IIETA International Information and Engineering Technology Association
Advancing the World of Information and Engineering

Mathematical Modelling of Engineering Problems
Vol. 10, No. 2, April, 2023, pp. 433-441
Journal homepage: http://iieta.org/journals/mmep

# Two-Step Hybrid Block Method with Generalized Two Off-Step Points Within Each Step for Solving Second Order Initial Value Problems

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https://doi.org/10.18280/mmep.100207

**Received:** 18 October 2022
**Accepted:** 25 January 2023

***Keywords:***
*two-step, hybrid block method, generalized off-step points, second order initial value problems*

**ABSTRACT**

This article develops a new two-step hybrid block method for the numerical solution of second order initial value problems with better accuracy. Two off-step points are introduced in generalized form and the resulting block method is developed using interpolation approach with interpolation points at one on-step point and one off-step point. The hybrid points are given in a generalized form to give room for flexibility of the choice of hybrid points which will give more information on which points produces the best solutions. The resultant order seven block method obtained satisfied all basic properties such as order, zero stability, consistency and convergence and produces better accuracy than existing numerical methods for solving second-order initial value problems. Thus, justifying the adoptability of the new block method for solving second-order initial value problems.

## 1. INTRODUCTION

This article focuses on solving second order initial value problems (IVPs) in the form:

$$y'' = f(x, y, y'),\ y(a) = \eta_0, y'(a) = \eta_1 \tag{1}$$

using hybrid block methods with generalized off-step points.

IVPs are a class of ordinary differential equations (ODEs) where a dependent variable and one independent variable are related, and the order being second order results from the value of the highest derivative in the differential equation [1-3]. Numerical approaches are adopted to solve differential equations generally when the exact solution does not exist or is difficult to obtain [4], and for IVPs in the form of Eq. (1), the conventional implementation procedure to solve using a numerical approach was its reduction from a second order IVP to a system of two first order IVPs. This reduction approach is discussed by Awoyemi [5] and said to be saddled with the disadvantage of high computational time and low accuracy. This led to another implementation procedure using predictor-corrector procedure to directly solve IVPs. Studies [6-10] utilised the predictor-corrector procedure. Shokri [6] developed a two-step block method, Panopoulos et al. [7] developed an eight-step symmetric embedded predictor-corrector method, Ndanusa and Tafida [8] introduced predictor-corrector methods of high order, Biasa et al. [9] obtained a predictor-corrector block iteration method, and a predictor-corrector linear multistep method was developed in Kayode and Adegboro [10]. Although, the predictor-corrector procedure was a direct procedure instead of using the concept of reduction, further studies discovered that block methods gave better solutions that predictor-corrector because the order of the predictors are less than the correctors and this impacts the accuracy negatively [11, 12].

Block methods were first proposed by Milne [13] and Sarafyan [14] to obtain starting values for predictor-corrector methods. Adesanya [15] described the formulation of block methods as evaluation of a multistep method at different grid and/or off-points to generate a system of numerical schemes that can be applied to produce approximate solutions at the grid and/or off-grid points simultaneously.

Block methods developed using only grid points are seen in studies [16-21] but the consideration of off-grid points in addition have been seen to result in better accuracy. The use of both grid and off-points in block methods led to the name hybrid block methods. Some studies that have developed hybrid block methods to obtain solutions for models in the form of Eq. (1) include Olukunle and Felix [22] where a one-step hybrid scheme was developed, Shokri et al. [23] which proposed a symmetric two-step semi-hybrid scheme, Ehigie et al. [24] with a generalised two-Step continuous linear multistep method of hybrid type, Fasasi et al. [25] developing a one-step continuous hybrid block method, and Gebremedhin and Jena [26] where the approximate solution of certain ODEs were obtained using a hybrid block approach.

Specifically, this article considers an extension of the previous study by Mansor et al. [27] where a two-step hybrid block method with one generalized off-step point was proposed. In their work, the authors considered one off-step point within each step to find the direct solution of the second order IVPs. Although, the results obtained had good accuracy, there is still room for improvement as only one off-step point between two-step intervals was considered in deriving the method. Therefore, an improvement is introduced in this article by introducing two off-step points within each step for a two-step block method (See Figure 1). The aim of introducing more off-step points is to increase the order of the method, thus, improving the accuracy of the solution. The off-step points are also introduced in a generalised form, such that various suitable values within each interval can be selected as off-step points.

Therefore, this article derives a new block method using interpolation and collocation approach as discussed in the next section. Section 3 tests the basic properties of a numerical method for the developed method, Section 4 shows the results obtained from selected numerical examples, and Section 5 concludes the article.

## 2. DEVELOPMENT OF THE TWO-STEP HYBRID BLOCK METHOD WITH GENERALIZED TWO OFF-STEP POINTS WITHIN EACH STEP

The following power series polynomial:

$$y(x)=\sum_{j=0}^{i+c-1} a_j\left(\frac{x-x_n}{h}\right)^j \tag{2}$$

is used as an approximate solution of Eq. (1) where $x\in(x_n, x_{n+2}]$ for $n$=0, 2, 4, ..., $N$-2 with $h=x_{\delta+1}-x_\delta$ and $\delta$=0, 1, 2, ..., N in the interval [a, b]. $i$ and $c$ denote the number of interpolation and collocation points respectively.

The interpolation-collocation strategy is illustrated in Figure 1 below where two off-step points within each step are denoted by $p$, $q$, $r$, and $s$ for $0<p<q<1<r<s<2$.

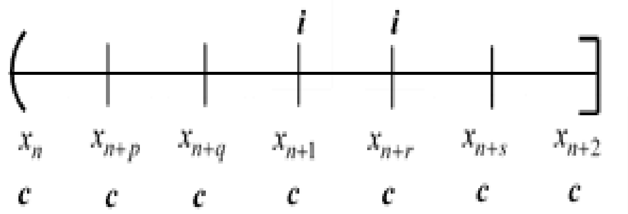


**Figure 1.** Interpolation and collocation points for two-step hybrid block method with generalized two off-step points within each step

Referring to Figure 1, $i$=2 and $c$=7. Substituting these values in Eq. (2) gives:

$$y(x)=\sum_{j=0}^{8} a_j\left(\frac{x-x_n}{h}\right)^j \tag{3}$$

whose second derivative is:

$$y''(x)=f(x,y,y')=\sum_{j=2}^{8} a_j\frac{j(j-1)}{h^2}\left(\frac{x-x_n}{h}\right)^{j-2} \tag{4}$$

Interpolating Eq. (2) at $x_{n+\sigma}$ ($\sigma$=1, $r$) yields:

$$y_{n+\sigma}=\sum_{j=0}^{8} a_j\left(\frac{x-x_n}{h}\right)^j \tag{5}$$

and collocating Eq. (4) at all points $x_{n+\sigma}$ ($\sigma$=0, $p$, $q$, 1, $r$, $s$, 2) produces:

$$f_{n+\sigma}=\sum_{j=2}^{8} a_j\frac{j(j-1)}{h^2}\left(\frac{x-x_n}{h}\right)^{j-2} \tag{6}$$

which form nine simultaneous equations involving the unknown coefficients $a_j$'s. The values of $a_j$'s are obtained using Gaussian elimination method and then substituted back in Eq. (2) to produce a continuous implicit hybrid two-step scheme with generalised two off-step points as below:

$$\begin{aligned} y(x)=&\sum_{j=0,p}\alpha_j(x)y_{n+j}\\ &+\sum_{j=0}^{3}\beta_j(x)f_{n+j}+\sum_{j=p,q,r}\beta_j(x)f_{n+j}. \end{aligned} \tag{7}$$

which is then evaluated at the non-interpolating points, $x_{n+\mu}$ ($\mu$=0, $p$, $q$, $s$, 2).

Differentiating Eq. (7) once, we get:

$$\begin{aligned} y'(x)=&\sum_{j=0,p}\frac{\partial}{\partial x}\alpha_j(x)y_{n+j}\\ &+\sum_{j=0}^{3}\frac{\partial}{\partial x}\beta_j(x)f_{n+j}+\sum_{j=p,q,r}\frac{\partial}{\partial x}\beta_j(x)f_{n+j}. \end{aligned} \tag{8}$$

Evaluating Eq. (8) at $x_n$ leads to the formation of a discrete implicit hybrid two-step scheme which can be represented in matrix form as:

$$\begin{aligned} A^{2[2]_2}Y^{2[2]_2}=&B_1^{2[2]_2}R_1^{2[2]_2}+B_2^{2[2]_2}R_2^{2[2]_2}\\ &+h^2\left[D^{2[2]_2}R_3^{2[2]_2}+E^{2[2]_2}R_4^{2[2]_2}\right]. \end{aligned} \tag{9}$$

Multiplying Eq. (9) by the inverse of $A^{2[4]_2}$ yields:

$$\begin{aligned} I_{6\times 6}Y^{2[2]_2}=&\bar{B}_1^{2[2]_2}R_1^{2[2]_2}+\bar{B}_2^{2[2]_2}R_2^{2[2]_2}\\ &+h^2\left[\bar{D}^{2[2]_2}R_3^{2[2]_2}+\bar{E}^{2[2]_2}R_4^{2[2]_2}\right]. \end{aligned} \tag{10}$$

which is equivalent to:

$$\begin{aligned} y_{n+p}=&y_n+hpy'_n+h^2[\bar{D}_{11}f_n+\bar{E}_{11}f_{n+p}+\bar{E}_{12}f_{n+q}\\ &+\bar{E}_{13}f_{n+1}+\bar{E}_{14}f_{n+r}+\bar{E}_{15}f_{n+s}+\bar{E}_{16}f_{n+2}]\\ y_{n+q}=&y_n+hqy'_n+h^2[\bar{D}_{21}f_n+\bar{E}_{21}f_{n+p}+\bar{E}_{22}f_{n+q}\\ &+\bar{E}_{23}f_{n+1}+\bar{E}_{24}f_{n+r}+\bar{E}_{25}f_{n+s}+\bar{E}_{26}f_{n+2}]\\ y_{n+1}=&y_n+hy'_n+h^2[\bar{D}_{31}f_n+\bar{E}_{31}f_{n+p}+\bar{E}_{32}f_{n+q}\\ &+\bar{E}_{33}f_{n+1}+\bar{E}_{34}f_{n+r}+\bar{E}_{35}f_{n+s}+\bar{E}_{36}f_{n+2}]\\ y_{n+r}=&y_n+hry'_n+h^2[\bar{D}_{41}f_n+\bar{E}_{41}f_{n+p}+\bar{E}_{42}f_{n+q}\\ &+\bar{E}_{43}f_{n+1}+\bar{E}_{44}f_{n+r}+\bar{E}_{45}f_{n+s}+\bar{E}_{46}f_{n+2}]\\ y_{n+s}=&y_n+hsy'_n+h^2[\bar{D}_{51}f_n+\bar{E}_{51}f_{n+p}+\bar{E}_{52}f_{n+q}\\ &+\bar{E}_{53}f_{n+1}+\bar{E}_{54}f_{n+r}+\bar{E}_{55}f_{n+s}+\bar{E}_{56}f_{n+2}] \end{aligned}$$

$$y_{n+2} = y_n + 2hy'_n + h^2[\bar{D}_{61} f_n + \bar{E}_{61} f_{n+p} + \bar{E}_{62} f_{n+q} + \bar{E}_{63} f_{n+1} + \bar{E}_{64} f_{n+r} + \bar{E}_{65} f_{n+s} + \bar{E}_{66} f_{n+2}].$$

Combining Eq. (8) at $x_{n+\mu}$ ($\mu$=$p$, $q$, 1, $r$, $s$, 2) and Eq. (10) produces the following first derivative of the main block:

$$\begin{aligned}
y'_{n+p} &= y'_n + h[D'_{11} f_n + E'_{11} f_{n+p} + E'_{12} f_{n+q} + E'_{13} f_{n+1} + E'_{14} f_{n+r} + E'_{15} f_{n+s} + E'_{16} f_{n+2}] \\
y'_{n+q} &= y'_n + h[D'_{21} f_n + E'_{21} f_{n+p} + E'_{22} f_{n+q} + E'_{23} f_{n+1} + E'_{24} f_{n+r} + E'_{25} f_{n+s} + E'_{26} f_{n+2}] \\
y'_{n+1} &= y'_n + h[D'_{31} f_n + E'_{31} f_{n+p} + E'_{32} f_{n+q} + E'_{33} f_{n+1} + E'_{34} f_{n+r} + E'_{35} f_{n+s} + E'_{36} f_{n+2}] \\
y'_{n+r} &= y'_n + h[D'_{41} f_n + E'_{41} f_{n+p} + E'_{42} f_{n+q} + E'_{43} f_{n+1} + E'_{44} f_{n+r} + E'_{45} f_{n+s} + E'_{46} f_{n+2}] \\
y'_{n+s} &= y'_n + h[D'_{51} f_n + E'_{51} f_{n+p} + E'_{52} f_{n+q} + E'_{53} f_{n+1} + E'_{54} f_{n+r} + E'_{55} f_{n+s} + E'_{56} f_{n+2}] \\
y'_{n+2} &= y'_n + h[D'_{61} f_n + E'_{61} f_{n+p} + E'_{62} f_{n+q} + E'_{63} f_{n+1} + E'_{64} f_{n+r} + E'_{65} f_{n+s} + E'_{66} f_{n+2}].
\end{aligned}$$

The coefficients $\{\bar{D}_{11}, \bar{D}_{21}, \ldots, \bar{D}_{61}\}$, $\{\bar{E}_{11}, \bar{E}_{12}, \ldots, \bar{E}_{66}\}$, $\{D'_{11}, D'_{21}, \ldots, D'_{61}\}$, and $\{E'_{11}, E'_{12}, \ldots, E'_{66}\}$ are obtained with respect to the value of the off-step selected (See Appendix).

The next section discusses the properties of the block method in its generalized form, while in Section 4, comparison will be made with the methods in previous studies for some numerical examples. This is to show the advantage of the improvement introduced in this article over the existing studies.

## 3. PROPERTIES OF THE METHOD

### Order of the Method

The linear difference operator $L$ associated with Eq. (10) is defined as:

$$L[y(x);h] = I_{6\times 6} Y^{2[2]_2} - \bar{B}_1^{2[2]_2} R_1^{2[2]_2} + \bar{B}_2^{2[2]_2} R_2^{2[2]_2} - h^2 \left[ \bar{D}^{2[2]_2} R_3^{2[2]_2} + \bar{E}^{2[2]_2} R_4^{2[2]_2} \right]. \tag{11}$$

where, $y(x)$ is an arbitrary test function continuously differentiable on $a$. Expanding the components of $Y^{2[2]_2}$ and $R_4^{2[2]_2}$ in Taylors series about $x$=$x_n$ and collecting the terms in powers of $h$ gives:

$$L[y(x);h] = \bar{C}_0^{2[2]_2} y(x) + \bar{C}_1^{2[2]_2} hy'(x) + \bar{C}_2^{2[2]_2} h^2 y''(x) + \ldots \tag{12}$$

**Definition 3.1** Hybrid block method (10) and associated linear operator in (11) are said to be of order $d$ if $\bar{C}_0^{2[2]_2} = \bar{C}_1^{2[2]_2} = \bar{C}_2^{2[2]_2} = \ldots = \bar{C}_{d+1}^{2[2]_2} = 0$ and $\bar{C}_{d+2}^{2[2]_2} \neq 0$ with error vector constants $\bar{C}_{d+1}^{2[2]_2}$ [28].

**Definition 3.2** A linear multistep method is consistent if it has order d≥1 [28].

Expanding all terms in Eq. (11) using Taylor series about $x_n$, the order of method is found to be $[7, 7, 7, 7, 7, 7]^T$ which implies that the new two-step hybrid block method (10) is consistent since its order is greater than 1.

### Zero Stability

The new two-step hybrid block method in (10) is zero-stable if no root of the first characteristic polynomial $\pi(\omega) = \left|\omega I_{6\times 6} - \bar{B}_1^{2[2]_2}\right|$ is having a modulus greater than one and every root of modulus one is simple, where $I_{6\times 6}$ is the identity matrix and $\bar{B}_1^{2[2]_2}$ is the coefficients matrix of $y_n$ function.

By setting determinant $\pi(\underline{\omega})$=0, then $\pi(\omega)=\omega^5(\omega-1)=0$ which implies $\omega$=0, 0, 0, 0, 0, 1. Hence, the newly developed method is zero-stable.

### Consistency and Convergence

**Theorem 3.1** Consistency and zero stability are sufficient conditions for a linear multistep method to be convergent as seen in study [28].

Based on the above theorem, the developed method satisfies the properties for consistency and zero-stability as shown above, and thus is convergent.

## 4. NUMERICAL EXAMPLES

**Problem 1**

$y'' + \left(\frac{6}{x}\right) y' + \left(\frac{4}{x^2}\right) y = 0, y(1) = 1, y'(1) = 1$ with $h = \frac{1}{320}$.

Exact Solution: $y(x) = \frac{5}{3x} - \frac{4}{x^2}$.

Source: Studies [29] and [30].

**Problem 2**

$y''$-$y'$=0, $y(0)$=1, $y'(0)$=-1 with $h = \frac{1}{10}$.

Exact Solution: $y(x)$=1-$e^x$.

Source: Studies [31] and [32]

**Problem 3**

$y''$-2$y^3$=0, $y(1)$=1, $y'(1)$=-1 with $h = \frac{1}{10}$.

Exact Solution: $y(x) = \frac{1}{x}$.

Source: Study [33]

Tables 1, 2, and 3 have shown the results obtained for solving Problems 1, 2, and 3 respectively, which include linear and nonlinear examples, and the comparison of these results with existing studies. For Problem 1, the solution using the newly developed two-step hybrid block was compared with the one-step hybrid block method developed in Abdelrahim [29] and the continuous implicit hybrid one-step method from Anake [30]. The results in Table 1 show the new method having distinctly better accuracy than Abdelrahim [29] and Anake [30]. For Problems 2 and 3, the existing approaches [27, 31-33] whose solutions were compared with the new hybrid block method compared closely to the new hybrid block method, however there was still a clear improvement in the solutions obtained by the new hybrid block method.

**Table 1.** Comparison of absolute errors obtained by new method with the previous studies for solving Problem 1

| $x$ | Absolute Error [29] | Absolute Error [30] | Absolute Error New Method $p = 4/5,\ q = 9/10,\ r = 101/100,\ q = 11/10.$ |
|---|---|---|---|
| 1.0031 | 2.66121e-12 | 0.77010e-12 | 1.55431e-15 |
| 1.0063 | 3.04912e-11 | 0.71780e-09 | 4.44089e-15 |
| 1.0094 | 9.77780e-11 | 0.19177e-08 | 6.66134e-15 |
| 1.0125 | 1.97586e-10 | 0.35709e-08 | 1.59872e-14 |
| 1.0156 | 3.28364e-10 | 0.56571e-08 | 3.70815e-14 |
| 1.0188 | 4.88623e-10 | 0.81569e-08 | 5.46230e-14 |
| 1.0219 | 6.76930e-10 | 0.11051e-07 | 8.81517e-14 |
| 1.0250 | 8.91909e-10 | 0.14323e-07 | 1.18794e-13 |
| 1.0281 | 1.13224e-09 | 0.17953e-07 | 1.62093e-13 |
| 1.0313 | 1.39664e-09 | 0.21925e-07 | 2.03837e-13 |

**Table 2.** Comparison of absolute errors obtained by new method with the previous studies for solving Problem 2

| $x$ | Absolute Error [31] | Absolute Error [32] | Absolute Error New Method $p = 1/4,\ q = 3/4,\ r = 5/4,\ q = 7/4.$ |
|---|---|---|---|
| 0.1 | 2.508826e-13 | 5.7260e-06 | 2.498002e-16 |
| 0.2 | 6.493175e-11 | 6.6391e-06 | 2.903233e-14 |
| 0.3 | 1.683146e-09 | 7.0283e-06 | 6.858236e-12 |
| 0.4 | 1.700635e-08 | 7.4539e-06 | 1.444056e-11 |
| 0.5 | 1.025454e-07 | 7.8935e-06 | 3.112266e-11 |
| 0.6 | 2.558711e-06 | 8.1942e-06 | 4.960188e-11 |
| 0.7 | 5.273300e-06 | 8.1810e-06 | 8.016476e-11 |
| 0.8 | 8.275935e-06 | 8.1810e-06 | 1.139946e-10 |
| 0.9 | 1.161667e-16 | 8.1730e-06 | 1.637674e-10 |
| 1.0 | 1.542187e-05 | 8.1650e-06 | 2.188389e-10 |

**Table 3.** Comparison of absolute errors obtained by new method with the previous studies for solving Problem 3

| $x$ | Absolute Error [27] | Absolute Error [33] | Absolute Error New Method $p = 1/4,\ q = 3/4,\ r = 5/4,\ q = 7/4.$ |
|---|---|---|---|
| 1.1 | 8.076931e-08 | 1.37360e-06 | 2.810252e-11 |
| 1.2 | 2.966144e-07 | 1.55450e-06 | 4.390854e-11 |
| 1.3 | 6.900244e-07 | 2.25690e-06 | 2.088394e-08 |
| 1.4 | 1.141509e-06 | 2.38050e-06 | 4.254005e-08 |
| 1.5 | 1.670007e-06 | 2.63360e-06 | 6.852856e-08 |
| 1.6 | 2.253526e-06 | 2.90400e-06 | 9.633702e-08 |
| 1.7 | 2.903171e-06 | 2.95520e-06 | 1.269875e-07 |
| 1.8 | 3.616883e-06 | 3.07970e-06 | 1.602706e-07 |
| 1.9 | 4.402479e-06 | 3.25070e-06 | 1.966604e-07 |
| 2.0 | 5.262784e-06 | 3.28141e-06 | 2.363168e-07 |

## 5. CONCLUSIONS

This study has successfully developed a new two-step hybrid block method with two off-step points within each step for directly solving second order initial value problems of ODEs. The effect of introducing additional off-step points was considered to justify the property that the introduction of more off-step points increases order and thus improves accuracy. The new method was tested on some existing second initial value problems available in literature. These problems included both linear and nonlinear IVPs which covers both types of second order IVPs. The numerical results indicate that the new method produces better accuracy than the previous methods when solving the same linear and nonlinear second order IVPs as considered by existing studies.


## ACKNOWLEDGMENT

This research was supported by Ministry of Higher Education (MOHE) of Malaysia through Fundamental Research Grant Scheme (FRGS/1/2017/STG06/UUM/01/1).


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**APPENDIX**

$$\overline{D}_{16} = \frac{-p^2}{1680qrs}(5p^5 - 560qrs - 8p^4(3 + q + r + s) + 14p^3(2 + 3s + r(3 + s) + q(3 + r + s)) - 28p^2(2s + r(2 + 3s) + q(2 + 3s + r(3 + s))) + 70p(2rs + q(2s + r(2 + 3s)))),$$

$$\overline{D}_{26} = \frac{q^2}{1680prs}(2p(4q^4 + 280rs - 7q^3(3 + r + s) + 14q^2(2 + 3s + r(3 + s)) - 35q(2s + r(2 + 3s))) + q(-5q^4 - 140rs + 8q^3(3 + r + s) - 14q^2(2 + 3s + r(3 + s)) + 28q(2s + r(2 + 3s)))),$$

$$\overline{D}_{36} = \frac{1}{1680pqrs}(11 - 20r - 20s + 42rs + q(42r - 20 + 42s - 112rs) + 2p(21r - 10 + 21s - 56rs + 7q(3 - 8r - 8s + 35rs))),$$

$$\overline{D}_{46} = \frac{r^2}{1680pqs}(-2p(r(70s - 4r^3 + 7r^2(3 + s) - 14r(2 + 3s)) + 7q(r^3 - 40s - 2r^2(3 +$$

$s) + 5r\,(2 + 3s))) + r(r(-5r^3 + 56s\ + 8r^2\,(3 + s)\ - 14r\,(2 + 3s)) + 2q(4r^3 - 70s - 7r^2(3 + s) + 14r(2 + 3s)))),$

$\bar{D}_{56} = \frac{s^2}{1680pqr}\,(2p\,(7q\,(r\,(40 - 15s + 2s^2) - s\,(10 - 6s + s^2)) + s(-7r(10 - 6s + s^2) + s(28 - 21s + 4s^2)))\ + s\,(-2q(s\,(21s - 28 - 4s^2) + 7r\,(10 - 6s\ + s^2)) + s(s(24s - 28 - 5s^2) + r(56 - 42s + 8s^2)))),$

$\bar{D}_{66} = \frac{2}{105pqrs}(4(r + s - 2) + q(4 - 7rs) + p(4 - 7rs + 7q(5rs - r - s))),$

$\bar{E}_{11} = \frac{p^2}{840\,(2-3p+p^2)(p-q)(p-r)(p-s)}\,(15\,p^5 - 280qrs\ - 20\,p^4\,(3 + q + r + s) + 28p^3(2 + 3s + r\,(3 + s) + q\,(3 + r + s))\ - 42p^2\,(2s\ + r\,(2 + 3s)\ + q\,(2 + 3s\ + r\,(3 + s))) + 70p(2rs + q(2s + r(2 + 3s)))),$

$\bar{E}_{12} = \frac{p^4}{840\,(p-q)q(2-3q+q^2)(q-r)(q-s)}(5p^4 + 140rs - 8p^3(3 + r + s) + 14p^2\,(2 + 3s + r(3 + s)) - 28p(2s + r(2 + 3s))),$

$\bar{E}_{13} = \frac{p^4}{840(p-1)(q-1)(r-1)(s-1)}\,(5p^4 + 140qrs - 8p^3\,(2 + q + r + s) + 14p^2(2s + r(2 + s) + q(2 + r + s)) - 28p(2rs + q(2s + r(2 + s)))),$

$\bar{E}_{14} = \frac{-p^4}{840r(r-p)(r-q)(2-3r+r^2)(r-s)}\,(5p^4 + 140qs - 8p^3\,(3 + q + s)\ + 14p^2\,(2 + 3s + q(3 + s)) - 28p(2s + q(2 + 3s))),$

$\bar{E}_{15} = \frac{p^4}{840\,(q-s)\,s(s-p)\,(s-r)(\,2-3s+s^2)}(\,5p^4\ + 140qr - 8p^3\,(3 + q + r) + 14p^2(2 + 3r + q(3 + r)) - 28p(2r + q(2 + 3r))),$

$\bar{E}_{16} = -\,\frac{p^4}{1680(p-2)(q-2)(r-2)(s-2)}(\,5p^4 + 70qrs - 8p^3\,(1 + q + r + s) + 14p^2\,(r + s + rs + q(1 + r + s)) - 28p(rs + q(r + s + rs))),$

$\bar{E}_{21} = \frac{q^4}{840p(2-3p+p^2)(p-q)(p-r)(p-s)}(8q^3\,(3 + r + s) - 5q^4 - 140rs - 14q^2\,(2 + 3s + r\,(3 + s)) + 28q(2s + r(2 + 3s))),$

$\bar{E}_{22} = \frac{q^2}{840(p-q)(2-3q+q^2)(q-r)(q-s)}(2p\,(10\,q^4 + 140rs - 14q^3(3 + r\ + s) + 21q^2\,(2\ + 3s + r\,(3 + s)) - 35q\,(2s + r(2 + 3s))) + q\,(20q^3(3 + r + s) - 15q^4 - 140rs - 28q^2(2 + 3s + r(3 + s)) + 42q(2s + r(2 + 3s)))),$

$\bar{E}_{23} = \frac{q^4}{840\,(p-1)(q-1)(r-1)(s-1)}(q\,(\,5q^3 - 56rs\ - 8q^2(2 + r\ + s)\ + 14q\,(2s\ + r(2 + s))) - 2p(4q^3 - 70rs - 7q^2(2 + r + s) + 14q(2s + r(2 + s)))),$

$\bar{E}_{24} = \frac{q^4}{840r(-p+r)(r-q)(2-3r+r^2)(r-s)}\,(q(56s - 5q^3 + 8q^2\,(3 + s) - 14q\,(2 + 3s)) + 2p(4q^3 - 70s - 7q^2(3 + s) + 14q(2 + 3s))),$

$\bar{E}_{25} = \frac{q^4}{840\,(r-s)s(s-p)(s-q)(2-3s+s^2)}(q\,(5q^3\ - 56r - 8q^2(3 + r\,) + 14q\,(2 + 3r)) - 2p(4q^3 - 70r - 7q^2(3 + r) + 14q(2 + 3r))),$

$\bar{E}_{26} = \frac{q^4}{1680(p-2)(q-2)(r-2)(s-2)}\,(q\,(\,28rs\ - 5q^3 + 8q^2\,(\,1 + r + s\,) - 14q\,\,(\,r + s\ + rs\,)) + 2p(4q^3 - 35rs - 7q^2(1 + r + s) + 14q(r + s + rs))),$

$\bar{E}_{31} = \frac{1}{840\,p\,(2\,-3p\,+\,p^2\,)\,(p-q)\,\,(p-r\,)\,(\,p-s\,)}(\,11\ - 20s\ + r\,(\,42\,s\ - 20\,)\ + q(42s\ - 20 - 14r(8s - 3))),$

$\bar{E}_{32} = \frac{1}{840\,(p-q\,)\,q\,\,(\,2\,-3q\,+q^2\,)(\,q\,-r\,)(\,q\,-\,s)}(r\,(\,20\ - 42s) - 11\ + 20s\ + 2p\,(10\ - 21s + 7r(8s - 3))),$

$\bar{E}_{33} = \frac{1}{840(p-1)(q-1)(r-1)(s-1)}\,(\,25\ - 36r\ - 36s + 56rs + q\,(56\,r - 36 + 56\,s - 98rs) + p(-36 + 56r + 56s - 98rs + 14q(4 - 7r - 7s + 15rs))),$

$\bar{E}_{34} = \frac{1}{840r\,(r-p)\,(r-q)(2-3r+r^2)(r-s)}(11 - 20s\ + q(42s - 20) + p(42s - 20 - 14q(8s - 3))),$

$\bar{E}_{35} = \frac{1}{840(r-s)s(s-p)(s-q)(2-3s+s^2)}(q(20\ - 42r) - 11 + 20r + 2p(10 - 21r + 7q(8r - 3))),$

$\bar{E}_{36} = \frac{1}{1680(p-2)(q-2)(r-2)(s-2)}(8r - 5 + 8s - 14rs + 2q(4 - 7r - 7s + 14rs) - 2p(7r - 4 + 7s - 14rs + 7q(1 - 2r - 2s + 5rs))),$

$\bar{E}_{41} = \frac{r^4}{840p(2-3p+p^2)(p-q)(p-r)(p-s)}(r(56s\ - 5r^3 + 8r^2(3\ +\ s)\ - 14r\,(2\ + 3s)) + 2q(4r^3 - 70s - 7r^2(3 + s) + 14r(2 + 3s))),$

$\bar{E}_{42} = \frac{r^4}{840(p-q)q(2-3q+q^2)(q-r)(q-s)}(r(5r^3 - 56s\ - 8r^2(3\ +\ s)\ + 14r\,(2 + 3s)) - 2p(4r^3 - 70s - 7r^2(3 + s) + 14r(2 + 3s))),$

$\bar{E}_{43} = \frac{r^4}{840\,(p-1)(q-\,1)(r-1)(s-1)}\,(2p\,(\,7q\,(\,r^2\ + 10s\ - 2r\,(\,2 + s\,)) + r\,(\,7r\,(\,2 + s\,) - 4r^2 - 28s)) + r(r(5r^2 + 28s - 8r(2 + s)) + q(-8r^2 - 56s + 14r(2 + s)))),$

$\bar{E}_{44} = \frac{r^2}{840\,(p-r)(q-r)(2-\,3r\,+r^2)(r-s)}(2\,p\,(r\,(70\,s - 10\,r^3 + 14\,r^2\,(\,3 + s\,) - 21r\,(\,2 + 3s\,)) + 7q\,(\,2r^3 - 20s - 3r^2\,(\,3 + s) + 5r\,(\,2 + 3s\,))) + r\,(\,-2q(\,10r^3 - 70s - 14\,r^2\,(\,3 + s\,) + 21r(2 + 3s)) + r(15r^3 - 84s - 20r^2(3 + s) + 28r(2 + 3s)))),$

$\bar{E}_{45} = \frac{r^4}{840(r-s)s(-p+s)(s-q)(2-3s+s^2)}(2p(r(21r - 4r^2 - 28) + 7q(10 - 6r + r^2)) + r(q(42r - 56 - 8r^2) + r(28 - 24r + 5r^2))),$

$\bar{E}_{46} = \frac{r^4}{1680\,(p-2)(q-2)(r-2)(s-2)}\,(-2\,p\,(\,7q\,(\,r^2+5s\,-\,2r\,(1+s)) + r\,(7r\,(\,1+s\,) - 4r^2 - 14s)) + r(2q(4r^2+14s-7r(1+s)) + r(-5r^2-14s+8r(1+s)))),$

$\bar{E}_{51} = \frac{s^4}{840p(2-3p+p^2)(p-q)(p-r)(p-s)}\,(7r(10-6s+s^2) - 2q(s(21s-4s^2-28\,)) + s(s(24s-28-5s^2) + r(56-42s+8s^2))),$

$\bar{E}_{52} = \frac{s^4}{840\,(p-q)q(2-3q+q^2)(q-r)(q-s)}\,(2p\,(s\,(\,21s-4s^2-28)\,+\,7r\,(10-6s+s^2)) + s(r(-56+42s-8s^2) + s(28-24s+5s^2))),$

$\bar{E}_{53} = \frac{s^4}{840\,(p-1)(q-1)(r-1)(s-1)}\,(-2p\,(\,7q\,(\,2r(\,s-5) - (\,s-4\,)s) + s\,(-7r(s-4) + 2s(2s-7))) + s(2q(7r(-4+s) + 2(7-2s)s) + s(r(28-8s) + s(-16+5s)))),$

$\bar{E}_{54} = \frac{s^4}{840r(r-p)(-q+r)(2-3r+r^2)(r-s)}\,(-2p\,(s(\,21s-4s^2-28) + 7q(\,10-6s\,+s^2))\,+\,s(s(-28+24s-5s^2) + q(56-42s+8s^2))),$

$\bar{E}_{55} = \frac{s^2}{840\,(r-s)(s-p)(s-q\,)(2-3s+s^2)}\,(s\,(\,s\,(s\,(\,60s\,-15s^2\,-56) + 4r\,(\,21-\,21s\,+\,5s^2\,)) - 2q(\,7r(10-9s+2s^2) - 2s(21-21s+5s^2))) + 2p(7q(s(9s-10-2s^2) + r\,(20-15s+3s^2)) + s(-7r(10-9s+2s^2) + 2s(21-21s+5s^2)))),$

$\bar{E}_{56} = \frac{s^4}{1680(-2+p)(-2+q)(-2+r)(-2+s)}\,(s\,(-2q(7r(s-2) + (7-4s)s) + s((8-5s)s + 2r(4s-7))) + 2p(-7q(r(5-2s) + (-2+s)s) + s(-7r(-2+s) + s(-7+4s)))),$

$\bar{E}_{61} = -\frac{4}{105p(2-3p+p^2)(p-q)(p-r)(p-s)}\,(q(-4+7rs) - 4(-2+r+s)),$

$\bar{E}_{62} = \frac{4h^2}{105(p-q)q(2-3q+q^2)(q-r)(q-s)}\,(p(-4+7rs) - 4(-2+r+s)),$

$\bar{E}_{63} = \frac{4}{105(p-1)(q-1)(r-1)(s-1)}\,(-4(8r-10+q\,(8+7r(s-1) - 7s) + 8s-7rs) + p(-4(8+7r(-1+s) - 7s) + 7q(4-4s+r(-4+5s)))),$

$\bar{E}_{64} = -\frac{4h^2}{105r(-p+r)(-q+r)(2-3r+r^2)(r-s)}\,(p(-4+7qs) - 4(-2+q+s)),$

$\bar{E}_{65} = \frac{4h^2}{105(r-s)s(-p+s)(-q+s)(2-3s+s^2)}\,(p(-4+7qr) - 4(-2+q+r)),$

$\bar{E}_{66} = \frac{-2}{105(p-2)(q-2)(r-2)(s-2)}\,(24r-\,40+q(\,24+7r(s-2) - 14s\,) + 24s-14rs + p(24+7r(-2+s) - 14s + 7q(-2+r+s))),$

$\acute{D}_{16} = \frac{p}{840qrs}\,(\,420\,qrs\,-\,10\,p^5+\,14\,p^4\,(3\,+q+\,r\,+\,s) -\,21\,p^3\,(2+\,3s+\,r\,(\,3+\,s\,) + q\,(3+r\,+s))\,+35p^2\,(2s\,+r(2\,+3s) + q\,(2\,+3s+\,r\,(3\,+s)))\,\,-70p\,(\,2rs+\,q(2s+r(2+3s)))),$

$\acute{D}_{26} = \frac{q}{840prs}\,\,(7p(2q^4\,+60rs\,-3q^3(3\,+r\,+s)\,+\,5q^2\,(2\,+\,3s+\,r\,(3\,+s)) - 10q(2s+r(2+3s))) + q(14q^3\,(3+r+\,s) -\,10q^4 -\,140rs - 21q^2(2+3s+r(3+s)) + 35q(2s+r(2+3s)))),$

$\acute{D}_{36} = \frac{1}{840pqrs}\,(\,18\,-28r\,-28s\,+49rs\,-7q\,(4\,-7r\,-7s\,+15rs) + 7p\,(\,7r-4+7s-15rs+q(7-15r-15s+50rs))),$

$\acute{D}_{46} = \frac{r^2}{840pqs}\,(r(70s-10r^3+14r^2(3+s) - 21r(2+3s)) + 7q(2r^3-20s-9r^2-\,3r^2s+5r\,(2+3s)))\,\,-7p\,(r(\,20s-2r^3\,+\,3r^2(3+\,s) - 5r(2\,+3s)) + q\,(3r^3-60s-5r^2(3+s) + 10r(2+3s)))),$

$\acute{D}_{56} = \frac{s}{840pqr}\,(7p\,\,(q\,(s\,(15s-\,20-3s^2)\,\,+5r\,(12-6s+s^2)) + s(r(-20+15s-3s^2)\,\,+ s(10\,-9s+2s^2)))\,\,+s(-7q\,(s(-10\,+9s-2s^2) + r(20-15s+3s^2)) + s(7r(10-9s+2s^2) - 2s(21-21s+5s^2)))),$

$\acute{D}_{66} = \frac{1}{105pqrs}\,(7q(r+s-2) - 2(24+7r(s-2) - 14s) + 7p(q(5rs-2) - 2(r+s-2))),$

$\acute{E}_{11} = \,\frac{p}{420(2-3p+p^2)(p-q)(p-r)(p-s)}\,(60p^5-\,420\,qrs\,-70\,p^4\,(\,3\,+\,q\,+\,r\,+\,s\,) + 84p^3(2+3s+r\,\,(\,3+s\,)\,\,+\,\,q\,(\,3+r+s)) - 105\,p^2\,(\,2s+r\,(\,2+3s\,) +\,q\,(2+\,3s+r(3+s))) + 140p(2rs+q(2s+r(2+3s)))),$

$\acute{E}_{12} = \frac{p^3}{420\,(p-\,q)\,q\,(2-\,3q+\,q^2)\,(q-\,r)\,(q-\,s)}\,(10\,p^4+\,140\,r\,s\,-\,14\,p^3(\,3\,+\,r\,\,+\,s\,) + 21p^2\,(2\,+3s+r(3+s)) - 35p(2s+r(2+3s))),$

$\acute{E}_{13} = \frac{p^3}{420\,(p-1)(q-1)(r-1)(s-1)}\,(10p^4+\,140qrs\,-\,14p^3(2\,+q\,+r+\,s) + 21p^2(2s+r(2+s) + q(2+r+s)) - 35p(2rs+q(2s+r(2+s)))),$

$\acute{E}_{14} = \frac{p^3}{420r(r-p)(r-q)(2-3r+r^2)(r-s)}\,(14p^3(3+q+s) - 10p^4-140qs-21p^2(\,2+3s+q(3+s)) + 35p(2s+q(2+3s))),$

$\acute{E}_{15} = \frac{p^3}{420(r-s)s(s-p)(s-q)(2-3s+s^2)}\,(10p^4+140qr\,-\,14p^3(3+q+r) + 21p^2\,(2+3r+q(3+r)) - 35p(2r+q(2+3r))),$

$\acute{E}_{16} = \frac{p^3}{840\,(p-2)(q-2)(r-2)(s-2)}\,(14p^3\,(\,1+\,q+\,r+\,s) - 10p^4-70qrs-21p^2(r+s+rs+q(1+r+s)) + 35p(rs+q(r+s+rs))),$

$$\acute{E}_{21} = \frac{q^3}{420p(2-3p+p^2)(p-q)(p-r)(p-s)}(14q^3(3+r+s) - 10q^4 - 140rs - 21q^2(2+3s+r(3+s)) + 35q(2s+r(2+3s))),$$

$$\acute{E}_{22} = \frac{q}{420(p-q)(2-3q+q^2)(q-r)(q-s)}(7p(10q^4 + 60rs - 12q^3(3+r+s) + 15q^2(2+3s+r(3+s)) - 20q(2s+r(2+3s))) + q(-60q^4 - 280rs + 70q^3(3+r+s) - 84q^2(2+3s+r(3+s)) + 105q(2s+r(2+3s)))),$$

$$\acute{E}_{23} = \frac{q^3}{420(p-1)(q-1)(r-1)(s-1)}(-7p(2q^3 - 20rs - 3q^2(2+r+s) + 5q(2s+r(2+s))) + q(10q^3 - 70rs - 14q^2(2+r+s) + 21q(2s+r(2+s)))),$$

$$\acute{E}_{24} = \frac{q^3}{420r(r-p)(r-q)(2-3r+r^2)(r-s)}(q(70s - 10q^3 + 14q^2(3+s) - 21q(2+3s)) + 7p(2q^3 - 20s - 3q^2(3+s) + 5q(2+3s))),$$

$$\acute{E}_{25} = \frac{q^3}{420(r-s)s(s-p)(s-q)(2-3s+s^2)}(q(10q^3 - 70r - 14q^2(3+r) + 21q(2+3r)) - 7p(2q^3 - 20r - 3q^2(3+r) + 5q(2+3r))),$$

$$\acute{E}_{26} = \frac{q^3}{840(p-2)(q-2)(r-2)(s-2)}(q(35rs - 10q^3 + 14q^2(1+r+s) - 21q(r+s+rs)) + 7p(2q^3 - 10rs - 3q^2(1+r+s) + 5q(r+s+rs))),$$

$$\acute{E}_{31} = \frac{2}{420p(2-3p+p^2)(p-q)(p-r)(p-s)}(9 - 14s + \frac{49r}{2}(s - \frac{4}{7}) - \frac{7q}{2}(4 - 7s + r(15s-7))),$$

$$\acute{E}_{32} = \frac{2}{420(p-q)q(2-3q+q^2)(q-r)(q-s)}(14s - 9 - \frac{49r}{2}(s - \frac{4}{7}) + \frac{7p}{2}(4 - 7s + r(15s-7))),$$

$$\acute{E}_{33} = \frac{1}{420(p-1)(q-1)(r-1)(s-1)}(80 - 98r - 98s + 126rs - 7q(14 - 18r - 18s + 25rs) + 7p(-14 + 18r + 18s - 25rs + 18q - 25qs - 25qr + 40qrs)),$$

$$\acute{E}_{34} = \frac{2}{420r(r-p)(r-q)(2-3r+r^2)(r-s)}(9 - 14s + \frac{7q}{2}(7s - 4) - \frac{7p}{2}(4 - 7s + q(15s-7))),$$

$$\acute{E}_{35} = \frac{2}{420(r-s)s(s-p)(s-q)(2-3s+s^2)}(14r - 9 - \frac{7q}{2}(7r - 4) + \frac{7p}{2}(4 - 7r + q(15r-7))),$$

$$\acute{E}_{36} = \frac{1}{840(p-2)(q-2)(r-2)(s-2)}(14r - 10 + 14s - 21rs + 7q(2 - 3r - 3s + 5rs) - 7p(-2 + 3r + 3s - 5rs + q(3 - 5r - 5s + 10rs))),$$

$$\acute{E}_{41} = \frac{r^3}{420p(2-3p+p^2)(p-q)(p-r)(p-s)}(r(70s - 10r^3 + 14r^2(3+s) - 21r(2+3s)) + 7q(2r^3 - 20s - 3r^2(3+s) + 5r(2+3s))),$$

$$\acute{E}_{42} = \frac{r^3}{420(p-q)q(2-3q+q^2)(q-r)(q-s)}(-7p(2r^3 - 20s - 3r^2(3+s) + 5r(2+3s)) + r(10r^3 - 70s - 14r^2(3+s) + 21r(2+3s))),$$

$$\acute{E}_{43} = \frac{2r^3}{420(p-1)(q-1)(r-1)(s-1)}(r(2r(5r^2 + 21s - 7r(2+s)) - 7q(2r^2 + 10s - 3r(2+s))) + 7p(q(3r^2 + 20s - 5r(2+s)) + r(-2r^2 - 10s + 3r(2+s)))),$$

$$\acute{E}_{44} = \frac{r}{420(p-r)(q-r)(2-3r+r^2)(r-s)}(7p(r(40s - 10r^3 + 12r^2(3+s) - 15r(2+3s)) + q(12r^3 - 60s - 15r^2(3+s) + 20r(2+3s))) + r(-7q(10r^3 - 40s - 12r^2(3+s) + 15r(2+3s)) + 2r(30r^3 - 105s - 35r^2(3+s) + 42r(2+3s)))),$$

$$\acute{E}_{45} = \frac{r^3}{420(r-s)s(s-p)(s-q)(2-3s+s^2)}(7p(r(-10 + 9r - 2r^2) + q(20 - 15r + 3r^2)) + r(-7q(10 - 9r + 2r^2) + 2r(21 - 21r + 5r^2))),$$

$$\acute{E}_{46} = \frac{r^3}{840(p-2)(q-2)(r-2)(s-2)}(-7p(q(3r^2 + 10s - 5r(1+s)) + r(3r(1+s) - 2r^2 - 5s)) + r(7q(2r^2 + 5s - 3r(1+s)) + r(-10r^2 - 21s + 14r(1+s)))),$$

$$\acute{E}_{51} = \frac{s^3}{420p(2-3p+p^2)(p-q)(p-r)(p-s)}(-7q(s(9s - 10 - 2s^2) + r(20 - 15s + 3s^2)) + s(7r(10 - 9s + 2s^2) - 2s(21 - 21s + 5s^2))),$$

$$\acute{E}_{52} = \frac{s^3}{420(p-q)q(2-3q+q^2)(q-r)(q-s)}(7p(s(9s - 10 - 2s^2) + r(20 - 15s + 3s^2)) + s(-7r(10 - 9s + 2s^2) + 2s(21 - 21s + 5s^2))),$$

$$\acute{E}_{53} = \frac{s^3}{420(p-1)(q-1)(r-1)(s-1)}(-7p(q(5r(s-4) + (10-3s)s) + s(r(10-3s) + 2(s-3)s)) + s(7q(-2(s-3)s + r(-10+3s)) + 2s(-7r(s-3) + s(-14+5s)))),$$

$$\acute{E}_{54} = \frac{s^3}{420r(r-p)(r-q)(2-3r+r^2)(r-s)}(-7p(s(9s - 2s^2 - 10) + q(20 - 15s + 3s^2)) + s(7q(10 - 9s + 2s^2) - 2s(21 - 21s + 5s^2))),$$

$$\acute{E}_{55} = \frac{s}{420(q-s)(s-p)(s-r)(2-3s+s^2)}(7p(q(15r(s-2)^2 + s(-40 + 45s - 12s^2)) + s(r(-40 + 45s - 12s^2) + 2s(15 - 18s + 5s^2))) + s(2s(7r(15 - 18s + 5s^2) - 3s(28 - 35s + 10s^2)) - 7q(-2s(15 - 18s + 5s^2) + r(40 - 45s + 12s^2)))),$$

$$\acute{E}_{56} = \frac{s^3}{840(-2+p)(-2+q)(-2+r)(-2+s)}(s(s(2(7-5s)s + 7r(2s-3)) + 7q(r(5-3s) + s(-3+2s))) + 7p(q(5r(-2+s) + (5-3s)s) + s(r(5-3s) + s(-3+2s)))),$$

$\acute{E}_{61} = -\frac{4}{105p(2-3p+p^2)(p-q)(p-r)(p-s)}\ (24 + 7r(-2 + s) - 14s + 7q(-2 + r + s)),$

$\acute{E}_{62} = \frac{4}{105(p-q)q(2-3q+q^2)(q-r)(q-s)}(24 + 7r(-2 + s) - 14s + 7p(-2 + r + s)),$

$\acute{E}_{63} = \frac{4}{105\ (p-1)\ (q-1)\ (r-1)\ (s-1)}\ (\ 80\ -\ 56r -\ 56s\ +\ 42rs\ +\ 7p\ (\ 6r -\ 8 + q\ (\ 6 + 5r(s-1) - 5s) + 6s - 5rs) - 7q(8 - 6r - 6s + 5rs)),$

$\acute{E}_{64} = -\frac{4}{105r(-p+r)(-q+r)(2-3r+r^2)(r-s)}(24 + 7q(-2 + s) - 14s + 7p(-2 + q + s)),$

$\acute{E}_{65} = \frac{4}{105(r-s)s(-p+s)(-q+s)(2-3s+s^2)}(24 + 7q(-2 + r) - 14r + 7p(-2 + q + r)),$

$\acute{E}_{66} = \frac{1}{105(p-2)(q-2)(r-2)\ (s-2)}(7p\ (q\ (18 + 5r\ (s-2)\ - 10s) + 2(r(9 - 5s) - 16 + 9s)) + 2(200 - 112s + 7r(9s - 16) - 7q(16 - 9s + r(5s - 9))))$