1] \times [0,1] \times \mathbb{R} \rightarrow \mathbb{R} \)

are continuous functions satisfying:
\[ |Y(v,\zeta,h_1) - Y(v,\zeta,h_2)| \leq e_2 |h_1 - h_2|, v, \zeta \in [0,1], h_1, h_2 \in \mathbb{R} \quad (28) \]

\[ |\Psi(v,\zeta,h_1) - \Psi(v,\zeta,h_2)| \leq e_3 |h_1 - h_2| \quad (29) \]

with \( e_2, e_3 > 0 \), \( 0 < 1 \), \( \Lambda \in C^1[0,1] \) satisfies:
\[ |\Lambda(v) + D_{\alpha}^\gamma \Lambda(v) - g(v,\Lambda(v))| \leq \rho(\Omega,\omega,\omega) \quad (30) \]

\[ \rho(\Omega,\omega,\omega) \leq \left[ \frac{\int_0^\xi E_{1-a,1}(-(v-\zeta)^{1-a}) \left[ g(v,\zeta,w(\zeta)) - g(s,w(\zeta)) \right] d\zeta}{\zeta(v)} \right] \]

\[ + \sup_{v \in [0,1]} \left[ \int_0^\xi \int_0^\tau E_{1-a,1}(-(v-\zeta)^{1-a}) \left[ Y(\zeta,\tau,\omega(\tau)) - Y(\zeta,\tau,w(\tau)) \right] d\tau d\zeta \right] \zeta(v) \]

\[ + \sup_{v \in [0,1]} \left[ \int_0^\xi \int_0^\tau E_{1-a,1}(-(v-\zeta)^{1-a}) \left[ \Psi(\zeta,\tau,\omega(\tau)) - \Psi(\zeta,\tau,w(\tau)) \right] d\tau d\zeta \right] \zeta(v) \]

\[ \leq e_1 \sup_{v \in [0,1]} \left[ \int_0^\tau E_{1-a,1}(-(v-\zeta)^{1-a}) \left[ |\omega(s) - w(\zeta)| \right] d\tau \right] \zeta(v) \]

\[ + e_2 \sup_{v \in [0,1]} \left[ \int_0^\tau E_{1-a,1}(-(v-\zeta)^{1-a}) \left[ |\omega(\tau) - w(\tau)| \right] d\tau \right] \zeta(v) \]

\[ + e_3 \sup_{v \in [0,1]} \left[ \int_0^\tau E_{1-a,1}(-(v-\zeta)^{1-a}) \left[ |\omega(\tau) - w(\tau)| \right] d\tau \right] \zeta(v) \]

\[ \leq c_1 \xi \rho(\omega,w) + c_2 \xi \rho(\omega,w) + c_3 \xi \rho(\omega,w) = (e_1 + e_2 + e_3) \xi \rho(\omega,w) \quad (35) \]

From the hypothesis \( (e_1 + e_2 + e_3) \xi < 1 \), \( \Omega \) is strictly contractive.

Here, it can assume that \( \Lambda(v) \in C^1[0,1] \) satisfies Eq. (30). By Eqs. (30), (26) and Lemma 3.1, then:

\[ |\Lambda(v) - \eta - \int_0^\xi E_{1-a,1}(-(v-\zeta)^{1-a}) \left[ g(\zeta,\Lambda(\zeta)) + \int_0^\tau Y(\zeta,\tau,\Lambda(\tau)) d\tau + \int_0^\tau \Psi(\zeta,\tau,\Lambda(\tau)) d\tau \right] d\zeta| \leq |\Lambda(v) - \eta - \Lambda(v)| \leq \xi \zeta(v) \quad (36) \]

The equations for proving Eq (36) are the same as for proving Lemma 3.1. From \( \Omega \) definition and Eq. (36), it can conclude:
\[ |\Omega(v) - \Lambda(v)| \leq \xi \zeta(v) \quad (37) \]

Then, by \( \rho \) definition, it get:
\[ \rho(\Omega,\Lambda) \leq \xi < 1 < \infty \quad (38) \]

Let \( C^1[0,1] = \{ \psi \in C^1[0,1] : \rho(\Omega,\psi) < \infty \} \). By Banach fixed-point theorem, there is a unique solution \( \Theta \in C^1[0,1] \) such that \( \Theta = \Omega \Theta = \Theta \), that means \( \Theta \) is a solution of the system (33). Therefore, \( \Theta \) is the solution of the system (1). Then:
\[ \rho(\Lambda, \Theta) \leq \frac{1}{1 - (e_1 + e_2 + e_3)^2} \rho(\Omega \Lambda, \Lambda) \leq \frac{\xi}{1 - (e_1 + e_2 + e_3)^2} \xi \] (39)

From \( \rho \) definition, the Eq. (33) holds. This completes the proof.

3.2 Semi-U-H-R and U-H stabilities for the system (5)

The study will investigate in this segment the stabilities of U-H and semi-U-H-R in \( C^1[0,1] \) for the system in Eq. (5).

**Theorem 3.3** Assume that a function \( \zeta \) is continuous non-decreasing defined as \( \zeta: [0,1] \rightarrow (0, \infty) \) and there is \( \xi \in [0,1] \) satisfying:

\[ \int_0^\omega E_{1-\alpha,1} \left( -(v - \zeta)^{-1} \right) \zeta(\zeta) d\zeta \leq \xi \zeta(\zeta) \] (40)

Let \( e_1, e_2, e_3 > 0 \) for \( (e_1 + e_2 + e_3) \xi \leq 1 \). Assume that \( h: [0,1] \times \mathbb{R} \rightarrow \mathbb{R} \) and \( \gamma, \psi: [0,1] \times [0,1] \times \mathbb{R} \rightarrow \mathbb{R} \) are continuous functions satisfying:

\[ \begin{cases} |g(v, h_1, g(v, h_2))| \leq e_1 |h_1 - h_2|, v \in [0,1], h_1, h_2 \in \mathbb{R} \\ |\gamma(v, c, h_1) - \gamma(v, c, h_2)| \leq e_2 |h_1 - h_2|, v, c \in [0,1], h_1, h_2 \in \mathbb{R} \\ |\psi(v, c, h_1) - \psi(v, c, h_2)| \leq e_3 |h_1 - h_2|, v, c \in [0,1], h_1, h_2 \in \mathbb{R} \end{cases} \] (41)

If \( \Lambda \in C^1[0,1] \) satisfies:

\[ \left| \Lambda(v) + \mathcal{D} \frac{\partial}{\partial \tau} \Lambda(v) - g(v, \Lambda(v)) \right| \leq \theta, v \in [0,1] \]

with \( \theta > 0 \), then there is \( \Theta(v) \) solution of the system (5) satisfies:

\[ \frac{\partial \Theta(v)}{\partial v} \leq \left| \Lambda(v) - \Theta(v) \right| \leq \theta \left| \int_0^\omega E_{1-\alpha,1} \left( -(v - \zeta)^{-1} \right) d\zeta \right|, v \in [0,1] \]

This means that under above conditions, the system (5) has the semi-U-H-R stability.

**Proof.** Consider \( \Omega: C^1[0,1] \rightarrow C^1[0,1] \), defined by:

\[ (\Omega \omega)(v) = \eta + \int_0^\omega E_{1-\alpha,1} \left( -(v - \zeta)^{-1} \right) \left[ g(s, \omega(s)) + \int_0^s \gamma(\zeta, \tau, \omega(\tau)) d\tau + \int_0^\tau \psi(\zeta, \tau, \omega(\tau)) d\tau \right] d\zeta \] (44)

where, \( v \in [0,1] \), \( \omega \in C^1[0,1] \).

For any \( \omega, w \in C^1[0,1] \), then it has:

\[ \rho(\Omega \omega, \Omega \omega) \leq (e_1 + e_2 + e_3) \xi \rho(\omega, w) \] (45)

From \( (e_1 + e_2 + e_3) \xi < 1 \), then \( \Omega \) is strictly contractive in \( (C^1[0,1], \rho) \).

Next, suppose that \( \Lambda(v) \in C^1[0,1] \) satisfies Eq. (42). By Eq. (42) and Lemma 3.1 it can get:

\[ \left| \Lambda(v) - \eta - \int_0^\omega E_{1-\alpha,1} \left( -(v - \zeta)^{-1} \right) \left[ g(s, \Lambda(s)) + \int_0^s \gamma(\zeta, \tau, \Lambda(\tau)) d\tau + \int_0^\tau \psi(\zeta, \tau, \Lambda(\tau)) d\tau \right] d\zeta \right| \leq \theta \left| \int_0^\omega E_{1-\alpha,1} \left( -(v - \zeta)^{-1} \right) d\zeta \right|, v \in [0,1] \]

From the continuity of Mittag-Leffler function, we have \( \left| \int_0^\omega E_{1-\alpha,1} \left( -(v - \zeta)^{-1} \right) d\zeta \right| \) is a continuous nonnegative function. From the Eqns. (44) and (46), then:

\[ \left| (\Omega \Lambda)(v) - \Lambda(v) \right| \leq \theta \left| \int_0^\omega E_{1-\alpha,1} \left( -(v - \zeta)^{-1} \right) d\zeta \right| \]

(47)

Since \( \zeta \) is a continuous function, it can get:

\[ \rho(\Omega \Lambda, \Lambda) \leq \sup_{v \in [0,1]} \frac{\left| (\Omega \Lambda)(v) - \Lambda(v) \right|}{\zeta(v)} \leq \sup_{v \in [0,1]} \frac{\theta \left| \int_0^\omega E_{1-\alpha,1} \left( -(v - \zeta)^{-1} \right) d\zeta \right|}{\zeta(v)} < \infty \]

(48)

Let \( C^1[0,1] = \{ y \in C^1[0,1]: \rho(\Omega \Lambda, y) < \infty \} \).

Applying the Banach fixed-point theorem, thus there is a solution \( \Theta \in C^1[0,1] \) such that \( \Omega \Theta = \Theta \). That means \( \Theta(v) \) is a unique solution of the system (5).

From the Banach fixed-point theorem and Eq. (48), then:

\[ \rho(\Lambda, \Theta) \leq \frac{1}{1 - (e_1 + e_2 + e_3)^2} \rho(\Omega \Lambda, \Lambda) \leq \theta \left| \int_0^\omega E_{1-\alpha,1} \left( -(v - \zeta)^{-1} \right) d\zeta \right| \]

(49)

Thus, by the definition of \( \rho \), then:

\[ \left| \Lambda(v) - \Theta(v) \right| \leq \theta \left| \int_0^\omega E_{1-\alpha,1} \left( -(v - \zeta)^{-1} \right) d\zeta \right| \]

(50)

where, \( \zeta(v) \left| \int_0^\omega E_{1-\alpha,1} \left( -(v - \zeta)^{-1} \right) d\zeta \right| \) is a continuous non-negative function. This completes the proof.

**Remark 3.4** For any \( v \in [0,1] \), \( \int_0^\omega E_{1-\alpha,1} \left( -(v - \zeta)^{-1} \right) d\zeta \) is real number convergent series. Then, there exists \( N > 0 \), such that:

\[ \left| \int_0^\omega E_{1-\alpha,1} \left( -(v - \zeta)^{-1} \right) d\zeta \right| < N \]

(51)

**Theorem 3.5** Assume that \( e_1, e_2, e_3 \) are constants for which \( e_1 > 0, e_2 > 0, e_3 > 0, \xi \leq 1, (e_1 + e_2 + e_3) \xi < 1 \). Assume that \( g, \gamma, \psi \) are continuous functions, such that:

\[ \begin{cases} |g(v, h_1, g(v, h_2))| \leq e_1 |h_1 - h_2|, v \in [0,1], h_1, h_2 \in \mathbb{R} \\ |\gamma(v, c, h_1) - \gamma(v, c, h_2)| \leq e_2 |h_1 - h_2|, v, c \in [0,1], h_1, h_2 \in \mathbb{R} \\ |\psi(v, c, h_1) - \psi(v, c, h_2)| \leq e_3 |h_1 - h_2|, v, c \in [0,1], h_1, h_2 \in \mathbb{R} \end{cases} \]

Let \( \xi: [0,1] \rightarrow (0, \infty) \) be a continuous non-decreasing function, and satisfies:

\[ \left| \int_0^\omega E_{1-\alpha,1} \left( -(v - \zeta)^{-1} \right) \zeta(\zeta) d\zeta \leq \xi \zeta(\zeta) \right| \]

(52)

If \( \Lambda \in C^1[0,1] \) satisfies (4), with \( \theta > 0 \), then there is a solution \( \Theta(v) \) of the system (5) such that:

\[ |\Lambda(v) - \Theta(v)| \leq \frac{\theta \left| \int_0^\omega E_{1-\alpha,1} \left( -(v - \zeta)^{-1} \right) d\zeta \right|}{\zeta(\xi, v \in [0,1])} \]

(53)

**Proof.** Since \( \zeta \) is a continuous non-decreasing function,

\[ \zeta(\xi) \leq \xi \zeta(\xi) \]

(54)

By theorem 3.3, Eqns. (43) and (51), then it can obtain:
Theorem 3.5 shows that the system (5) has the U-H stability.

3.3 Illustrative example

Example 1. Let’s assume a fractional Volterra-Fredholm system as follows

\[
\begin{align*}
\theta'(v) + D_0^\alpha \Theta(v) &= \frac{1}{100} v \cos \Theta(v) + \Theta(v) \sin v + \frac{1}{50} \sin \Theta(\zeta) + \frac{1}{50} \int_0^v \cos \Theta(\zeta) d\zeta, \\
\Theta(0) &= 0
\end{align*}
\]

By comparison with the system (5), it can get:

\[
\begin{align*}
\alpha &= \frac{1}{2}, \\
\gamma(v, \Theta(v)) &= \frac{1}{100} [v \cos \Theta(v) + \Theta(v) \sin v] Y(v, \zeta, \Theta(\zeta)) = \frac{1}{50} \sin \Theta(\zeta), \\
\Psi(v, \zeta, \Theta(\zeta)) &= \frac{1}{50} \sin \Theta(\zeta)
\end{align*}
\]

Then:

\[
\begin{align*}
|g(v, h_1) - g(v, h_2)| &\leq \frac{1}{50} |h_1 - h_2|, h_1, h_2 \in \mathbb{R}, v \in [0, 1] \\
|Y(v, \zeta, \Theta(\zeta)) - Y(v, \zeta, \Theta(\zeta))| &\leq \frac{1}{50} |h_1 - h_2|, h_1, h_2 \in \mathbb{R}, v, \zeta \in [0, 1] \\
|\Psi(v, \zeta, \Theta(\zeta)) - \Psi(v, \zeta, \Theta(\zeta))| &\leq \frac{1}{50} |h_1 - h_2|, h_1, h_2 \in \mathbb{R}, v, \zeta \in [0, 1]
\end{align*}
\]

Let \( \zeta(v) = e^v \), it can obtain:

\[
\int_0^v E_{1-\xi} (-(v - \zeta))^\frac{3}{4} e^v d\zeta < e^v - 1 < \frac{3}{4} e^v, v \in [0, 1]
\]

Here, it has \( e_1 = e_2 = e_3 = 1 \), \( \xi = \frac{3}{4}, \) and \( (e_1 + e_2 + e_3) \xi = 0.045 < 1 \).

It can see that all the conditions in Theorems 3.2 and 3.3 are satisfied. Then, the system (56) is U-H stability, U-H-R stability and semi-U-H-R stability.

4. CONCLUSIONS

The objective of this study was to provide and demonstrate a novel stability theorem for the nonlinear Volterra-Fredholm integro-differential equation with Caputo fractional derivative utilising the weighted space method and fixed-point technique. The study specifically examines the H-U-R stability and semi-U-H-R stability results.

Besides, a class of nonlinear fractional Volterra-Fredholm integro-differential equations with initial conditions is discussed. By means of the Banach fixed-point techniques and weighted space, stability of the fractional nonlinear Volterra-Fredholm system has been tested. An illustrative example that demonstrates the applicability of the results has been included.

Discussing U-H-Mittag-Leffler stability [22] and finite-time stability [23] for the -Hilfer fractional Volterra-Fredholm integro-differential equations with time-varying delay terms would be a delightful extension of the current results. This will be the focus of future research.

REFERENCES


