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Analysis Simple Step Stress Model under Competing Extension Weibull Failure Distribution Based on Progressive Type-II Censoring

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ABSTRACT

Accelerated life testing (ALT), a procedure utilized in reliability analysis, allows testing units to be subject to increasingly elevated grades of stress during an experiment. Step-stress tests are a subclass of accelerated tests in which the stress levels rise consecutively at prearranged cycles, consequently, the researcher might find out results more swiftly than in ordinary working settings about the parameter of the lifetime distribution. Moreover, there are frequently multiple fatal causes for a test element's failure, for instance, technical or electric. These causes are recognized as "competing risks". The purpose of the analysis is to assess simple step stress accelerated life testing (SS-ALT) with competing Risks originating from the extension of Weibull distribution by applying a progressive Type-II censoring scheme. In this case, under the assumption of a cumulative exposure model, the authors successfully obtained the Bayes estimates (BEs) and maximum likelihood estimators (MLEs) of the undetermined average parameters of the various causes. For Bayesian computations, the squared error loss functions are considered. Additionally, the estimators' asymptotic variance-covariance matrix was created. Additionally, credible intervals and asymptotic confidence intervals (CIs) are provided. For a large sample size, the CIs of the unidentified parameters are developed. A numerical study is also involved to exhibit the accuracy and variability of various estimators for several sample sizes. An example is being used to exemplify the inference method that's also considered here. This study concludes that the mean lengths of credible intervals and asymptotic confidence intervals get shorter as the number of failures rises. The credible interval technique is suggested, nevertheless.

1. INTRODUCTION

Due to continual improvement in manufacturing design, it is more difficult to obtain information about the lifetime of products or materials with high reliability at the time of testing under normal conditions. This makes lifetime testing under these conditions very costive and takes a long time. To get information about the lifetime distribution of these materials, a sample of these materials is subjected to more severe operation conditions than normal ones. These conditions are called stresses which may be in the form of temperature, voltage, pressure, vibration, cycling rate, load, etc. This kind of testing is called an accelerated life test (ALT), where products are put under stresses higher than usual to yield more failure data in a short time. Furthermore, it has a wide spread use in (i) Materials, which include metals, plastics, rubber and elastics, concrete and cement, ceramics, and building materials. (ii) Products, which include semiconductors, microelectronics, capacitors, electrical devices, and mechanical components. (iii) Degradation mechanism, life fatigue, creep, and cracking. For more details about the above uses of ALT, see studies [1-3].

There are mainly three ALT methods. The first method is called the constant stress accelerated life test (CS-ALT); the second one is referred to as the step stress accelerated life test (SS-ALT) and the third is the progressive stress accelerated life test (PS-ALT). The first method is used when the stress remains unchanged so that if the stress is weak, the test has to last for a long time. But, the other two methods can reduce the testing time and save a lot of manpower, material sources, and money.

The major assumption of ALT is that the observable behavior under accelerated conditions can be related to the behavior under normal use conditions through a mathematical model called an acceleration model. Thus, life tests conducted under accelerated conditions can be used to make inferences about the behavior of a device in normal use conditions. So, it is necessary to consider the relationship between one or more than one parameter of the failure distribution and the accelerated conditions. The main difficulty of ALT lies in using the failure data obtained at higher conditions to predict the reliability, mean life or other quantities under normal use conditions. The acceleration model is then used to extrapolate the reliability performance to the normal use conditions. Where these different types of methods met a lot of researchers and for more details, you can look at the researches [4-15].

Regarding time and expense reductions, censoring is commonly utilized in life tests. There are various patterns of censoring. Type-I and type-II censoring, whose the first is censored at a particular time and the last is censored at a stable number, are the two most often used censoring techniques. Furthermore, at distinct stages of the research, it can be vital to eradicate a number of test units for a diversity of reasons. Progressive censoring will be concluded from this. Utilizing the resources available is exceedingly efficient and effective with progressive censoring Type-II. Consistent with the research [16]. It enables be summarized as putting n identical units through a life test at zero time. The surviving items t_h, R_h are randomly omitted when the time of h^{th}(h=1,2,…, m-1) failure occurs. While waiting for the m th failure t_m to be noticed, the test is terminated, and the remaining $R_m=n-m-\sum_{h=1}^{m-1} R_h$ elements are all omitted, where $m(m<n)$ and R_h are pre-fixed. Conventional Type-II censoring is a special case when $R_1=R_2=\cdots=R_{m-1}=0$ and $R_m=n-m$. Many authors have investigated competing risk data under this censoring strategy from other distributions; for more data [17-19], and others.

Based upon a reliability analysis, more than one fatal risk factor frequently contributes to a product's failure, as the inner structure and outer environmental are mutually complex. For instance, shaft or bearing failures may be linked to a bearing assembly failure. In actuality, the failure causes could be dependent or independent. Research [20] indicates that there is a difficulty with the underlying model's identifiability, even though a dependent risk temple would be further actual. The hypothesis of s-independent risks [21, 22], cannot be tested without the knowledge of the covariates. consequently, the failure causes are naturally expected to be independent of the objectives of exploring a competing risk model. One way to conceptualize a multi-component series system is as a prototype of an independent competing risk. Many investigators have explored competing risk models under the assumption that competing failure causes are independent. The analysis of ALT when more than one cause of failure is expressible was stated by Klein and Basu [23, 24]. The SS-ALT and CS-ALT in part have drawn the consideration of many researchers in realistic uses by employing life tests of several types with data from competing risks through different censored schemes and more clarifications can be seen in researches [25-41].

The Weibull model is superb at creating real phenomena with monotonous failure rates. However, the Weibull model should not be used for data with non-monotonous failure rates. The bathtub-shaped failure rate is one of the more practical non-monotonous failure rate functions, and it is used in a variety of literary contexts. For instance, in reliability engineering, it is observed that the lifecycle of an electronic constituent has a failure rate function with a bathtub shape, and also bio-analysis for the human death rate.

Chen [42] who examined a lifetime distribution with two-parameter with neither rising nor bathtub-shaped failure rates, inspected both possibilities. Its cumulative distribution function (CDF) is

$$F(t)=1-\exp\left(\vartheta\left(1-e^{t^{\eta}}\right)\right); t>0,\ \ \vartheta,\eta>0 \tag{1}$$

where, η>0 is the shape parameter and ϑ>0 is the scale parameter. The equivalent survival function is

$$S(t)=\exp\left(\vartheta\left(1-e^{t^{\eta}}\right)\right);\quad t>0 \tag{2}$$

The probability density function (pdf) is

$$f(t)=\vartheta\eta t^{\eta-1}\exp\left(\vartheta\left(1-e^{t^{\eta}}\right)\right);\ t>0,\ \ \vartheta,\eta>0 \tag{3}$$

This article could be designed as exhibited further down. The extension Weibull distribution is presented as a lifetime model in Section 2, along with a description of the model. The ML method is employed in Section 3 to represent point estimates of the parameters for the expansion Weibull distribution under simple SS-ALT using progressively censored competing risks data. The asymptotic variance and covariance matrix are explored in section 4. In Section 5, the Bayesian approach for estimating the unknown parameters is derived. The simulation studies and the illustrative example used to illustrate the theoretical findings are explained in Section 6. Finally, section 7 deals with the findings and conclusions.

2. DESCRIPTION OF THE MODEL

Suppose that S_0 points to the normal level of stress. Assume n identical units starting on SS-ALT beneath the premier level of stress S_1(S_1> S_0). As well as information on which risk factor triggered each failure, the number of times it occurred successively is recorded. At a pre-specific time $\tau\in(0,\infty)$, there is a rise in the level of stress from S_1 up to S_2 and the life test maintains until the m th (m is pre-determined) failure is noticed. At the h^{th} ($1\leq h\leq m$-1) failure time, R_h of the surviving elements are omitted, where R_h is pre-specific and $R_m=n-m-\sum_{h=1}^{m-1} R_h$.

1. Only a single of two independent causes compete for failures with lifetimes T_1 and T_2 could cause a product to fail. Thus, the failure time for products T=min(T_1, T_2).
2. The lifetime of the c^{th} failure cause T_{lc} follows an extension Weibull distribution with parameter scale ϑ_{lc}, the recognized shape parameter η_{lc} and l, c=1,2, under the level of stress S_l.
3. Under several levels of stress, the failure mechanisms are the same, i.e., $\eta_{1c}=\eta_{2c}=\eta$.
4. Following Nelson's cumulative exposure model (CEM) [3], the cumulative distribution exposure function of random variable T for simple SS-ALTs with one stress level change appears as the following:

$$F(t)=\begin{cases} F_1(t) & 0<t<\tau \\ F_2(t-\tau-u_1) & t\geq\tau \end{cases}$$

Consequently, the pdf and CDF of the lifetime T_{lc} could be calculated according to the formula:

$$F_c(t)=\begin{cases} 1-e^{\left[\vartheta_{1c}\left(1-e^{t^{\eta_c}}\right)\right]}; & 0<t<\tau \\ 1-e^{\left[\vartheta_{2c}\left(1-e^{t^{\eta_c}-\tau^{\eta_c}}\right)+\vartheta_{1c}\left(1-e^{\tau^{\eta_c}}\right)\right]}; & t\geq\tau \end{cases} \tag{4}$$

and

$$f_c(t)=\begin{cases} \eta_c\,\vartheta_{1c}t^{\eta_c-1}e^{\left[t^{\eta_c}+\vartheta_{1c}\left(1-e^{t^{\eta_c}}\right)\right]}; & 0<t<\tau \\ \eta_c\,\vartheta_{2c}t^{\eta_c-1}e^{\left[\left(t^{\eta_c}-\tau^{\eta_c}\right)+\vartheta_{2c}\left(1-e^{t^{\eta_c}-\tau^{\eta_c}}\right)+\vartheta_{1c}\left(1-e^{\tau^{\eta_c}}\right)\right]}; & t\geq\tau \end{cases} \tag{5}$$

5. The log-linear accelerated function (AF) of the c^{th} cause of failure:

$$\log\vartheta_{lc}=a_c+b_c\varphi(s_l) \tag{6}$$

whereas, a_c, b_c>0 are unidentified parameters, The provided decline function of the level of stress s is known as $\varphi(s)$. In this research, the Arrhenius model is applied, so $\varphi(s)=s^{-1}$.

For c=1,2. Since we will observe only the smaller of T_1 and T_2, let T =min(T_1, T_2) refer to the overall failure time of a test unit. Then, its CDF and PDF are easily achieved to be:

$$F_T(t) = 1 - \big(1 - G_1(t)\big)\big(1 - G_2(t)\big)$$

$$F_T(t) = \begin{cases} 1 - e^{\left[\vartheta_{11}\left(1-e^{t^{\eta_1}}\right)+\vartheta_{12}\left(1-e^{t^{\eta_2}}\right)\right]}; & 0 < t < \tau \\ 1 - e^{\left[\vartheta_{21}\left(1-e^{t^{\eta_1}-\tau^{\eta_1}}\right)+\vartheta_{11}\left(1-e^{\tau^{\eta_1}}\right)+\vartheta_{22}\left(1-e^{t^{\eta_2}-\tau^{\eta_2}}\right)+\vartheta_{12}\left(1-e^{\tau^{\eta_2}}\right)\right]}; & t \geq \tau \end{cases} \quad (7)$$

$$f_T(t) = \begin{cases} \left[\eta_1\vartheta_{11}t^{\eta_1-1}e^{(t^{\eta_1})} + \eta_2\vartheta_{12}t^{\eta_2-1}e^{(t^{\eta_2})}\right] \\ e^{\left[\vartheta_{11}\left(1-e^{t^{\eta_1}}\right)+\vartheta_{12}\left(1-e^{t^{\eta_2}}\right)\right]}; \quad 0 < t < \tau \\ \left[\eta_1\vartheta_{21}t^{\eta_1-1}e^{(t^{\eta_1}-\tau^{\eta_1})} + \eta_2\vartheta_{22}t^{\eta_2-1}e^{(t^{\eta_2}-\tau^{\eta_2})}\right] \\ e^{\left[\vartheta_{21}\left(1-e^{t^{\eta_1}-\tau^{\eta_1}}\right)+\vartheta_{11}\left(1-e^{\tau^{\eta_1}}\right)+\vartheta_{22}\left(1-e^{t^{\eta_2}-\tau^{\eta_2}}\right)+\vartheta_{12}\left(1-e^{\tau^{\eta_2}}\right)\right]}; \quad t \geq \tau \end{cases} \quad (8)$$

Let ξ be the indicator of the failure cause, then we derive the joint PDF of (T, ξ) as:

$$f_{T,\xi}(t) = g_c(t)\big(1 - G_{c'}(t)\big)$$

$$f_{T,\xi}(t) = \begin{cases} \eta_c\vartheta_{1c}t^{\eta_c-1}e^{(t^{\eta_c})}e^{\left[\vartheta_{11}\left(1-e^{t^{\eta_1}}\right)+\vartheta_{12}\left(1-e^{t^{\eta_2}}\right)\right]}; & 0 < t < \tau \\ \eta_c\vartheta_{2c}t^{\eta_c-1}e^{[t^{\eta_c}-\tau^{\eta_c}]} \\ e^{\left[\vartheta_{21}\left(1-e^{t^{\eta_1}-\tau^{\eta_1}}\right)+\vartheta_{11}\left(1-e^{\tau^{\eta_1}}\right)+\vartheta_{22}\left(1-e^{t^{\eta_2}-\tau^{\eta_2}}\right)+\vartheta_{12}\left(1-e^{\tau^{\eta_2}}\right)\right]}; & t \geq \tau \end{cases} \quad (9)$$

for c, c'=1,2 and $c \neq c'$.

3. MAXIMUM LIKELIHOOD ESTIMATION

Recognizing that there are N_1 failures prior to changing the stress time τ. When we point out η_{1c} and η_{2c} that signifies the sum of failures associated with a failure caused c under the level of stress s_1 and s_2, in turn, so $N_1=n_{11}+n_{12}$ is the sum of failures under the level of stress s_1 and m- N_1= N_2 $n_{21}+n_{22}$ is the sum of failures under the level of stress s_2. Given that the resultant cause of failure takes place contained by each failure time, let $\xi=(\xi_1,\xi_2,\ldots,\xi_m)$characterize the observed failure cause indicator series relating to the recorded failure time $t=(t_1, t_2,\ldots, t_m)$. The likelihood function is then constructed using the progressively censoring scheme R_1, R_2 …, R_m and presupposition 4 as shown by Balakrishnan and Aggarwala [16].

$$L\left(\frac{\psi_c}{t}\right) \propto \prod_{h=1}^{N_1} f_1(t_h, \xi_h)[1 - F(t_h)]^{R_h} \prod_{h=N_1+1}^{m} f_2(t_h, \xi_h)[1 - F(t_h)]^{R_h} \quad (10)$$

Then

$$L(\psi_c/t) \propto U_1U_2 \exp\left[\sum_{h=1}^{N_1} t_h^{\eta_c} + \sum_{h=N_1+1}^{m}\left(t_h^{\eta_c} - \tau^{\eta_c}\right) + \sum_{c=1}^{2}\vartheta_{1c}(U_{1c} + U_{2c}) + \sum_{c=1}^{2}\vartheta_{2c}U_{3c}\right] \quad (11)$$

where,
$U_1 = \prod_{i,c=1}^{2}\left[\eta_c^{n_{ic}}\,\vartheta_{ic}^{n_{ic}}\right]$, $U_2 = \prod_{h=1}^{m} t_h^{\eta_c-1}$,
$U_{1c} = \sum_{h=1}^{N_1}(1 + R_h)\left(1 - e^{t_h^{\eta_c}}\right)$,
$U_{2c} = \sum_{h=N_1+1}^{m}(1 + R_h)\left(1 - e^{\tau^{\eta_c}}\right)$ and
$U_{3c} = \sum_{h=N_1+1}^{m}(1 + R_h)\left(1 - e^{\left(t_h^{\eta_c}-\tau^{\eta_c}\right)}\right)$.

Utilizing the likelihood function (11), henceforth the MLE of $\psi_c=(\vartheta_{1c}, \vartheta_{2c})$ and c=1,2. Thus, to estimate ϑ_{lc}, we may fairly presume that $\vartheta_{lc}\geq 1$(l, c=1,2), i.e., At least one failure must be observed under each failure caused by each level of stress. The MLEs of the parameters for Eq. (11) are got by maximizing the logarithm of the likelihood function stated as:

$$\begin{aligned}\log L \propto \textstyle\sum_{i,c=1}^{2} n_{ic}(\log\eta_c + \log\vartheta_{ic}) \\ +(\eta_c - 1)\textstyle\sum_{h=1}^{m}\log(t_h) + \sum_{h=1}^{N_1} t_h^{\eta_c} \\ +\textstyle\sum_{h=N_1+1}^{m}\left(t_h^{\eta_c} - \tau^{\eta_c}\right) + \sum_{c=1}^{2}\vartheta_{1c}(U_{1c} + U_{2c}) \\ +\textstyle\sum_{c=1}^{2}\vartheta_{2c}U_{3c}\end{aligned} \quad (12)$$

The log-likelihood function's first partial derivative with respect to the parameters ψ_c= (η_c, ϑ_{1c}, ϑ_{2c}) and c=1,2 respectively as follows:

$$\begin{aligned}\frac{\partial\log L}{\partial\eta_c} = \hat{\eta}_c^{-1}\textstyle\sum_i^2 n_{ic} + Q^* + \sum_{h=N_1+1}^{m} Q_{hc} \\ -\hat{\vartheta}_{1c}[Q_{1c} + Q_{2c}] - \hat{\vartheta}_{2c}Q_{3c} = 0\end{aligned} \quad (13)$$

$$\frac{\partial\log L}{\partial\vartheta_{1c}} = n_{1c}\hat{\vartheta}_{1c}^{-1} + U_{1c} + U_{2c} = 0 \quad (14)$$

and

$$\frac{\partial\log L}{\partial\vartheta_{2c}} = n_{2c}\hat{\vartheta}_{2c}^{-1} + U_{3c} = 0 \quad (15)$$

where, $Q^* = \sum_{h=1}^{m}\log(t_h) + \sum_{h=1}^{N_1} t_h^{\hat{\eta}_c}\log t_h$,
$Q_{hc} = \left(t_h^{\hat{\eta}_c}\log t_h - \tau^{\hat{\eta}_c}\log\tau\right)$,
$Q_{1c} = \sum_{h=1}^{N_1}(1 + R_h)\left(e^{t_h^{\hat{\eta}_c}}t_h^{\eta_c}\log(t_h)\right)$,
$Q_{2c} = \sum_{h=N_1+1}^{m}(1 + R_h)\left(e^{\tau^{\hat{\eta}_c}}\tau^{\hat{\eta}_c}\log\tau\right)$ and
$Q_{3c} = \sum_{h=N_1+1}^{m}(1 + R_h)\left(e^{\left(t_h^{\hat{\eta}_c}-\tau^{\hat{\eta}_c}\right)}\left(t_h^{\hat{\eta}_c}\log t_h - \tau^{\hat{\eta}_c}\log\tau\right)\right)$.

As of (14) and (15), the MLEs of ϑ_{1c} and ϑ_{2c} are easily acquired as:

$$\hat{\vartheta}_{1c} = -n_{1c}[U_{1c} + U_{2c}]^{-1} \quad (16)$$

$$\hat{\vartheta}_{2c} = -n_{2c}[U_{3c}]^{-1} \quad (17)$$

The nonlinear Eq. (13) have difficult closed-form solutions. Therefore, a numerical approach ought to be utilized to resolve these concurrent equations to finding $\hat{\eta}_c$; c=1,2.

4. ASYMPTOTIC VARIANCE AND COVARIANCE'S OF ESTIMATION

The asymptotic Fisher information matrix of the parameter's MLE could be approximated by numerically inverting the asymptotic Fisher information matrix I_c. It is made up of the negative second and mixed partial derivatives of the likelihood function's actual logarithm as defined by the MLE. It can be computed utilizing the following matrix:

$$I_c=\begin{bmatrix} I_{c11} & I_{c12} & I_{c13}\\ I_{c21} & I_{c22} & I_{c23}\\ I_{c31} & I_{c32} & I_{c33}\end{bmatrix} = \begin{bmatrix} \frac{-\partial^2 \log L}{\partial \eta_c^2} & \frac{-\partial^2 \log L}{\partial \eta_c \partial \vartheta_{1c}} & \frac{-\partial^2 \log L}{\partial \eta_c \partial \vartheta_{2c}}\\ \frac{-\partial^2 \log L}{\partial \vartheta_{1c} \partial \eta_c} & \frac{-\partial^2 \log L}{\partial \vartheta_{1c}^2} & \frac{-\partial^2 \log L}{\partial \vartheta_{1c} \partial \vartheta_{2c}}\\ \frac{-\partial^2 \log L}{\partial \vartheta_{2c} \partial \eta_c} & \frac{-\partial^2 \log L}{\partial \vartheta_{1c} \partial \vartheta_{2c}} & \frac{-\partial^2 \log L}{\partial \vartheta_{2c}^2}\end{bmatrix}$$

The components of the noticed Fisher information matrix I_{c11}, I_{c22}, I_{c33}, I_{c12}, and I_{c13} are obtained as the following:

$$I_{c11}=\frac{-\partial^2 \log L}{\partial \eta_c^2}=\eta_c^{-2}\sum_i^2 n_{ic}-E^*-E^{**}+\vartheta_{1c}E_{1c}+\vartheta_{2c}E_{2c},$$

$$I_{c22}=\frac{-\partial^2 \log L}{\partial \vartheta_{1c}^2}=n_{1c}\vartheta_{1c}^{-2},\ I_{c33}=\frac{-\partial^2 \log L}{\partial \vartheta_{2c}^2}=n_{2c}\vartheta_{2c}^{-2},$$

$$I_{c12}=\frac{-\partial^2 \log L}{\partial \eta_c \partial \vartheta_{1c}}=I_{c21}=\frac{-\partial^2 \log L}{\partial \vartheta_{1c} \partial \eta_c}=Q_{1c}+Q_{2c}$$

$$I_{c13}=\frac{-\partial^2 \log L}{\partial \eta_c \partial \vartheta_{2c}}=I_{c31}=\frac{-\partial^2 \log L}{\partial \vartheta_{2c} \partial \eta_c}=-Q_{3c} \text{ and}$$

$$I_{c23}=\frac{-\partial^2 \log L}{\partial \vartheta_{1c} \partial \vartheta_{2c}}=I_{c32}=\frac{-\partial^2 \log L}{\partial \vartheta_{2c} \partial \vartheta_{1c}}=0$$

where, $E^*=\sum_{h=1}^{N_1} t_h^{\eta_c}(\log t_h)^2$,
$E^{**}=\sum_{h=N_1+1}^{m}\left(t_h^{\eta_c}(\log t_h)^2-\tau^{\eta_c}(\log\tau)^2\right)$,
$E_{1c}=\sum_{h=1}^{N_1}(1+R_h)\left[e^{t_h^{\eta_c}}t_h^{\eta_c}(\log t_h)^2\left(1+t_h^{\eta_c}\right)\right] + \sum_{h=N_1+1}^{m}(1+R_h)\left[e^{\tau^{\eta_c}}\tau^{\eta_c}(\log\tau)^2(1+\tau^{\eta_c})\right]$,
$E_{2c}=\sum_{h=N_1+1}^{m}(1+R_h)\left\{e^{(t_h^{\eta_c}-\tau^{\eta_c})}\left[t_h^{\eta_c}\log t_h\left(\log t_h+\log t_h\, t_h^{\eta_c}-\tau^{\eta_c}\log\tau\right)-\log\tau\,\tau^{\eta_c}\left(\log\tau+t_h^{\eta_c}\log t_h-\tau^{\eta_c}\log\tau\right)\right]\right\}$.

Substitute the MLEs $\hat{\vartheta}_{1c}$ for ϑ_{1c}, $\hat{\vartheta}_{2c}$ for ϑ_{2c} and c=1,2. It is feasible to acquire the noticed Fisher information matrix $\hat{I}_c$. when this matrix is inverted and symbolized by $\hat{V}_c=\hat{I}_c^{-1}$.

One might obtain the convergent CIs of the parameters on the asymptotic distribution of the MLEs of the units of the vector of unidentified parameters $\psi_c=(\vartheta_{1c},\vartheta_{2c})$, c=1,2. The asymptotic distribution of the MLEs $\left(\hat{\psi}_c-\psi_c/\sqrt{var\ (\hat{\psi}_c)}\right), c=1,2$ is known to be approximated by a standard normal distribution, whereas $var\ (\hat{\psi}_c)$ is assessed as the asymptotic variance, then, the estimated 100(1-γ)% two-sided CI for $\psi_c=(\vartheta_{1c},\vartheta_{2c})$, c=1,2 are achieved, therefore;

$$p\left[\hat{\psi}_c-z_{\gamma/2}\sqrt{var\ (\hat{\psi}_c)}\le\psi_c\le\hat{\psi}_c+z_{\gamma/2}\sqrt{var\ (\hat{\psi}_c)}\right]\cong\alpha$$

whereas, $z_{\gamma 2}$ is the 100(1-γ)% standard normal percentile.

5. BAYESIAN ESTIMATION

In this section, established on data of competing risks, the Bayesian estimation employing square error loss functions is obtained based on a simple step-stress model with type II progressive censoring. One could propose utilizing independently distributed gamma priors with known parameters ϑ_{1c}, ϑ_{2c} and η_{1c} where c=1,2 of the extension Weibull distribution (EWD) as:

$$\pi_1(\eta_c)=\eta_c^{-1};\ 0<\eta_c<1$$

and

$$\pi_2(\vartheta_{lc})=[\Gamma(a_{lc})]^{-1}b_{lc}{}^{a_{lc}}\,\vartheta_{lc}^{a_{lc}-1}\exp(-b_{lc}\vartheta_{lc});\quad \vartheta_{lc},a_{lc},b_{lc}>0,\qquad l,c=1,2$$

where, the hyper-parameters a_{lc}, b_{lc} and l, c=1,2 are elected to mirror prior knowledge of the unknown parameters and the parametric space of ϑ_{1c} and ϑ_{2c} should be $K_{\vartheta_{lc}}$={ϑ_{lc}, $\vartheta_{1c}\le\vartheta_{2c}$}, c=1,2.

Hence, the jointly prior densities of ψ_c= (η_c, ϑ_{1c}, ϑ_{2c}), c=1,2 can then be written as:

$$\pi(\psi_c)\propto\eta_c^{-1}\vartheta_{1c}^{a_{1c}-1}\vartheta_{2c}^{a_{2c}-1}e^{(-b_{1c}\vartheta_{1c}-b_{2c}\vartheta_{2c})}I_{(\vartheta_{1c}\le\vartheta_{2c})};\ c=1,2 \tag{18}$$

For the noticed data t obtained from a life test experiment's type II progressive censoring with four independent the extension Weibull distribution ϑ_{lc} and l, c=1,2 and from Eq. (11) of the likelihood function and Eq. (18) of prior distribution the equivalent posterior density of ψ_c=(η_c, ϑ_{1c},ϑ_{2c}), c=1,2 and is given by:

$$\pi(\psi_c|\underline{t})\propto L(\psi_c|\underline{t}).\pi(\eta_1,\eta_2,\vartheta_{11},\vartheta_{12},\vartheta_{21},\vartheta_{22})$$

The posterior density function is given by:

$$\pi(\psi_c|\underline{t})\propto \left[\eta_c^{\sum_{l=1}^{2}n_{lc}-1}\right]\left[\prod_{l=1}^{2}\vartheta_{lc}^{n_{lc}+a_{lc}-1}\right]\left(\prod_{h=1}^{m}t_h^{\eta_c-1}\right) \exp\left[\sum_{h=1}^{N_1}t_h^{\eta_c}+\sum_{h=N_1+1}^{m}\left(t_h^{\eta_c}-\tau^{\eta_c}\right)+\vartheta_{1c}(U_{1c}+U_{2c}-b_{1c})+\vartheta_{2c}(U_{3c}-b_{2c})\right]I_{(\vartheta_{1c}\le\vartheta_{2c})};\ c=1,2 \tag{19}$$

By integrating $\pi(\psi_c|\underline{t})$ with regard to ϑ_1 and ϑ_{2c}, the marginal posterior density function of η_c is displayed to be proportional to:

$$m(\eta_c|t)\propto\eta_c^{\sum_{l=1}^{2}n_{lc}-1}\left(\prod_{h=1}^{m}t_h^{\eta_c-1}\right)\Gamma(n_{1c}+a_{1c}) \Gamma(n_{2c}+a_{2c})\exp\left[\sum_{h=1}^{N_1}t_h^{\eta_c}+\sum_{h=N_1+1}^{m}\left(t_h^{\eta_c}-\tau^{\eta_c}\right)\right] [(U_{1c}+U_{2c}-b_{1c})^{n_{1c}+a_{1c}}(U_{3c}-b_{1c})^{n_{2c}+a_{2c}}]^{-1}$$

To achieve the most suitable estimator using Bayes' technique, one must select a loss function that corresponds with all of the potential estimators. In this section the estimates are obtained for two various kinds of loss functions, explicitly, squared error loss function (SELF) as an illustrative of the first type, and LINEX loss function (LLF) as an exemplar of another type. The SELF is improper when there is an exaggeration or underestimation. In this situation, LLF could be utilized as an alternate option for an estimate the parameters. Additionally, it is helpful when exaggeration and underestimation are both serious issues. The loss functions that are currently researched could be shown as follows, assuming that ω_c is an estimator for the unknown parameter ψ_c.

- SELF: From "$(\omega_c-\psi_c)^2$" and Bayes estimate "$E_{\psi_c}(\psi_c|\underline{t})$".
- LLF: From "$\exp[u(\omega_c-\psi_c)]-u(\omega_c-\psi_c)-1, u\neq 0$" and Bayes estimate "$-\frac{1}{u}\ln\left[E_{\psi_c}\left(\exp\left((-u\psi_c)\middle|\underline{t}\right)\right)\right]$".

Referring to Eq. (19), one could observe the difficulty of computing integrals, followed by the incapability to acquire the closed form for the joint posterior that allows us to calculate Bayes estimations of the unknown parameters $\psi_c=(\eta_c, \vartheta_{1c},\vartheta_{2c})$, c=1,2. As a result, we will use the MCMC approach to obtain these estimates, which allows us to generate simulated samples from the parameter posterior distributions. These generated samples will be used to calculate the interval and point estimation of unknown parameters. The mechanism of this approach is based on the computation of conditional posterior functions wherein a conditional distribution of $(\vartheta_{1c}|\eta_c)$ and $(\vartheta_{2c}|\eta_c)$ is gamma distribution with respective PDF:

$$\pi_1^*(\vartheta_{1c}|\eta_c) = D_{1c}\,\vartheta_{1c}^{n_{1c}+a_{1c}-1}\exp[-\vartheta_{1c}(b_{1c}-U_{1c}-U_{2c})] \simeq Gamma(n_{1c}+a_{1c}, b_{1c}-U_{1c}-U_{2c})$$

$$\pi_2^*(\vartheta_{2c}|\eta_c) = D_{2c}\,\vartheta_{1c}^{n_{2c}+a_{2c}-1}\exp[-\vartheta_{2c}(b_{2c}-U_{3c})] \simeq Gamma(n_{2c}+a_{2c}, b_{2c}-U_{3c})$$

where, $D_{1c}=(b_{1c}-U_{1c}-U_{2c})^{n_{1c}+a_{1c}}/\Gamma(n_{1c}+a_{1c})$ and $D_{1c}=(b_{2c}-U_{3c})^{n_{2c}+a_{2c}}/\Gamma(n_{2c}+a_{2c})$.

Gilks and Wild [43] proposed a simple and applies solution to this problem. We can sample from a full conditional distribution that ought to be log-concave utilizing technique. As a result, we must establish whether the condition is satisfied. Given ϑ_{1c}, ϑ_{2c} and c=1,2 the logarithm of the density of conditional posterior function of η_c is:

$$\log\pi(\eta_c|\vartheta_{1c},\vartheta_{2c}) \propto \left(\sum_{l=1}^{2} n_{lc}-1\right)\log\eta_c + (\eta_c-1)\sum_{h=1}^{m}\log t_h + \sum_{h=1}^{N_1} t_h^{\eta_c} + \sum_{h=N_1+1}^{m}\left(t_h^{\eta_c}-\tau^{\eta_c}\right) + \vartheta_{1c}(U_{1c}+U_{2c}-b_{1c}) + \vartheta_{2c}(U_{3c}-b_{2c}) \tag{20}$$

Compute the second derivative of (20) as:

$$\partial^2\log\pi(\eta_c|\vartheta_{1c},\vartheta_{2c})/\partial\eta_c^2 = -\eta_c^{-2}\left(\sum_{l=1}^{2} n_{lc}-1\right) + E^* + E^{**} - \vartheta_{1c}E_{1c} - \vartheta_{2c}E_{2c}$$

It is clear that $\pi_1^*(\vartheta_{1c}|\eta_c)$ and $\pi_2^*(\vartheta_{2c}|\eta_c)$ are gamma distributions. Therefore, employing a gamma generator, samples of ϑ_{1c} and ϑ_{2c} can be generated. Furthermore, $\pi(\eta_c|\vartheta_{1c}, \vartheta_{2c})$ cannot be directly shrunk for drawing samples utilizing standard techniques. For this type of situation, acquiring Bayes' estimate for η_c, we can employ one of the very well algorithms in the MCMC approach, the Metropolis-Hastings (MH) algorithm model, which was published in the study by Metropolis et al. [44]. To employ this algorithm, we must first presume a suggestion function for a sample from it. Throughout this technique, we can choose either to employ a non-symmetric or a symmetric suggestion distribution to reduce the rejection rate as much as potential. Because the marginal distribution of η_c is unknown, the normal distribution is designated as a symmetric proposal distribution. The Metropolis-Hastings steps are employed by the Gibbs sampler to update η_c, whereas ϑ_{1c} and ϑ_{2c} are updated directly from their full conditionals. The following is a hybrid algorithm that utilizes Gibbs sampling steps to update the parameters ϑ_{1c} and ϑ_{2c} with MH steps to update η_c and create the associated HPD credible intervals. Typically, we select the MLEs $\hat{\vartheta}_{1c}, \hat{\vartheta}_{2c}$ and c=1,2 as primary values.

Step 1: Given ϑ_{1c}, ϑ_{2c}, according to the sampling algorithm suggested by Gilks and Wild [43] of the adaptive rejection, generate η_c.

Step 2: Based upon η_c from step 1, generate ϑ_{1c} from $Gamma(n_{1c}+a_{1c}, b_{1c}-U_{1c}-U_{2c})$ and ϑ_{2c} from $Gamma(n_{2c}+a_{2c},b_{2c}-U_{3c})$.

Step 3: Corroborate that $\vartheta_{1c}\leq\vartheta_{2c}$, if not, return to step 2. If this is the case, repeat 1,000 times for steps 1 to 3.

Signify the *kth* Gibbs sampler by $\left(\eta_c^{(k)}, \vartheta_{1c}^{(k)}, \vartheta_{2c}^{(k)}, k=1,2,\ldots,1000\right)$. Before stationary, iterations (N times in total) are discarded, therefore, the Bayesian estimates of $\eta_c, \vartheta_{1c}, \vartheta_{2c}$ and c=1, 2 are:

$\eta_{cBS}=\sum_{k=N+1}^{1000}\eta_c^{(k)}/(1000-N)$, and

$\vartheta_{lcBS}=\sum_{k=N+1}^{1000}\vartheta_{lc}^{(k)}/(1000-N)$, for l, c=1, 2.

Step 4: Calculate all conceivable *100(1-γ)%* credible intervals of the form $\left[\eta_{c(p)},\eta_{c(p+100(1-\gamma))}\right]$ and $\left[\vartheta_{lc(p)},\vartheta_{lc(p+100(1-\gamma))}\right]$ where *l, c=1, 2;p=1, 2, ..., 1000-100(1-γ)* respectively, by sorting the Gibbs sampler $\eta_c^{(k)}, \vartheta_{1c}^{(k)}, \vartheta_{2c}^{(k)}, k=1,2,\ldots,1000$ in ascending order.

Step 5: Calculate the lengths of each credible interval, then select the lowest interval to serve as the HPD credible interval of η_c, ϑ_{1c}, ϑ_{2c} and c=1,2.

6. SIMULATION STUDY AND ILLUSTRATIVE EXAMPLE

The goal of this section is to compare the performance of the different estimation methods introduced in the preceding sections. A real given dataset is used for illustration purposes; furthermore, a simulation study is used to evaluate the behavior of the suggested methods and to test the statistical performances of the estimators given a Progressive Censoring Type-II scheme under step-stress for extension Weibull distribution in the presence of competing risks. The R statistical software program was used to accomplish the calculations. Computing MLEs and HPD intervals in the program R is done by employing the *bbmle* and *HDInterval* packages.

6.1 Simulation study

In this sub-section, to analyze the accuracy of estimation methods, including MLE and Bayesian estimation, a Monte Carlo simulation study is employed, under progressive Type-II under step-stress for extension Weibull distribution in presence of competing causes of risks. For the MLEs, 1,000 observations are generated from the extension Weibull distribution based on the following assumptions:

1. Parameters are given by: ϑ_{11}=0.5, ϑ_{12}=0.75, ϑ_{21}=1.25, ϑ_{22}=1.5, η_1=1.75, η_2=2.
2. Sample sizes of n=40, n=80, and n=100.
3. The number of stages of progressive censoring: m=30, 40, 60, 80.
4. Removed items R_i are assumed at different sample sizes n and number of stages m as shown in Table 1.

Table 1. Numerous patterns for removing items from life tests at different stages

n	m	Censoring Schemes				
		S_1	S_2	S_3	S_4	S_5
60	30	$(30, 0^{*29})$	$(15, 0^{*28}, 15)$	$(0^{*14}, 15, 15, 0^{*14})$	$(1^{*30}, 0^{*0})$	$(0^{*29}, 30)$
	40	$(20, 0^{*39})$	$(10, 0^{*38}, 10)$	$(0^{*19}, 10, 10, 0^{*19})$	$(1^{*20}, 0^{*20})$	$(0^{*39}, 10)$
80	40	$(40, 0^{*39})$	$(20, 0^{*38}, 20)$	$(0^{*19}, 20, 20, 0^{*19})$	$(1^{*40}, 0^{*0})$	$(0^{*39}, 40)$
	60	$(20, 0^{*59})$	$(10, 0^{*58}, 10)$	$(0^{*29}, 10, 10, 0^{*29})$	$(1^{*20}, 0^{*40})$	$(0^{*59}, 20)$
100	60	$(40, 0^{*59})$	$(20, 0^{*58}, 20)$	$(0^{*29}, 20, 20, 0^{*29})$	$(1^{*40}, 0^{*20})$	$(0^{*59}, 40)$
	80	$(20, 0^{*79})$	$(10, 0^{*78}, 10)$	$(0^{*39}, 10, 10, 0^{*39})$	$(1^{*20}, 0^{*60})$	$(0^{*79}, 20)$

Here, $(5^{*3}, 0)$, for instance, implies that the censorship scheme utilized is (5,5,5,0).

Suppose that the levels of accelerated temperature S_1=270°F, S_2=320°F and the utilized temperature is S_0=210°F. The lifetime of the two failure causes follows an expansion of Weibull distribution with recognized shape parameter η_1, η_2, respectively. The amount of the parameter was selected to be ϑ_{11}, ϑ_{12}, ϑ_{21}, ϑ_{22}. To clarify a specific scenario underneath each cause of failure the increase of stress level in our case of the study with the extension Weibull model will be achieved by increasing the rate of the scale parameter ϑ_c, which will be reflected in shrinking the main time to failure.

Before continuing, first, the progressively censored Type-II is created through competing risks data utilizing the extension Weibull cumulative exposure model (CEM) for constant m, (R_1, R_2…, R_m, n), as shown below:

Step 1: Based on the algorithm proposed by Balakrishnan and Sandu [45], generating two samples that are progressively censored W_1, W_2…, W_m and U_1,U_2,…U_m from Uniform distribution (0,1).

Step 2: Calculating t_{11h} and t_{12h} using $U_h = 1 - \exp\left[\vartheta_{11}\left(1 - e^{t_{11h}^{\eta_1}}\right)\right]$ and $W_h = 1 - \exp\left[\vartheta_{12}\left(1 - e^{t_{12h}^{\eta_2}}\right)\right]$, the minimum of ($t_{11h}$, t_{12h}) is registered as t_h^*, while the corresponding minimum index comes out of this condition ($t_{11h} < t_{12h}$, $set\ \xi_h^* = 1$; else set $\xi_h^* = 2$) for $1 \le h \le N_1$.

Step 3: Let's assume that various values of stress changing time τ as follows: $\tau = \left(mean(t_h^*) + median(t_h^*)\right)/2$.

Step 4: Find N_1 such that $t_{N_1}^* < \tau < t_{N_1+1}^*$. Hence, put $t_h = t_h^*$ and $\xi_h = \xi_h^*$ for $1 \le h \le N_1$.

Step 5: Generating t_{21h} and t_{22h} utilizing:
$U_h = 1 - \exp\left[\vartheta_{21}\left(1 - e^{t_{21h}^{\eta_1} - \tau^{\eta_1}}\right) + \vartheta_{11}\left(1 - e^{\tau^{\eta_1}}\right)\right]$ and
$W_h = 1 - \exp\left[\vartheta_{22}\left(1 - e^{t_{22h}^{\eta_2} - \tau^{\eta_2}}\right) + \vartheta_{12}\left(1 - e^{\tau^{\eta_2}}\right)\right]$, for $N_1 + 1 \le h \le m$. The minimum values of (t_{21h}, t_{22h}) assigned as t_h^*.

Step 6: Setting the value of $t_h = t_h^*$ and $\xi_h = \xi_h^*$ for $N_1 + 1 \le h \le m$. Then $t_1, t_2, \ldots, t_m$ are the required order observation and $\xi = [\xi_1, \xi_2, \ldots, \xi_m]$ the vector of the indices.

MLEs and related 95% asymptotic confidence intervals (ACIs) are produced based on the generated data. On deriving MLEs, be aware that the initial assume values are regarded as true parameter values.

We compute Bayesian estimates using informative priors for the Bayesian estimation method using (18) as the value for all hyperparameters. Such values of informative priors are plugged-in to evaluate the required estimates. Through implementing the MH algorithm, the MLEs are used as initial guess values, as well as the corresponding variance-covariance matrix I_c of $(\hat{\psi})$, where ψ=(ϑ_{11}, ϑ_{12}, ϑ_{21}, ϑ_{22}, η_1, η_2). In the end, 1500 burn-in samples were deleted from the total of 6,000 generated samples by the posterior density and produced Bayes estimates under two loss functions, namely: squared error loss (SEL) function LINEX at τ=-1.5, 1.5 Also, HPD interval estimates have been computed according to the technique of Chen and Shao [46].

All the average estimates for methods are reported in Tables 3 to 8 for different combinations of sample size n and number of stages m. Further, the first column donates the average estimates (Avg.) and in the second column, related means square errors (MSEs). For confidence intervals, we have asymptotic confidence intervals for MLEs and HPD for Bayesian estimates based on MCMC which are reported in Tables 9 to 14 for different combinations of sample size n and number of stages m. Further, the first column represents the lower bound confidence interval, the second column represents the upper bound of CI, the third column represents the average interval lengths (AILs), and in the last column, related coverage probabilities (CPs) in percentage (%).

According to the tabulated values, one can indicate that:

1. As n raises and m fixed, Avg. estimates for MLE and BE using MCMC tend to gradually converge for the initial parameter values.
2. When the sample size increases, the MSE and average of MLE and BE of the considered parameters decrease.
3. The MLE and Bayesian estimates for Scheme 5 have good statistical properties than other Schemes.
4. As small as the sample size, Bayesian estimates can be offered.
5. As n fixed and m increases, AIL for MLE and BE using MCMC tend to decrease.
6. MSE for BE under LINEX with (τ=1.5) is smaller than other BE methods for all parameters expect.
7. For fixed sample size and m small, the AIL for BE is shorter than the AIL for MLE using MCMC, but in most cases, for m increasing and fixed sample size, the AIL for MLE is shorter than the AIL for BE.
8. For fixed m and n increasing, the AIL for MLE and BE using MCMC tend to decrease.
9. For the same sample size, the average estimate for MLE tends to the initial values of the parameter with m small.
10. For fixed n and m small, the MSE of MLE decreases.
11. All most cases, the point estimates of the scale parameters for BEs are better than for MLEs and the opposite is true for the shape parameter.

Also, a comparison of Bayesian estimation under MCMC for all different combinations of samples size n and number of stages m in the case of Scheme 3 (S_3) can be shown graphically the graphs of MCMC estimates for ϑ_{11}, ϑ_{12}, ϑ_{21}, ϑ_{22} using the MH algorithm and η_1 and η_2 using Gibbs sampling algorithm in cases of informative priors are the plotting of estimates, histogram of estimates, and cumulative mean of estimates. These graphs can be shown in Figure 1 to Figure 3. Also, one can conclude the convergence of estimates under different sample sizes n and number of stages m.

6.2 Illustrative example

In this section, we simulate progressive Type-II censored samples under step stress for extension Weibull distribution in presence of a competing failure model. The dataset is generated with the following selections of parameters: ϑ_{11}=0.75, ϑ_{12}=0.80, ϑ_{21}=0.65, ϑ_{22}=0.70, η_1=2.0, and η_2=2.10 and sample size n=50, number of stages m=30 with censoring scheme:

$$R_i = (2,0,2,0,2,0,2,0,2,0,0,0,0,0,0,0,0,0,0,0,0,2,0,2,0,2,0,2,0,2).$$

Also, the assumed stress level is τ=0.5. The progressively censored Type-II data are given in the following Table 2.

From this dataset, the estimates (Est.) and standard error (St.Er) of MLEs and BEs using MCMC of the parameters are derived. Also, ACIs and HPD intervals are computed. The results are presented in Table 15 for Est. and St.Ers and in Table 16 for Cis (lower, upper, interval length (IL)).

The graphs of MCMC estimates for ϑ_{11}, ϑ_{12}, ϑ_{21}, ϑ_{22}, η_1 and η_2 using the MH algorithm can be shown in Figures 4 to 6 which indicates the normality of posterior samples generated using the MH algorithm and utilizing MCMC.

Table 2. Data of the progressively censored Type-II for the illustrative example

The first level of stress	(0.1742090, 2) (0.2031259, 1) (0.2315669, 1) (0.2759920, 1) (0.2877801, 1) (0.3045739, 1) (0.3457952, 1) (0.3508297, 1) (0.3648599, 1) (0.4635089, 1)
The second level of stress	(0.7515350, 2) (0.7987907, 2) (0.8096729, 2) (0.8502840, 2) (0.8565298, 1) (0.8593854, 1) (0.8725126, 1) (0.8737639, 1) (0.8747619, 1) (0.8775376, 1) (0.9101309 1) (0.9281317, 1) (0.9294583, 1) (0.9319161, 1) (0.9823121, 1) (1.0200595, 1) (1.0678518, 1) (1.0796584, 1) (1.0945277, 1) (1.1951210, 2)

The first element represents failure time and the second element represents the failure cause.

Table 3. Avg. estimated values and MSEs of the ML and BE using MCMC for different schemes of progressive Type-II censoring step-stress for extension Weibull distribution at n=60 and m=30

Sch.	**Parm.**	**MLE**		**BE MCMC: SEL**		**BE MCMC: LINEX**			
						τ=-1.5		**τ=1.5**	
		Avg.	MSE	Avg.	MSE	Avg.	MSE	Avg.	MSE
S_1	ϑ_{11}	0.3629	0.0207	0.3357	0.0288	0.3292	0.0309	0.3426	0.0267
	ϑ_{12}	0.3730	0.1448	0.3467	0.1652	0.3387	0.1716	0.3552	0.1586
	ϑ_{21}	1.2385	0.0275	1.1466	0.0350	1.0284	0.0660	1.3410	0.0784
	ϑ_{22}	1.1036	0.1861	1.0193	0.2560	0.9262	0.3476	1.1575	0.1585
	η_1	0.9362	0.6697	0.9749	0.6085	0.9481	0.6500	1.0040	0.5649
	η_2	1.0171	0.9772	1.0603	0.8947	1.0326	0.9465	1.0903	0.8401
S_2	ϑ_{11}	0.4409	0.0062	0.4033	0.0120	0.3948	0.0135	0.4124	0.0105
	ϑ_{12}	0.4525	0.0917	0.4147	0.1154	0.4055	0.1215	0.4245	0.1091
	ϑ_{21}	2.0345	0.6859	1.9390	0.5395	1.7121	0.2578	2.3654	1.4249
	ϑ_{22}	1.9388	0.2649	1.8426	0.1824	1.6442	0.0663	2.1837	0.6085
	η_1	1.0923	0.4454	1.1542	0.3686	1.1069	0.4252	1.2082	0.3099
	η_2	1.1413	0.7528	1.2050	0.6485	1.1586	0.7224	1.2581	0.5698
S_3	ϑ_{11}	0.3311	0.0294	0.3128	0.0359	0.3082	0.0376	0.3176	0.0342
	ϑ_{12}	0.5035	0.0650	0.4777	0.0785	0.4659	0.0847	0.4905	0.0721
	ϑ_{21}	1.1781	0.2045	1.1521	0.1849	1.0398	0.1700	1.3411	0.3655
	ϑ_{22}	0.9829	0.4289	0.9456	0.4540	0.8726	0.5033	1.0529	0.4327
	η_1	1.0842	0.4547	1.0801	0.4591	1.0442	0.5076	1.1188	0.4101
	η_2	1.2664	0.5535	1.2766	0.5375	1.2398	0.5908	1.3157	0.4838
S_4	ϑ_{11}	0.4002	0.0118	0.3784	0.0234	0.3683	0.0218	0.3997	0.2040
	ϑ_{12}	0.4715	0.0811	0.4965	0.7273	0.4371	0.1385	0.5964	2.2996
	ϑ_{21}	2.0791	0.7487	2.0115	0.6452	1.7817	0.3232	2.4575	1.7950
	ϑ_{22}	1.8916	0.2060	1.8397	0.2331	1.6519	0.0646	2.1967	2.6084
	η_1	1.0718	0.4701	1.1064	0.4259	1.0654	0.4793	1.1550	0.3716
	η_2	1.1806	0.6831	1.2170	0.6304	1.1747	0.6973	1.2753	0.7254
S_5	ϑ_{11}	0.5518	0.0072	0.5089	0.0043	0.4951	0.0038	0.5236	0.0052
	ϑ_{12}	0.5518	0.0438	0.5081	0.0628	0.4944	0.0692	0.5229	0.0563
	ϑ_{21}	2.8508	2.6734	2.7456	2.3323	2.4121	1.4165	2.4604	0.2327
	ϑ_{22}	2.8509	1.9355	2.7525	1.6630	2.4167	0.9036	2.4886	0.3228
	η_1	1.0986	0.4385	1.1635	0.3584	1.1076	0.4247	1.2288	0.2897
	η_2	1.0986	0.8267	1.1604	0.7188	1.1047	0.8133	1.2252	0.6175

Table 4. Avg. estimated values and MSEs of the ML and BE using MCMC for different schemes of progressive Type-II censoring step-stress for extension Weibull distribution at n=60 and m=40

Sch.	**Parm.**	**MLE**		**BE MCMC: SEL**		**BE MCMC: LINEX**			
						τ=-1.5		**τ=1.5**	
		Avg.	MSE	Avg.	MSE	Avg.	MSE	Avg.	MSE
S_1	ϑ_{11}	0.3892	0.0136	0.3651	0.0195	0.3590	0.0212	0.3715	0.0179
	ϑ_{12}	0.3969	0.1263	0.3738	0.1431	0.3669	0.1484	0.3812	0.1377
	ϑ_{21}	1.3560	0.0344	1.2689	0.0231	1.1497	0.0262	1.4521	0.0844
	ϑ_{22}	1.2656	0.0783	1.1816	0.1236	1.0795	0.1931	1.3329	0.0666
	η_1	0.8742	0.7717	0.9108	0.7093	0.8898	0.7446	0.9334	0.6723

	η_2	0.9193	1.1736	0.9578	1.0927	0.9364	1.1373	0.9807	1.0460
S_2	ϑ_{11}	0.4091	0.0100	0.3791	0.0163	0.3732	0.0177	0.3852	0.0150
	ϑ_{12}	0.4134	0.1152	0.3837	0.1361	0.3775	0.1406	0.3902	0.1314
	ϑ_{21}	1.7902	0.3373	1.6945	0.2392	1.5147	0.0997	1.9987	0.6505
	ϑ_{22}	1.7244	0.0961	1.6321	0.0595	1.4676	0.0311	1.8979	0.2409
	η_1	0.9821	0.5966	1.0388	0.5136	1.0071	0.5588	1.0736	0.4663
	η_2	1.0120	0.9841	1.0688	0.8759	1.0373	0.9346	1.1034	0.8137
S_3	ϑ_{11}	0.3521	0.0226	0.3353	0.0279	0.3313	0.0293	0.3396	0.0266
	ϑ_{12}	0.4592	0.0881	0.4372	0.1030	0.4293	0.1076	0.4455	0.0983
	ϑ_{21}	1.3993	0.2220	1.3528	0.1898	1.2249	0.1284	1.5608	0.4178
	ϑ_{22}	1.3259	0.2300	1.2686	0.2314	1.1561	0.2466	1.4429	0.3035
	η_1	0.9734	0.6100	0.9924	0.5803	0.9668	0.6193	1.0196	0.5403
	η_2	1.0236	0.9614	1.0507	0.9093	1.0240	0.9600	1.0790	0.8570
S_4	ϑ_{11}	0.3484	0.0240	0.5273	1.3523	0.3939	0.2632	0.5502	1.2756
	ϑ_{12}	0.4201	0.1108	0.7327	3.2772	0.5134	0.5491	0.8500	1.7366
	ϑ_{21}	1.2498	0.0225	1.8732	1.2202	1.1313	1.1070	2.5090	1.0604
	ϑ_{22}	1.1521	0.1397	1.6107	0.6030	1.0096	0.5902	2.1884	0.1731
	η_1	0.9481	0.6478	1.7780	2.5824	1.1314	2.8737	2.0254	2.1397
	η_2	1.0082	0.9889	1.3636	2.9432	1.0327	2.0952	2.0950	2.08196
S_5	ϑ_{11}	0.4570	0.0041	0.4257	0.0078	0.4185	0.0088	0.4333	0.0068
	ϑ_{12}	0.4570	0.0881	0.4256	0.1075	0.4185	0.1120	0.4331	0.1028
	ϑ_{21}	2.2047	0.9737	2.0990	0.7770	1.8605	0.4113	2.5440	1.8374
	ϑ_{22}	2.2047	0.5589	2.0947	0.4124	1.8562	0.1671	2.5273	1.2076
	η_1	1.0184	0.5441	1.0824	0.4553	1.0438	0.5072	1.1255	0.4013
	η_2	1.0184	0.9725	1.0845	0.8482	1.0457	0.9195	1.1279	0.7722

Table 5. Avg. estimated values and MSEs of the ML and BE using MCMC for different schemes of progressive Type-II censoring step-stress for extension Weibull distribution at n=80 and m=40

Sch.	**Parm.**	**MLE**		**BE MCMC: SEL**		**BE MCMC: LINEX**			
						τ=-1.5		**τ=1.5**	
		Avg.	**MSE**	**Avg.**	**MSE**	**Avg.**	**MSE**	**Avg.**	**MSE**
S_1	ϑ_{11}	0.3694	0.0186	0.3481	0.0246	0.3428	0.0262	0.3537	0.0230
	ϑ_{12}	0.3778	0.1406	0.3571	0.1565	0.3508	0.1614	0.3637	0.1514
	ϑ_{21}	1.2576	0.0202	1.1842	0.0234	1.0848	0.0416	1.3264	0.0369
	ϑ_{22}	1.1311	0.1575	1.0620	0.2113	0.9837	0.2815	1.1679	0.1386
	η_1	0.9228	0.6894	0.9538	0.6393	0.9338	0.6712	0.9752	0.6061
	η_2	0.9957	1.0162	1.0297	0.9494	1.0094	0.9887	1.0514	0.9084
S_2	ϑ_{11}	0.4429	0.0054	0.4142	0.0095	0.4074	0.0106	0.4213	0.0084
	ϑ_{12}	0.4527	0.0909	0.4235	0.1091	0.4163	0.1137	0.4310	0.1043
	ϑ_{21}	2.0347	0.6699	1.9604	0.5531	1.7732	0.3099	2.2637	1.1197
	ϑ_{22}	1.9470	0.2561	1.8665	0.1866	1.7026	0.0801	2.1173	0.4696
	η_1	1.0862	0.4507	1.1350	0.3888	1.0989	0.4333	1.1752	0.3427
	η_2	1.1303	0.7685	1.1828	0.6806	1.1474	0.7383	1.2220	0.6198
S_3	ϑ_{11}	0.3239	0.0317	0.3107	0.0365	0.3073	0.0378	0.3142	0.0352
	ϑ_{12}	0.4971	0.0666	0.4776	0.0768	0.4685	0.0817	0.4872	0.0718
	ϑ_{21}	1.1052	0.1151	1.0950	0.1123	1.0132	0.1231	1.2128	0.1386
	ϑ_{22}	0.9101	0.4242	0.8824	0.4530	0.8329	0.5022	0.9466	0.4057
	η_1	1.0876	0.4456	1.0820	0.4528	1.0541	0.4904	1.1114	0.4148
	η_2	1.2773	0.5315	1.2855	0.5196	1.2573	0.5601	1.3152	0.4788
S_4	ϑ_{11}	0.3934	0.0126	0.3711	0.0179	0.3660	0.0192	0.3764	0.0166
	ϑ_{12}	0.4623	0.0852	0.4348	0.1020	0.4274	0.1064	0.4433	0.0991
	ϑ_{21}	2.0810	0.7315	2.0416	0.6693	1.8547	0.3947	2.3692	1.8543
	ϑ_{22}	1.8883	0.1855	1.8397	0.1496	1.6970	0.0649	2.0566	0.3963
	η_1	1.0663	0.4748	1.0967	0.4460	1.0626	0.4794	1.1267	0.3970
	η_2	1.1763	0.6869	1.2087	0.6352	1.1776	0.6847	1.2419	0.5846
S_5	ϑ_{11}	0.5493	0.0059	0.5166	0.0037	0.5059	0.0032	0.5278	0.0044
	ϑ_{12}	0.5493	0.0438	0.5171	0.0577	0.5065	0.0625	0.5283	0.0528
	ϑ_{21}	2.8325	2.5929	2.7607	2.3644	2.4830	1.5786	3.2832	4.3588
	ϑ_{22}	2.8325	1.8641	2.7575	1.6595	2.4823	1.0216	3.2661	3.3091
	η_1	1.1032	0.4300	1.1529	0.3687	1.1087	0.4217	1.2027	0.3140
	η_2	1.1032	0.8159	1.1534	0.7286	1.1095	0.8031	1.2027	0.6496

Table 6. Avg. estimated values and MSEs of the ML and BE using MCMC for different schemes of progressive Type-II censoring step-stress for extension Weibull distribution at n=80 and m=60

Sch.	**Parm.**	**MLE**		**BE MCMC: SEL**		**BE MCMC: LINEX**			
						τ=-1.5		**τ=1.5**	
		Avg.	**MSE**	**Avg.**	**MSE**	**Avg.**	**MSE**	**Avg.**	**MSE**

S_1	ϑ_{11}	0.4043	0.0101	0.3880	0.0136	0.3834	0.0146	0.3927	0.0126
	ϑ_{12}	0.4097	0.1169	0.3931	0.1286	0.3880	0.1322	0.3983	0.1249
	ϑ_{21}	1.4104	0.0421	1.3467	0.0264	1.2507	0.0133	1.4762	0.0760
	ϑ_{22}	1.3470	0.0400	1.2843	0.0630	1.1973	0.1045	1.3998	0.0337
	η_1	0.8456	0.8209	0.8715	0.7751	0.8575	0.7997	0.8862	0.7498
	η_2	0.8747	1.2698	0.9017	1.2102	0.8874	1.2415	0.9166	1.1777
S_2	ϑ_{11}	0.4066	0.0097	0.3860	0.0140	0.3818	0.0149	0.3903	0.0131
	ϑ_{12}	0.4088	0.1175	0.3877	0.1324	0.3833	0.1355	0.3921	0.1292
	ϑ_{21}	1.7094	0.2353	1.6390	0.1764	1.5076	0.0852	1.8287	0.3794
	ϑ_{22}	1.6664	0.0522	1.5971	0.0340	1.4731	0.0193	1.7733	0.1158
	η_1	0.9361	0.6665	0.9760	0.6038	0.9561	0.6345	0.9971	0.5719
	η_2	0.9546	1.0972	0.9947	1.0155	0.9749	1.0554	1.0158	0.9739
S_3	ϑ_{11}	0.3629	0.0194	0.3493	0.0233	0.3462	0.0243	0.3525	0.0224
	ϑ_{12}	0.4429	0.0969	0.4265	0.1079	0.4213	0.1112	0.4319	0.1046
	ϑ_{21}	1.4474	0.2020	1.4022	0.1637	1.3010	0.1105	1.5459	0.2950
	ϑ_{22}	1.4307	0.1696	1.3822	0.1600	1.2827	0.1588	1.5221	0.2184
	η_1	0.9332	0.6721	0.9529	0.6398	0.9359	0.6669	0.9706	0.6122
	η_2	0.9459	1.1161	0.9692	1.0676	0.9518	1.1035	0.9875	1.0307
S_4	ϑ_{11}	0.3673	0.0183	0.3536	0.0222	0.3500	0.0232	0.3574	0.0211
	ϑ_{12}	0.4108	0.1161	0.3953	0.1269	0.3901	0.1306	0.4007	0.1232
	ϑ_{21}	1.3179	0.0166	1.2731	0.0131	1.1869	0.0139	1.3891	0.0378
	ϑ_{22}	1.2448	0.0764	1.1992	0.1027	1.1220	0.1525	1.3017	0.0568
	η_1	0.8945	0.7344	0.9150	0.7002	0.8996	0.7259	0.9311	0.6737
	η_2	0.9325	1.1424	0.9567	1.0919	0.9404	1.1260	0.9737	1.0568
S_5	ϑ_{11}	0.4308	0.0060	0.4087	0.0096	0.4041	0.0104	0.4134	0.0088
	ϑ_{12}	0.4308	0.1031	0.4092	0.1174	0.4046	0.1205	0.4139	0.1142
	ϑ_{21}	1.9786	0.5693	1.8992	0.4587	1.7365	0.2644	2.1466	0.8714
	ϑ_{22}	1.9786	0.2675	1.9056	0.2020	1.7407	0.0858	2.1572	0.4997
	η_1	0.9785	0.6007	1.0244	0.5325	1.0005	0.5674	1.0502	0.4964
	η_2	0.9785	1.0490	1.0227	0.9613	0.9986	1.0084	1.0484	0.9122

Table 7. Avg. estimated values and MSEs of the ML and BE using MCMC for different schemes of progressive Type-II censoring step-stress for extension Weibull distribution at n=100 and m=60

Sch.	**Parm.**	**MLE**		**BE MCMC: SEL**		**BE MCMC: LINEX**			
						τ=-1.5		**τ=1.5**	
		Avg.	**MSE**	**Avg.**	**MSE**	**Avg.**	**MSE**	**Avg.**	**MSE**
S_1	ϑ_{11}	0.3842	0.0144	0.3683	0.0184	0.3642	0.0194	0.3725	0.0173
	ϑ_{12}	0.3901	0.1308	0.3749	0.1420	0.3703	0.1454	0.3797	0.1385
	ϑ_{21}	1.3256	0.0223	1.2668	0.0166	1.1849	0.0172	1.3735	0.0381
	ϑ_{22}	1.2279	0.0915	1.1738	0.1226	1.1047	0.1695	1.2611	0.0785
	η_1	0.8865	0.7492	0.9111	0.7076	0.8975	0.7304	0.9253	0.6840
	η_2	0.9369	1.1350	0.9619	1.0826	0.9481	1.1112	0.9764	1.0530
S_2	ϑ_{11}	0.4191	0.0077	0.3992	0.0113	0.3949	0.0121	0.4035	0.0105
	ϑ_{12}	0.4236	0.1078	0.4032	0.1216	0.3987	0.1247	0.4078	0.1185
	ϑ_{21}	1.8622	0.4042	1.8011	0.3339	1.6663	0.1966	1.9919	0.5976
	ϑ_{22}	1.7941	0.1170	1.7310	0.0836	1.6098	0.0360	1.8981	0.2041
	η_1	1.0232	0.5334	1.0614	0.4800	1.0388	0.5112	1.0855	0.4478
	η_2	1.0548	0.8992	1.0947	0.8261	1.0724	0.8666	1.1186	0.7839
S_3	ϑ_{11}	0.3354	0.0276	0.3258	0.0308	0.3232	0.0317	0.3284	0.0300
	ϑ_{12}	0.4749	0.0775	0.4616	0.0851	0.4556	0.0885	0.4678	0.0816
	ϑ_{21}	1.1647	0.0847	1.1473	0.0827	1.0810	0.0866	1.2334	0.0970
	ϑ_{22}	1.0375	0.2858	1.0158	0.3005	0.9650	0.3400	1.0798	0.2647
	η_1	1.0323	0.5200	1.0374	0.5125	1.0188	0.5392	1.0568	0.4855
	η_2	1.1337	0.7566	1.1445	0.7378	1.1250	0.7713	1.1648	0.7039
S_4	ϑ_{11}	0.3392	0.0264	0.3867	0.5193	0.3410	0.0590	0.4260	0.6208
	ϑ_{12}	0.3975	0.1253	0.5161	1.4882	0.4062	0.1539	0.5695	1.5474
	ϑ_{21}	1.6016	0.1443	1.7484	4.4111	1.4686	0.5586	2.0205	5.0282
	ϑ_{22}	1.4456	0.0193	1.5201	1.6944	1.2953	0.0800	1.8940	7.3793
	η_1	0.9737	0.6056	1.1568	2.3779	1.0243	1.6474	1.5549	1.5995
	η_2	1.0666	0.8745	1.2143	2.8129	1.0548	0.9438	1.4546	0.7479
S_5	ϑ_{11}	0.4859	0.0019	0.4642	0.0030	0.4586	0.0033	0.4700	0.0026
	ϑ_{12}	0.4859	0.0715	0.4647	0.0831	0.4591	0.0863	0.4705	0.0799
	ϑ_{21}	2.3918	1.3499	2.3262	1.2039	2.1391	0.8254	2.6164	1.9514
	ϑ_{22}	2.3919	0.8416	2.3335	0.7397	2.1437	0.4488	2.6248	1.3447
	η_1	1.0562	0.4877	1.0986	0.4313	1.0705	0.4681	1.1290	0.3934
	η_2	1.0562	0.8971	1.0966	0.8229	1.0683	0.8743	1.1272	0.7694

Table 8. Avg. estimated values and MSEs of the ML and BE using MCMC for different schemes of progressive Type-II censoring step-stress for extension Weibull distribution at n=100 and m=80

Sch.	Parm.	MLE		BE MCMC: SEL		BE MCMC: LINEX			
						τ=-1.5		τ=1.5	
		Avg.	MSE	Avg.	MSE	Avg.	MSE	Avg.	MSE
S_1	ϑ_{11}	0.4098	0.0088	0.3971	0.0113	0.3934	0.0121	0.4008	0.0106
	ϑ_{12}	0.4134	0.1140	0.4008	0.1228	0.3968	0.1255	0.4048	0.1200
	ϑ_{21}	1.4411	0.0513	1.3911	0.0346	1.3103	0.0156	1.4939	0.0792
	ϑ_{22}	1.3924	0.0265	1.3427	0.0396	1.2682	0.0658	1.4365	0.0238
	η_1	0.8334	0.8426	0.8538	0.8059	0.8431	0.8250	0.8649	0.7862
	η_2	0.8548	1.3143	0.8754	1.2675	0.8647	1.2917	0.8866	1.2427
S_2	ϑ_{11}	0.4054	0.0097	0.3894	0.0130	0.3861	0.0137	0.3927	0.0123
	ϑ_{12}	0.4067	0.1187	0.3905	0.1301	0.3871	0.1325	0.3939	0.1276
	ϑ_{21}	1.6607	0.1870	1.6058	0.1449	1.5033	0.0790	1.7418	0.2679
	ϑ_{22}	1.6295	0.0352	1.5744	0.0255	1.4769	0.0166	1.7014	0.0683
	η_1	0.9111	0.7066	0.9409	0.6578	0.9267	0.6809	0.9558	0.6341
	η_2	0.9243	1.1603	0.9551	1.0954	0.9409	1.1251	0.9700	1.0647
S_3	ϑ_{11}	0.3702	0.0173	0.3590	0.0204	0.3565	0.0211	0.3616	0.0197
	ϑ_{12}	0.4316	0.1036	0.4175	0.1130	0.4137	0.1154	0.4214	0.1106
	ϑ_{21}	1.5007	0.1988	1.4619	0.1679	1.3759	0.1152	1.5740	0.2679
	ϑ_{22}	1.5065	0.1342	1.4603	0.1228	1.3737	0.1134	1.5733	0.1680
	η_1	0.9055	0.7164	0.9234	0.6863	0.9109	0.7069	0.9362	0.6653
	η_2	0.9053	1.2014	0.9268	1.1548	0.9142	1.1818	0.9398	1.1271
S_4	ϑ_{11}	0.3807	0.0148	0.3694	0.0177	0.3664	0.0185	0.3726	0.0169
	ϑ_{12}	0.4101	0.1163	0.3982	0.1246	0.3942	0.1274	0.4023	0.1217
	ϑ_{21}	1.3603	0.0232	1.3193	0.0166	1.2466	0.0097	1.4110	0.0416
	ϑ_{22}	1.3018	0.0498	1.2626	0.0674	1.1952	0.1020	1.3475	0.0378
	η_1	0.8706	0.7757	0.8888	0.7442	0.8774	0.7638	0.9005	0.7242
	η_2	0.8989	1.2150	0.9181	1.1733	0.9061	1.1993	0.9305	1.1466
S_5	ϑ_{11}	0.4195	0.0073	0.4029	0.0103	0.3995	0.0109	0.4064	0.0096
	ϑ_{12}	0.4195	0.1100	0.4029	0.1213	0.3995	0.1237	0.4064	0.1189
	ϑ_{21}	1.8658	0.4024	1.8097	0.3372	1.6862	0.2095	1.9791	0.5673
	ϑ_{22}	1.8658	0.1570	1.8071	0.1185	1.6837	0.0530	1.9755	0.2621
	η_1	0.9501	0.6431	0.9833	0.5915	0.9664	0.6174	1.0011	0.5648
	η_2	0.9501	1.1056	0.9842	1.0357	0.9673	1.0701	1.0020	1.0001

Table 9. Interval estimates, AILs, and CP(%) values of the ML and BE using MCMC for different schemes of progressive Type-II step-stress for extension Weibull distribution at n=60 and m=30

Sch.	Parm.	Asy-CI				HPD			
		Lower	Upper	AIL	CP (%)	Lower	Upper	AIL	CP (%)
S_1	ϑ_{11}	0.2768	0.4490	0.1723	97.2	0.2433	0.4134	0.1701	96.6
	ϑ_{12}	0.2710	0.4750	0.2040	97.6	0.2480	0.4445	0.1964	97.3
	ϑ_{21}	0.9142	1.5628	0.6485	95.3	0.8733	1.4830	0.6097	97.5
	ϑ_{22}	0.7699	1.4373	0.6673	96.2	0.6950	1.3074	0.6124	95.7
	η_1	0.7683	1.1040	0.3357	97.1	0.8192	1.1567	0.3375	97.8
	η_2	0.8104	1.2239	0.4135	96.6	0.8582	1.2615	0.4033	96.8
S_2	ϑ_{11}	0.3386	0.5433	0.2047	96.7	0.3040	0.5088	0.2048	97.2
	ϑ_{12}	0.3415	0.5636	0.2220	96.4	0.3051	0.5189	0.2138	96.5
	ϑ_{21}	1.5141	2.5550	1.0409	96.9	1.4686	2.4419	0.9733	97.3
	ϑ_{22}	1.4114	2.4662	1.0548	97.4	1.3600	2.3445	0.9845	97.2
	η_1	0.8703	1.3143	0.4440	97.1	0.9270	1.3741	0.4471	96.9
	η_2	0.8984	1.3842	0.4858	96.7	0.9442	1.4268	0.4826	96.0
S_3	ϑ_{11}	0.2734	0.3888	0.1154	95.9	0.2624	0.3730	0.1105	96.8
	ϑ_{12}	0.3762	0.6308	0.2546	97.2	0.3306	0.5897	0.2590	95.8
	ϑ_{21}	0.3027	2.0535	1.7508	90.0	0.7458	2.3516	1.6057	95.1
	ϑ_{22}	0.1950	1.7708	1.5758	90.0	0.6290	2.0645	1.4355	95.5
	η_1	0.8742	1.2942	0.4200	98.4	0.8644	1.2708	0.4065	96.8
	η_2	1.0235	1.5094	0.4859	98.3	0.9992	1.4775	0.4783	96.7
S_4	ϑ_{11}	0.3161	0.4842	0.1681	96.4	0.2886	0.4689	0.1804	96.7
	ϑ_{12}	0.3543	0.5887	0.2344	96.3	0.3250	0.5653	0.2403	96.6
	ϑ_{21}	1.5932	2.5649	0.9717	96.1	1.5706	2.4748	0.9042	97.0
	ϑ_{22}	1.4417	2.3415	0.8998	96.0	1.3673	2.2548	0.8875	95.9
	η_1	0.8741	1.2695	0.3954	96.9	0.8999	1.2975	0.3976	96.8
	η_2	0.9690	1.3922	0.4233	97.0	1.0096	1.4586	0.4490	97.7
S_5	ϑ_{11}	0.4199	0.6838	0.2639	95.7	0.3956	0.6484	0.2528	97.1
	ϑ_{12}	0.4199	0.6838	0.2639	95.7	0.3912	0.6392	0.2480	96.8
	ϑ_{21}	2.1985	3.5032	1.3046	95.8	2.1178	3.3580	1.2402	96.0

	ϑ_{22}	2.1985	3.5032	1.3046	95.8	2.1797	3.3652	1.1855	96.6
	η_1	0.8652	1.3319	0.4667	96.8	0.9401	1.4123	0.4722	97.8
	η_2	0.8652	1.3319	0.4667	96.8	0.9559	1.4233	0.4675	98.0

Table 10. Interval estimates, AILs, and CP(%) values of the ML and BE using MCMC for different schemes of progressive Type-II step-stress for extension Weibull distribution at n=60 and m=40

Sch.	Parm.	Asy-CI				HPD			
		Lower	**Upper**	**AIL**	**CP (%)**	**Lower**	**Upper**	**AIL**	**CP (%)**
S_1	ϑ_{11}	0.3182	0.4601	0.1419	97.3	0.2953	0.4378	0.1425	97.6
	ϑ_{12}	0.3186	0.4752	0.1566	97.4	0.3007	0.4580	0.1573	98.1
	ϑ_{21}	1.0577	1.6542	0.5965	97.1	0.9923	1.5594	0.5671	97.2
	ϑ_{22}	0.9662	1.5649	0.5987	97.5	0.9167	1.4872	0.5705	98.0
	η_1	0.7419	1.0064	0.2646	97.4	0.7672	1.0448	0.2776	96.4
	η_2	0.7703	1.0683	0.2980	97.0	0.8061	1.1148	0.3088	97.1
S_2	ϑ_{11}	0.3279	0.4903	0.1624	97.0	0.3006	0.4627	0.1622	96.9
	ϑ_{12}	0.3278	0.4990	0.1712	97.0	0.2998	0.4672	0.1674	97.0
	ϑ_{21}	1.3722	2.2083	0.8361	96.3	1.3296	2.1015	0.7720	97.4
	ϑ_{22}	1.3048	2.1440	0.8392	96.3	1.2701	2.0452	0.7751	97.2
	η_1	0.8181	1.1462	0.3282	97.5	0.8738	1.2168	0.3430	97.3
	η_2	0.8376	1.1864	0.3488	97.3	0.8982	1.2549	0.3567	97.9
S_3	ϑ_{11}	0.2990	0.4053	0.1063	96.7	0.2836	0.3921	0.1085	97.3
	ϑ_{12}	0.3427	0.5757	0.2330	99.1	0.3185	0.5382	0.2197	97.0
	ϑ_{21}	0.5230	2.2757	1.7527	96.1	0.8883	2.2028	1.3145	95.9
	ϑ_{22}	0.4495	2.2023	1.7528	96.4	0.7913	2.0283	1.2370	95.3
	η_1	0.8101	1.1368	0.3267	98.3	0.8380	1.1465	0.3086	98.0
	η_2	0.8487	1.1984	0.3498	98.3	0.8716	1.2199	0.3483	97.4
S_4	ϑ_{11}	0.2873	0.4095	0.1222	97.2	0.0017	0.8938	0.8921	95.1
	ϑ_{12}	0.3328	0.5075	0.1747	97.9	0.0048	1.5740	1.5692	95.1
	ϑ_{21}	0.9554	1.5443	0.5889	95.2	0.0032	5.1306	5.1274	95.1
	ϑ_{22}	0.8841	1.4202	0.5361	95.9	0.0049	3.8354	3.8305	95.1
	η_1	0.8143	1.0818	0.2675	97.7	0.0055	3.9074	3.9019	95.0
	η_2	0.8670	1.1494	0.2825	98.0	0.0081	2.0450	2.0370	95.2
S_5	ϑ_{11}	0.3642	0.5498	0.1855	96.7	0.3380	0.5225	0.1844	97.4
	ϑ_{12}	0.3642	0.5498	0.1855	96.7	0.3395	0.5225	0.1830	97.1
	ϑ_{21}	1.7150	2.6943	0.9793	96.8	1.6774	2.5784	0.9010	97.4
	ϑ_{22}	1.7150	2.6943	0.9793	96.8	1.6525	2.5848	0.9322	97.3
	η_1	0.8341	1.2026	0.3686	97.2	0.8860	1.2688	0.3828	96.7
	η_2	0.8341	1.2026	0.3686	97.2	0.9076	1.2952	0.3876	98.2

Table 11. Interval estimates, AILs, and CP (%) values of the ML and BE using MCMC for different schemes of progressive Type-II step-stress for extension Weibull distribution at n=80 and m=40

Sch.	Parm.	Asy-CI				HPD			
		Lower	**Upper**	**AIL**	**CP (%)**	**Lower**	**Upper**	**AIL**	**CP (%)**
S_1	ϑ_{11}	0.2917	0.4471	0.1553	97.0	0.2737	0.4246	0.1509	97.1
	ϑ_{12}	0.2880	0.4676	0.1796	96.6	0.2774	0.4561	0.1788	98.4
	ϑ_{21}	0.9792	1.5361	0.5569	97.1	0.9196	1.4466	0.5270	96.8
	ϑ_{22}	0.8439	1.4183	0.5744	97.4	0.8034	1.3408	0.5374	97.4
	η_1	0.7829	1.0627	0.2798	96.2	0.8153	1.0954	0.2801	96.8
	η_2	0.8252	1.1662	0.3410	96.8	0.8602	1.1981	0.3379	96.6
S_2	ϑ_{11}	0.3516	0.5342	0.1826	96.4	0.3295	0.5063	0.1768	97.1
	ϑ_{12}	0.3548	0.5505	0.1957	96.4	0.3289	0.5202	0.1913	96.3
	ϑ_{21}	1.5788	2.4906	0.9118	97.0	1.5448	2.3890	0.8442	97.1
	ϑ_{22}	1.4819	2.4122	0.9303	97.1	1.4224	2.2947	0.8723	96.8
	η_1	0.8897	1.2827	0.3930	97.1	0.9461	1.3340	0.3879	97.1
	η_2	0.9154	1.3451	0.4297	96.5	0.9773	1.4157	0.4383	97.3
S_3	ϑ_{11}	0.2738	0.3741	0.1003	96.1	0.2666	0.3631	0.0965	97.1
	ϑ_{12}	0.3960	0.5983	0.2023	97.1	0.3789	0.5921	0.2132	98.6
	ϑ_{21}	0.5037	1.7068	1.2031	95.5	0.8083	1.3053	0.4970	95.4
	ϑ_{22}	0.3688	1.4513	1.0825	95.5	0.6460	1.0450	0.3989	95.3
	η_1	0.9251	1.2501	0.3249	98.7	0.9141	1.2268	0.3127	97.2
	η_2	1.0891	1.4656	0.3765	98.8	1.1183	1.4931	0.3749	99.7
S_4	ϑ_{11}	0.3242	0.4626	0.1385	96.4	0.3091	0.4439	0.1347	97.2
	ϑ_{12}	0.3658	0.5588	0.1930	96.7	0.3503	0.5334	0.1831	97.2
	ϑ_{21}	1.6843	2.4777	0.7934	97.0	1.6679	2.4458	0.7778	97.2
	ϑ_{22}	1.5230	2.2536	0.7307	96.8	1.4982	2.2085	0.7103	97.2
	η_1	0.8980	1.2346	0.3366	96.9	0.9424	1.2898	0.3474	98.3

	η_2	0.9957	1.3570	0.3613	96.7	1.0290	1.3914	0.3624	97.2
S_5	ϑ_{11}	0.4330	0.6656	0.2325	95.4	0.4121	0.6347	0.2226	96.6
	ϑ_{12}	0.4330	0.6656	0.2326	95.4	0.4126	0.6383	0.2257	96.7
	ϑ_{21}	2.2489	3.4161	1.1671	96.7	2.1897	3.3144	1.1246	96.4
	ϑ_{22}	2.2489	3.4161	1.1671	96.7	2.2537	3.3410	1.0873	97.2
	η_1	0.8914	1.3150	0.4236	97.2	0.9392	1.3607	0.4214	96.9
	η_2	0.8914	1.3150	0.4236	97.2	0.9497	1.3780	0.4283	97.9

Table 12. Interval estimates, AILs, and CP (%) values of the ML and BE using MCMC for different schemes of progressive Type-II step-stress for extension Weibull distribution at n=80 and m=60

Sch.	Parm.	Asy-CI				HPD			
		Lower	Upper	AIL	CP (%)	Lower	Upper	AIL	CP (%)
S_1	ϑ_{11}	0.3431	0.4655	0.1224	97.3	0.3250	0.4516	0.1266	97.7
	ϑ_{12}	0.3445	0.4749	0.1304	97.6	0.3306	0.4614	0.1308	97.9
	ϑ_{21}	1.1599	1.6610	0.5012	97.1	1.1062	1.6044	0.4982	97.1
	ϑ_{22}	1.0941	1.6000	0.5059	97.5	1.0377	1.5204	0.4827	96.2
	η_1	0.7390	0.9523	0.2133	97.6	0.7713	0.9891	0.2179	98.0
	η_2	0.7584	0.9910	0.2325	97.2	0.7895	1.0305	0.2410	98.3
S_2	ϑ_{11}	0.3447	0.4685	0.1238	97.5	0.3273	0.4495	0.1222	97.9
	ϑ_{12}	0.3446	0.4731	0.1285	97.5	0.3209	0.4508	0.1298	97.1
	ϑ_{21}	1.4038	2.0149	0.6111	96.8	1.3480	1.9411	0.5931	97.1
	ϑ_{22}	1.3595	1.9733	0.6137	97.1	1.3127	1.9008	0.5881	97.5
	η_1	0.8119	1.0602	0.2483	97.7	0.8482	1.1180	0.2698	97.7
	η_2	0.8252	1.0841	0.2589	97.5	0.8622	1.1350	0.2728	97.5
S_3	ϑ_{11}	0.3161	0.4097	0.0936	96.1	0.3044	0.4004	0.0960	96.8
	ϑ_{12}	0.3430	0.5427	0.1997	98.6	0.3263	0.5162	0.1899	96.8
	ϑ_{21}	0.6557	2.2392	1.5835	94.5	0.9218	2.1655	1.2437	95.2
	ϑ_{22}	0.6347	2.2267	1.5920	94.5	0.9025	2.1642	1.2617	95.2
	η_1	0.7945	1.0720	0.2776	98.1	0.8174	1.0772	0.2598	97.5
	η_2	0.8067	1.0852	0.2785	98.3	0.8147	1.0960	0.2813	97.1
S_4	ϑ_{11}	0.3142	0.4205	0.1063	96.0	0.3004	0.4098	0.1095	96.9
	ϑ_{12}	0.3464	0.4752	0.1288	96.2	0.3355	0.4693	0.1338	98.7
	ϑ_{21}	1.1030	1.5328	0.4298	95.6	1.0799	1.5179	0.4380	97.5
	ϑ_{22}	1.0365	1.4532	0.4167	95.4	0.9915	1.4196	0.4282	96.7
	η_1	0.7946	0.9944	0.1998	97.8	0.8174	1.0304	0.2130	98.7
	η_2	0.8259	1.0392	0.2133	97.7	0.8389	1.0643	0.2254	97.1
S_5	ϑ_{11}	0.3625	0.4991	0.1366	97.2	0.3411	0.4786	0.1375	97.1
	ϑ_{12}	0.3625	0.4991	0.1366	97.2	0.3388	0.4762	0.1373	97.0
	ϑ_{21}	1.5941	2.3631	0.7690	96.4	1.5291	2.2705	0.7414	96.7
	ϑ_{22}	1.5941	2.3631	0.7690	96.4	1.5310	2.3108	0.7798	96.8
	η_1	0.8330	1.1239	0.2909	97.9	0.8688	1.1747	0.3059	97.6
	η_2	0.8330	1.1239	0.2909	97.9	0.8577	1.1688	0.3112	96.7

Table 13. Interval estimates, AILs, and CP (%) values of the ML and BE using MCMC for different schemes of progressive Type-II step-stress for extension Weibull distribution at n=100 and m=60

Sch.	Parm.	Asy-CI				HPD			
		Lower	Upper	AIL	CP (%)	Lower	Upper	AIL	CP (%)
S_1	ϑ_{11}	0.3212	0.4473	0.1261	98.0	0.3073	0.4318	0.1245	97.5
	ϑ_{12}	0.3206	0.4595	0.1389	97.6	0.3043	0.4406	0.1364	97.5
	ϑ_{21}	1.0732	1.5779	0.5047	96.7	1.0013	1.5056	0.5043	96.0
	ϑ_{22}	0.9685	1.4873	0.5188	97.7	0.9281	1.4168	0.4887	96.8
	η_1	0.7697	1.0032	0.2335	97.9	0.7887	1.0205	0.2319	96.5
	η_2	0.8008	1.0730	0.2721	97.3	0.8415	1.1042	0.2627	97.8
S_2	ϑ_{11}	0.3537	0.4845	0.1308	96.6	0.3412	0.4697	0.1285	97.9
	ϑ_{12}	0.3548	0.4923	0.1375	96.8	0.3343	0.4767	0.1424	97.2
	ϑ_{21}	1.5254	2.1989	0.6735	96.7	1.4651	2.1332	0.6681	96.1
	ϑ_{22}	1.4520	2.1363	0.6843	96.9	1.4264	2.0823	0.6559	98.0
	η_1	0.8830	1.1634	0.2804	97.7	0.9210	1.2102	0.2892	97.8
	η_2	0.9043	1.2054	0.3010	97.6	0.9588	1.2633	0.3046	98.2
S_3	ϑ_{11}	0.2916	0.3792	0.0875	96.3	0.2821	0.3685	0.0865	96.4
	ϑ_{12}	0.3917	0.5582	0.1665	97.7	0.3643	0.5430	0.1787	97.9
	ϑ_{21}	0.6190	1.7104	1.0913	93.8	0.8651	1.9844	1.1193	95.2
	ϑ_{22}	0.5120	1.5630	1.0510	93.8	0.7837	1.8617	1.0780	95.7
	η_1	0.8940	1.1707	0.2767	98.2	0.8973	1.1658	0.2684	97.6
	η_2	0.9810	1.2863	0.3053	98.5	0.9910	1.2898	0.2988	98.0
S_4	ϑ_{11}	0.2937	0.3846	0.0909	96.0	0.2615	0.4839	0.2223	96.6

	ϑ_{12}	0.3339	0.4612	0.1273	95.8	0.2868	0.6525	0.3656	95.7
	ϑ_{21}	1.3194	1.8837	0.5643	96.7	1.1930	1.9534	0.7604	96.7
	ϑ_{22}	1.1946	1.6966	0.5020	96.6	1.0525	1.8074	0.7549	97.2
	η_1	0.8670	1.0805	0.2134	///97.7	0.8111	1.1666	0.3555	95.7
	η_2	0.9539	1.1794	0.2255	97.6	0.8409	1.2991	0.4581	97.4
S_5	ϑ_{11}	0.4054	0.5663	0.1609	96.7	0.3909	0.5517	0.1607	98.1
	ϑ_{12}	0.4054	0.5663	0.1609	96.7	0.3912	0.5516	0.1604	97.9
	ϑ_{21}	1.9706	2.8131	0.8425	96.9	1.9024	2.7379	0.8356	96.8
	ϑ_{22}	1.9706	2.8131	0.8425	96.9	1.9181	2.7482	0.8301	97.0
	η_1	0.8997	1.2127	0.3131	97.6	0.9319	1.2515	0.3196	96.7
	η_2	0.8997	1.2127	0.3131	97.6	0.9262	1.2503	0.3242	96.6

Table 14. Interval estimates, AILs, and CP (%) values of the ML and BE using MCMC for different schemes of progressive Type-II step-stress for extension Weibull distribution at n=100 and m=80

Sch.	Parm.	Asy-CI				HPD			
		Lower	Upper	AIL	CP (%)	Lower	Upper	AIL	CP (%)
S_1	ϑ_{11}	0.3583	0.4613	0.1029	97.2	0.3446	0.4520	0.1074	97.8
	ϑ_{12}	0.3593	0.4676	0.1083	97.6	0.3427	0.4576	0.1149	97.5
	ϑ_{21}	1.2029	1.6792	0.4763	96.5	1.1373	1.6201	0.4828	96.3
	ϑ_{22}	1.1533	1.6315	0.4782	96.6	1.0956	1.5755	0.4799	96.6
	η_1	0.7368	0.9299	0.1931	97.2	0.7602	0.9601	0.1998	97.6
	η_2	0.7521	0.9574	0.2053	96.9	0.7730	0.9885	0.2155	97.4
S_2	ϑ_{11}	0.3520	0.4589	0.1069	97.3	0.3364	0.4447	0.1083	97.2
	ϑ_{12}	0.3518	0.4615	0.1097	97.1	0.3268	0.4396	0.1128	96.1
	ϑ_{21}	1.3953	1.9262	0.5309	96.4	1.3579	1.8758	0.5180	97.2
	ϑ_{22}	1.3632	1.8958	0.5326	96.6	1.2933	1.8440	0.5506	96.3
	η_1	0.8060	1.0162	0.2102	97.1	0.8294	1.0478	0.2184	97.0
	η_2	0.8158	1.0327	0.2169	96.8	0.8363	1.0663	0.2300	97.0
S_3	ϑ_{11}	0.3284	0.4120	0.0836	96.7	0.3186	0.4059	0.0872	97.2
	ϑ_{12}	0.3390	0.5242	0.1852	99.2	0.3345	0.5006	0.1661	98.3
	ϑ_{21}	0.7778	2.2237	1.4459	93.4	1.0286	2.1988	1.1702	95.4
	ϑ_{22}	0.7882	2.2249	1.4366	93.4	1.0504	2.2068	1.1564	95.7
	η_1	0.7944	1.0166	0.2223	98.7	0.8127	1.0251	0.2124	97.5
	η_2	0.7972	1.0133	0.2161	98.5	0.8235	1.0312	0.2077	97.9
S_4	ϑ_{11}	0.3324	0.4289	0.0965	96.6	0.3208	0.4174	0.0966	96.6
	ϑ_{12}	0.3545	0.4657	0.1112	96.6	0.3452	0.4584	0.1132	97.4
	ϑ_{21}	1.1546	1.5660	0.4114	96.6	1.1225	1.5279	0.4054	97.2
	ϑ_{22}	1.1005	1.5032	0.4027	96.4	1.0727	1.4812	0.4085	97.8
	η_1	0.7766	0.9645	0.1880	98.1	0.7912	0.9778	0.1866	97.0
	η_2	0.8002	0.9975	0.1972	98.1	0.8190	1.0153	0.1964	97.6
S_5	ϑ_{11}	0.3642	0.4748	0.1106	97.3	0.3435	0.4567	0.1132	96.9
	ϑ_{12}	0.3642	0.4748	0.1106	97.3	0.3476	0.4609	0.1133	97.9
	ϑ_{21}	1.5671	2.1644	0.5973	96.9	1.5175	2.1039	0.5865	96.5
	ϑ_{22}	1.5672	2.1644	0.5973	96.9	1.5225	2.1222	0.5997	97.4
	η_1	0.8366	1.0637	0.2270	98.5	0.8704	1.1006	0.2301	97.4
	η_2	0.8366	1.0637	0.2270	98.5	0.8705	1.1098	0.2394	98.0

Table 15. Est. and St.Ers of the ML and BE using MCMC for the illustrative example

Parm.	MLE		BE MCMC: SEL		BE MCMC: LINEX	
	Est.	St.Er	Est.	St.Er	Est.: $\boldsymbol{v = -1.5}$	Est.: $\boldsymbol{v = 1.5}$
ϑ_{11}	0.37645	0.11855	0.34314	0.09630	0.33636	0.32027
ϑ_{12}	0.40752	0.12840	0.37572	0.11769	0.36586	0.36269
ϑ_{21}	1.90398	0.56882	2.11279	0.71536	1.83632	1.63905
ϑ_{22}	1.75120	0.47933	1.80192	0.57464	1.61408	1.60377
η_1	1.26986	0.35908	1.20095	0.33957	1.12060	1.02945
η_2	1.41021	0.35270	1.40928	0.34697	1.32175	1.30302

Table 16. Interval estimates and ILs values of the ML and BE using MCMC for the illustrative example

Par m.	Asy-CI			HPD		
	Lower	Upper	AIL	Lower	Upper	AIL
ϑ_{11}	0.14410	0.60880	0.46470	0.16710	0.53427	0.36716
ϑ_{12}	0.15587	0.65918	0.50331	0.16813	0.62265	0.45452
ϑ_{21}	0.78911	3.01884	2.22973	1.01234	3.00616	1.99382
ϑ_{22}	0.81172	2.69068	1.87896	0.82916	2.00659	1.17743
η_1	0.56609	1.97364	1.40755	0.53728	1.82395	1.28667
η_2	0.71893	2.10150	1.38257	0.71233	2.08963	1.37730

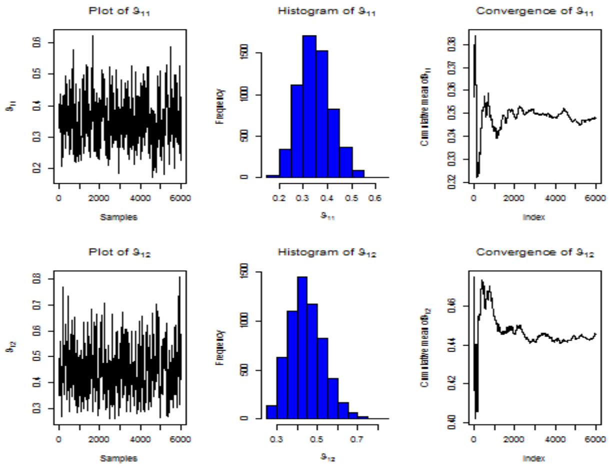

Figure 1. Convergence of MCMC estimates for ϑ_{11} and ϑ_{12} for n=80 and m=60

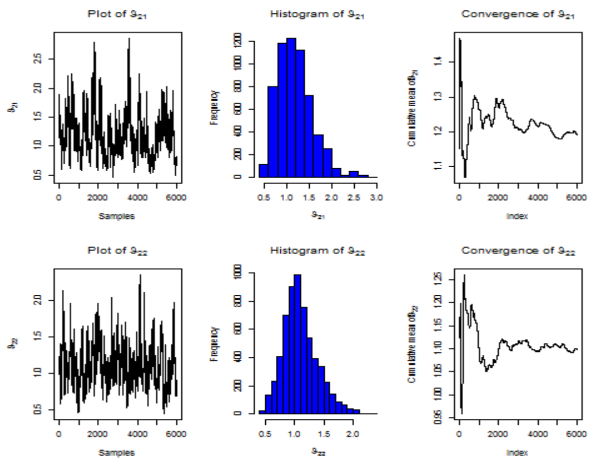

Figure 2. Convergence of MCMC estimates for ϑ_{21} and ϑ_{22} for n=80 and m=60

Nonetheless, the CIs of HPD is better than ACIs for MLE, this is because, the AIL of HPD is less than AIL for MLE. in addition to The BEs of the considered parameters based on LINEX loss function (τ=1.5) are smaller than that based on SE loss function.

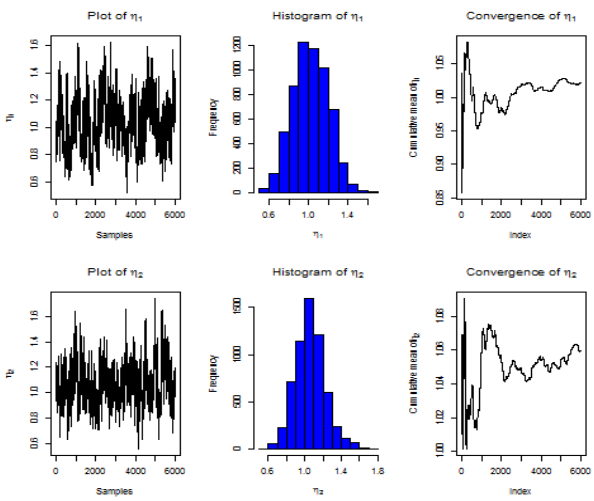

Figure 3. Convergence of MCMC estimates for η_1 and η_2 for n=80 and m=60

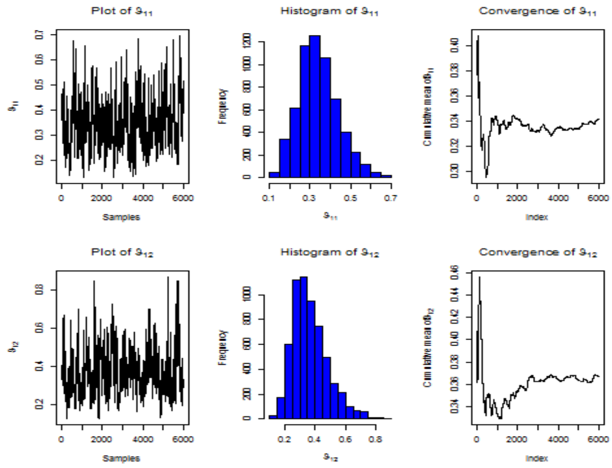

Figure 4. Convergence of MCMC estimates of ϑ_{11} and ϑ_{12} for the illustrative example

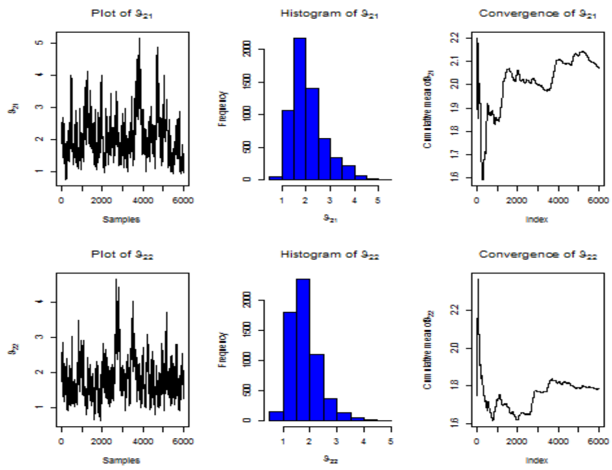

Figure 5. Convergence of MCMC estimates of ϑ_{21} and ϑ_{22} for the illustrative example

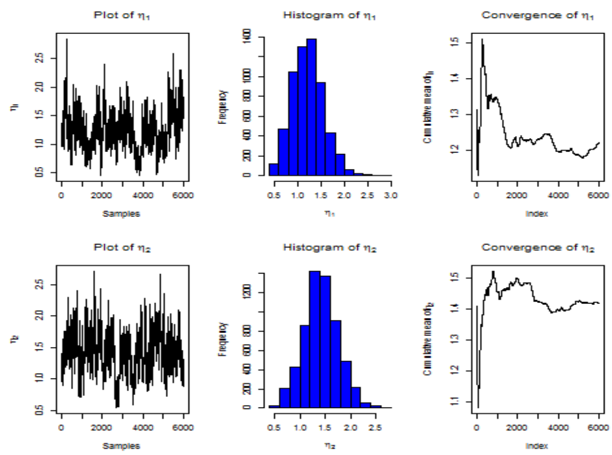

Figure 6. Convergence of MCMC estimates of η_1 and η_2 for the illustrative example

7. CONCLUSION

In this article, established on a cumulative exposure model under several progressively censoring type II, we studied a simple SS-ALT model with two independent failures competing for risks from extension Weibull distribution. We have derived MLEs and asymptotic confidence interval estimates for the unknown parameters of extension Weibull distribution. Also, we computed BEs and the corresponding HPD interval estimates under informative priors based on two different types of loss functions LINEX and squared error loss functions. We have then performed a simulation study to assess the performance of all these procedures and an explanatory instance has been offered to demonstrate all the methods of inference developed in this paper. An approximate CI and Bayesian credible interval for the unknown parameters are discussed when the sample size increases. Established on the outcomes of the simulation study, our recommendation for the BEs performs better than MLEs in terms of Average estimate and MSE. The MLEs and BEs for fixed sample size and pre-fixed number of failures increase, the MSEs and Average estimate for unknown parameter decrease. The MLE and Bayesian estimates for Scheme 5 have best properties than other Schemes. Nevertheless, MSE for BE under LINEX with (τ=1.5) is smaller than other BE methods for all parameters expected. Our recommendation for credible/confidence intervals, the average lengths of ACIs and credible intervals become smaller as the pre-fixed number of failures increases.

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