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Percolation Analysis in a Fractal Network with Stable Opinion Dynamics

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ABSTRACT

This paper presents the proposal of a stable non-consensus opinion model in an infinitely branched fractal network, analyzing how its percolation properties are affected and based on the Ising model that allows the stable coexistence of three states, forming two groups of agents that hold contrary opinions and a third group that assumes a state of indecision The model is structured in a Sierpinski folder S_3^1 in which its fractal attributes are characterized by the dimensions of Hausdorff (D_H), topological Hausdorff (D_{tH}) and the spectral dimension (d_s) since in these the values of the critical exponents of percolation are determined by the set of numbers of the dimensions (D_H, D_{tH}, d_s), rather than solely by spatial dimension (d). Our findings suggest that starting from a random distribution of agents to which initial conditions are given, and employing a stable opinion dynamic through numerical simulation to calculate the percolation threshold and its critical exponents, the kind of universality to which the model belongs is determined and how the fractal characteristics in an infinitely branched network affect its percolation properties.

1. INTRODUCTION

Many of the characteristics studied in the collective behavior of systems are independent of the attributes of each and the details of their interactions [1]. This makes it possible to model the dynamics of large groups using the tools of statistical physics [2]. The hypotheses of the opinion models are based on the observation that the agents tend to make when exchanging their opinions, and it is these interactions that are the cause of the change of opinion towards a consensus or non-consensus between them [1, 2]. One of the most used tools to analyze this type of phenomenon is the theory of percolation.

The theory of percolation has been known for several decades [3, 4] and is the simplest fundamental model in statistical physics. It is typical to approach percolation theory as the study of a variety of chaotic and random systems that exhibit criticality [5, 6]. These complex and self-organized systems present interconnectivity between their individuals either through links or sites and are characterized by being based on simple discrete models, but mainly by the importance of the study of their percolation threshold characterized by their critical exponents that are independent of the geometry of the system [7-10]. However, it has been found that percolation thresholds in cells with the same topological constraint depend on fine details of the Structure of the same [11-13] this may indicate that cells with different thresholds belong to different classes of universality.

This also implies that the work done to build universal formulas to predict percolation thresholds based on a small number of network characteristics with the spatial dimension and coordination number is not sufficient. Previously, Galam and Mauger [10] have provided estimates for various networks using the formula of the power law quantified by Eq. (1).

$$p_c = p_0 d^b q^{-a} \tag{1}$$

where, the d-dimensional lattices are determined by the different values of the constants p_0, a and b and by the topological restriction Eq. (2) [10].

$$q = (d-1)(Z_\infty - 1) \tag{2}$$

However, there are examples of networks with equal d and q, but with different percolation thresholds. Therefore, it seems likely that there are more classes of universality since Galam and Mauger specified that if there is a universal formula for percolation thresholds, it must be based on more information than d and q alone.

In addition, it can be used to identify unknown percolation thresholds, since although it may indicate that meshes with different thresholds belong to different classes of universality this also makes the spatial dimension and the coordination number not enough to build a universal law. However, this can be useful for testing the effects of topology and connectivity on the information percolation threshold in similar networks [11-15].

The rest of the article is structured as follows: section 2 addresses the main features of the sierpinsky folder and its related percolation network. Section 3 reports the numerical simulations performed in the sierpinsky folder. Section 4 discusses the fractal effects that resulted in the simulations performed. Finally, section 5 describes the relevant conclusions Figure 1.

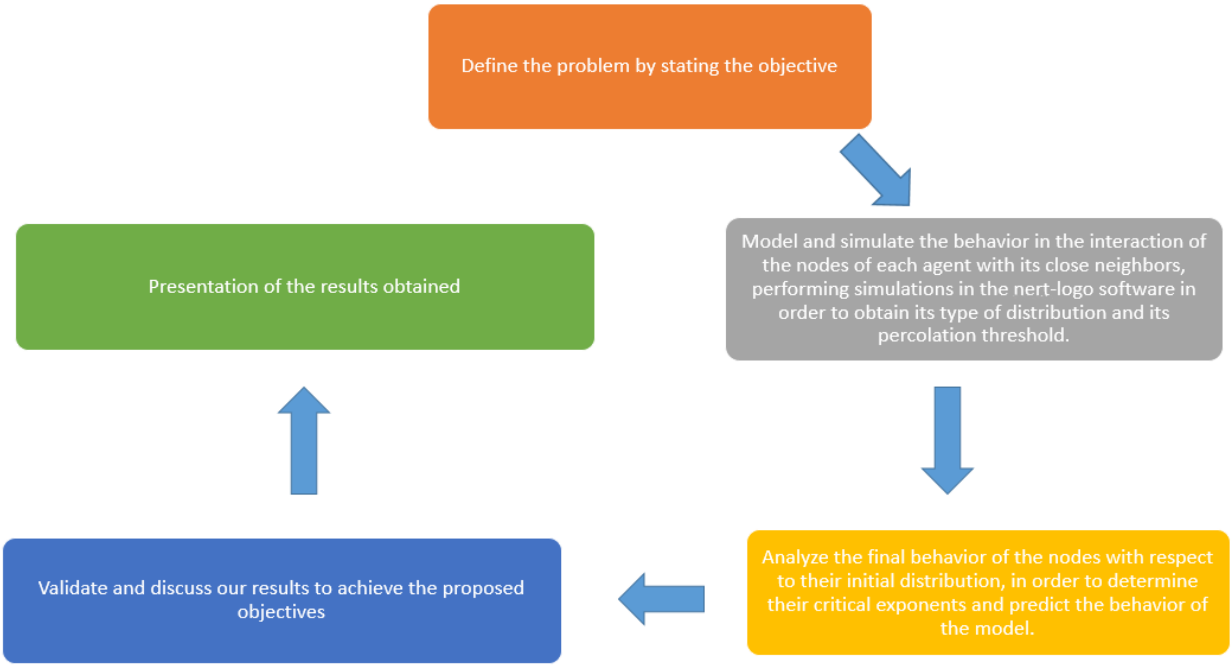

Figure 1. Algorithm representing the methodology of the system structure

2. INFINITELY RAMIFIED NETWORKS

Recent studies have shown that infinitely branched networks can be quantified by topology. Hausdorff dimension D_{tH} [12].

Infinitely branched networks are determined by their order, which is defined as the smallest number of links r_i at a point i that must be cut to isolate an arbitrarily large bounded set of points C_i connected to i. For the infinitely branched network, the number r_i grows with the size of the network L of C_i as $r_i \propto L^{Q_i}$, while for finitely branched networks all numbers r_i are finite [16].

To characterize fractal geometry, van der Marck [13] introduced a new dimension number D_{tH}, called the Hausdorff topological dimension this concept of dimension for metric spaces is defined by a combination of topological dimension and the Hausdorff dimension. Mandelbrot [16] defined that for a subset of a Euclidean space to be a fractal its topological dimension must be strictly smaller than its Hausdorff dimension. The value of the Topological Hausdorff dimension is always between the topological dimension and the Hausdorff dimension, in particular, this new dimension is a non-trivial lower estimate for the Hausdorff dimension.

That is, regular d-dimensional meshes are characterized by DtH=d, while, in general in fractal geometry, $D_{tH} \leq D_H \leq d$ [17], where D_H is the Hausdorff dimension that regulates the scale of number of sites in the network with the network size $N \propto L_{DH}$.

One of the most important fractal geometries is the Sierpinski carpet which is a generalization of the Cantor set in two dimensions [18-20]. It is universal for any compact object in the plane. Thus, any curve drawn in the plane with the intersections we want, however complicated, will be homeomorphic to a subset of the Sierpinski carpet. That can be built through an iterative process as follows. The square unit $[1,0]^2$ is divided into nxn sub-frames of equal size and the interior of m^2 sub-surfaces are removed. The same procedure is applied recursively to all remaining sub-surfaces indefinitely. The boundary area of the resulting network is equal to zero. Figure 1 shows two three- and four-step Sierpinski pre-fractal carpets Figure 2.

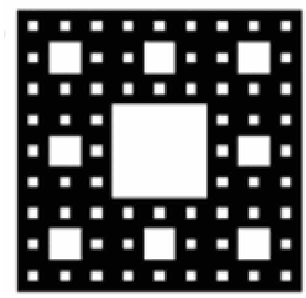

Figure 2. Sierpinski carpet pre-fractal S_3^1 3 steps

With the number of interactions k increases the number of sites, in the carpet N_k are increased as $N_k=(n^2-m^2)\ k$, while the size of the folder (measured in the number of sites k) increases as the size of the folder $L_k=n^k$. So the fractal dimension (Hausdorff) of is equal to Eq. (3).

$$D_H = \frac{\ln\,(n^2-m^2)}{\ln n} \tag{3}$$

where, m tells us the number of copies of similarity and n tells us the magnification factor.

The average number of coordination (average number of links per site) of increases with the number of iteration steps k as. Eq. (4) and (5).

$$Z_k = Z_\infty\left[1-\left(\frac{n}{n^2-m^2}\right)\right] \tag{4}$$

$$Z_k = Z_\infty\left[1-n^{-(D_H-1)k}\right] \tag{5}$$

where, Eq. (6) and (7).

$$Z_\infty = 4\left(\frac{n^2 - m^2}{n^2 - m^2 - n}\right) \tag{6}$$

$$Z_\infty = 4\frac{n^2 - m^2 - (n+m)}{n^2 - m^2 - n} \tag{7}$$

It is the average coordination number at the fractal boundary k→∞, while Z_k is the average coordination number after k iterations [20]. It is important to deduce the rapid convergence of Z_k to Z_∞ after some iterations.

From here we can see that the topological dimension of Hausdorff of the folder of sierpinky is equal to Eq. (8).

$$D_{tH} = \frac{ln(n-m)}{\ln n} \tag{8}$$

where, the second term on the right side equals the connectivity of folder 3 [21]. Of the Eqns. (3), (4) and (5) it follows that the average coordination number Z_∞ of Smn is related to D_H and D_{tH} as Eq. (9).

$$Z_\infty = 4\frac{1 - n^{1-2D_{tH}}}{1 - n^{1-2D_H}} \tag{9}$$

In addition, the phenomenological relationship for the spectral dimension Eq. (10) [21].

$$d_s = 2\frac{\ln(n^2 - m^2)}{\ln(n^2 + m^2)} \tag{10}$$

which provides good estimates for the standard Sierpinski folder. Consequently, the standard Sierpinski folder is characterized by Eq. (11).

$$D_{tH} < d_s < D_H < d = 2 \tag{11}$$

In this regard, it is pertinent to note that the rapid convergence of Z_k to Z_∞ allowing us to observe fractal effects in Sierpiski pre-fractal carpets after a few iterations (see, for example, Refs. [13, 21-28]).

3. NUMERICAL SIMULATIONS OF THE SYSTEM

To simulate the system, the pre-fractal Sierpinski carpet S_3^1 was changed in an interval of 1<N<567 nodes randomly distributed over the entire network in such a way that each node represents an agent which only interacts with its closest neighbors, while that said node is soon assigned any of two states, considering that the nodes with a third state are distributed in the rest of the population; a cycle is considered as the process carried out by N interactions between agents.

This system is based on a non-consensus NCO model that shows new states in which the stable relationship of a reduction in agents produces percolation of majority opinions. This stable state is reached from an initial random configuration after the application of a dynamic process according to the increase in the number of agents in a relatively short time.

It is intended to demonstrate that, when the population of an opinion is above a certain critical threshold, even a minority, a large group extension of a proportional size percolates to the total population. Using increasingly large simulations, it is shown that the phase transition in the NCO model belongs to the same universality class [29-31], since agents who have the same opinion form a group, where each member of the group will be part of a cluster holding its opinion, and that over time each cluster stabilizes at a percolating cluster saturation. The basic hypothesis of the proposed NCO model is that its formation is a process where an agent's opinion is influenced both by his own current opinion and by that of his nearby agents represented as nearest neighbors in the network [32].

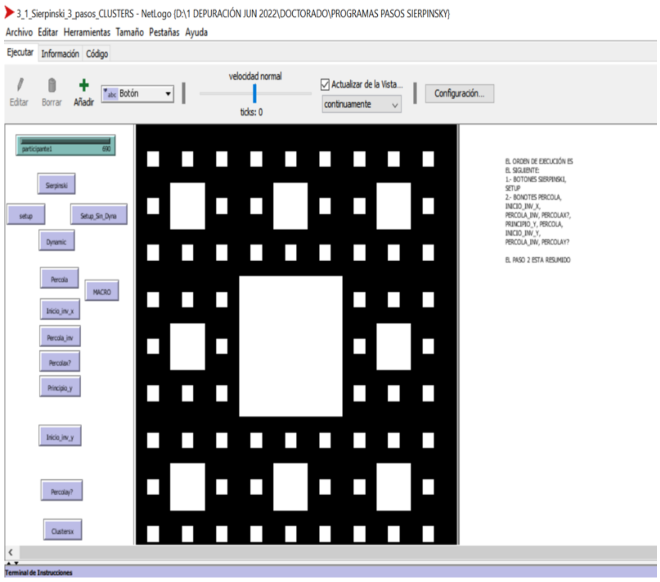

Figure 3. Sierpinski S31 folder design in Net logo

The system was developed in the Net logo Figure 3 simulator, carrying out for this purpose the design and programming of a two-dimensional Sierpinski S31 folder with N agents where the initial distribution of active Agents is always random, with active and non-active agents respectively. To which a dynamic is applied to analyze their behavior.

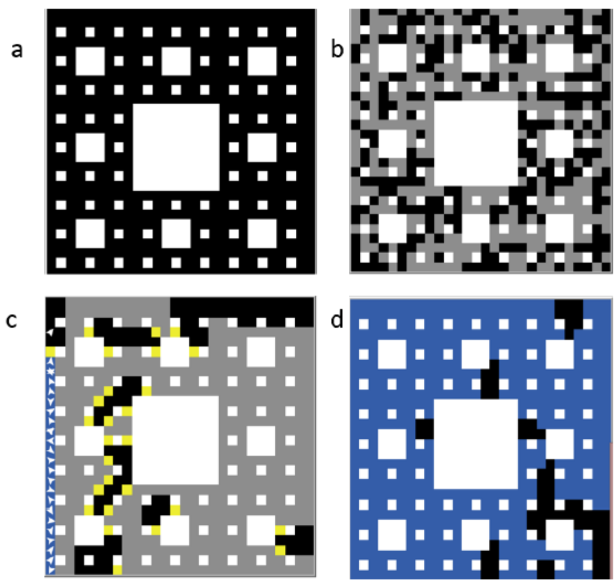

Figure 4. Evolution of the S_3^1 network at the percolation threshold with 375 participants a) Initial condition b) Application of dynamics c) Start of percolation d) Percolation of the network

The model was worked on in the Sierpinski carpet, its behavior analysis in the simulations was based on its borders to find the moment when the network reaches the percolation threshold, an average result of 1000 simulations is reported for each governed point. By the update rule of $1<a_i<500$ where a_i is the number of agents for each case.

Figure 4 illustrates the evolution of the simulation process for the particular case of a network S_3^1. In (a) the Sierpinski folder is shown in its initial position, in (b) the random spatial initial distribution is shown with 375 agents applying the dynamics to simulate the active and non-active agents, in (c) the moment in which the agents begin the initial behavior process at the network border, in (d) the final state of the configuration is shown when the percolation threshold is reached.

4. BEHAVIOR OF THE SYSTEM IN A STABLE OPINION DYNAMIC

The analysis of the distribution in the size of the cluster is very important since it is a very powerful tool in the investigation of percolation transitions. The simulations we perform take us to a constant scale of enlargement associated with a valid percolation transition under fractal dimensions since we are interested in the effect of fractal attributes on the filtration threshold in the Sierpinski carpet and universal checking as seen in the universal formula Eq. (12).

$$p_c = p_0 d^b q^{-a} \tag{12}$$

The results support the assumption for the distribution size distribution of the cluster, provides a correction scale that must be taken into account in the discussion of the scale functions [33], so to generalize the Eq. (1) in fractal networks, we need to estimate percolation thresholds in Sierpinski carpets with similar accuracy. It is interesting to note that there is no fixed percolation point on the Sierpinski carpet, so the average percolation number is given by Eq. (13).

$$p_A = p_c + \lambda L_k^{-\omega} \tag{13}$$

Which increases with the increase in the size of the folder Lk = nDH /k. Consequently, one can define an apparent percolation at the pA threshold (Zk) that is expected to decrease with the number of iteration steps k as [33], Eq. (14).

$$p_A = p_c + \lambda L_k^{-\omega} \tag{14}$$

where, p_C is the percolation threshold in is the fractal limit of $k \to \infty$, λ is the constant, while the scale exponent is found to be $\omega > 1/v$ [33]. And where the average coordination number converges to $Z\infty$ exponentially where the dependency size (10) becomes negligible with respect to the accuracy required after a few iterations.

On the other hand, it is found that for percolation in sierpinski's prefactal folder S_m^n with $3 \leq k \leq 6$ iterations (L_6=567 and Z_∞-Z_6<0.0089) [33]. This gives us the foundation that allows us to use Finite Size Scaling Analysis to estimate critical exponents and percolation threshold p_C with the required accuracy.

In the simulations, our amplitude search algorithm was determined by the spaces of the folders that were occupied one by one in random order starting with the empty state. The sites occupied in the transition formed contiguous clusters in the sense of with the first neighboring connectivity which determined a percolation pc=0.758 that provides a clear confirmation of a branch of the exponential function [34].

Based on this we can define S as the size of the cluster as a number of sites in it. On the other hand, the cluster expansion interval is defined as the cluster spanning the L_k size of the folder through either the horizontal or vertical directions, or both. [35, 36]. We calculate the interval expansion of probability as Eq. (15).

$$R_L(p) = 0.5\,(R_L^h + R_L^V) \tag{15}$$

where, $R_L^h\ y\ R_L^V$ are the probabilities that the folder cluster expansion intervals along the horizontal and vertical directions, respectively (See Figure 5).

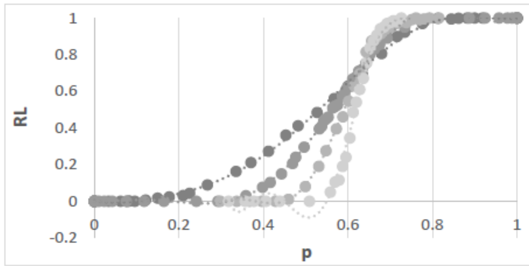

Figure 5. Probability of branching R_L compared to probability p for a Sierpinski carpet S_3^1 for 3<k<5

The limit of L=∞, the probability of expansion $R_L(p)$ is a step of the function that jumps from R_L ($p<pC$)=0 to $R_L(p>pC)$=1 that moves from close to zero to places close to one, where the theory of re-normalization defines the critical point as a fixed point RL (p_C)=p_C [37]. As the size of the limits depends on the grid of the finite size network where p_c(L) converges to p_C as Eq. (16).

$$|p_c(L) - p_c| \propto L^{-1/v} \tag{16}$$

where, v is the critical exponent that governs the correlation length ξ∝|p-pC|−ν [37]. A equality $R_L(pc)$=pc can fail due to system conditions however it can be calculated by limit pa (c, L) defined as Eq. (17).

$$R_L(P_A) = C \tag{17}$$

For 0<c<1 [37]. Where the dependence of the network size on the apparent critical concentration can be adjusted as Eq. (18).

$$p_c(L, C) = p_c + C_1 L^{-1/v} + C_2 L^{-2/v} \tag{18}$$

where, $c_1(C)$ and $c_2(C)$ are adjustment parameters (see Ref. [37]). Therefore, the probability expansion intevalos that are covered in the pre-fractal folders S_m^n (k) with sufficiently large k can be well collapsed into a single curve at coordinates, Eq. (19). (See Figure 6).

$$(R_L, X) = [p - p_a(k, C)]L^{1-v} \tag{19}$$

while, $pa(k, C)$ is defined as $R_L(P_A)$=C. After running the simulations, the values of $p_a(k, C)$ were determined for various values of C. We then used the collapsing probabilities of expansion.

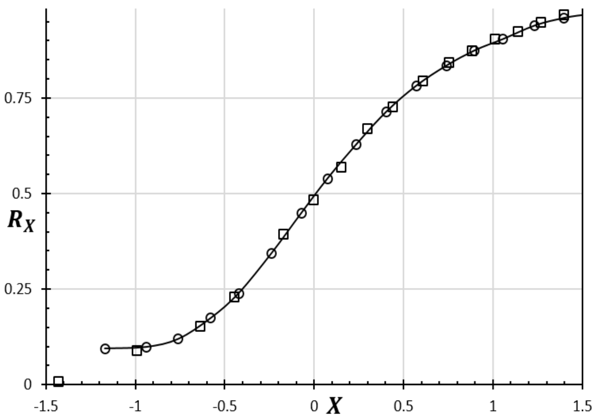

Figure 6. Collapse of R_L probability in a Sierpinski prefactal for pa defined by Eq. (19) with *C*=-0.01

5. DATA ANALYSIS AND RESULTS

Our findings suggest that starting from a random distribution of agents given initial conditions, and using stable opinion dynamics through numerical simulation to compute the percolation threshold and its critical exponents, the universality class at which that the model belongs to and how fractal features in an infinitely branched network affect its percolation properties.

For this system, the critical exponents, α, τ, were determined using the scaling method of the total number of clusters Eq. (20).

$$N_L p(c) \alpha L_k^{(2-\alpha)/v} \quad (20)$$

Such a relationship can be derived from the assumption that the scale for the cluster distribution size is usually written Eq. (21).

$$n_s(p) = s^{-\tau} f_{\pm}(s|p - p_c|^{1/\sigma}) \quad (21)$$

where, *f*± are scale functions associated with the two sides of the critical point, τ the Fisher exponent and σ another exponent [30]. From which it is seen that the cluster size distribution is Eq. (22).

$$n_s(p_c) \alpha s^{-\tau} \quad (22)$$

And the average size spanning the cluster is Eq. (23).

$$S(p_c) \alpha \sum s^2 \, n_s(p_c) \alpha L_k^{\gamma/v} \quad (23)$$

To make sure of the veracity between the numerical values of the critical exponents, we verify that the following relation Eq. (24).

$$\alpha + 2\beta + \gamma = 2 \quad (24)$$

Consequently, the percolation threshold p(c) and the critical exponents τ and γ are calculated using the following hyperscale relationships Eq. (25).

$$\gamma = D_{Hv} - 2v \quad (25)$$

And the Eq. (26).

$$D_c = D_{H} - \beta / v \quad (26)$$

From the above equations we can perform a calculation of the critical exponent τ is enough to confirm the hyperscale relationship [6] Eq. (27).

$$\tau = 1 + D_H / D_c \quad (27)$$

The values of obtained from the scaling behavior for a non-consensus network are reported in the Table 2.

The data found in the table for the Sierpinski S_n^m NO folder It does lead us to the fact that the irregular trellises and the percolation of bonds that occurs between them belongs to a kind of universality of random percolation, where the spatial dimension d was determined by the critical exponents, from which it is inferred that, if for two systems of the same spatial dimension d the critical exponents are different, then these systems belong to different kinds of universality [6, 31].

However, in their generalization the critical exponents in their universal form (Table 1) can also depend on other fundamental attributes of the system as can be seen in Table 2 and compared with the data reported in the references [37, 38] point out that the values of the critical exponents of random percolation are defined by the Dimensions (D_{tH}, D_H, d_s) that define the network, rather than just by *d*. Also, esot determines that the hyper dependence relations between the critical exponents found in Table 2 are governed by the Hausdorff dimension D_H.

Therefore, our findings indicate that percolation in infinitely branched Sierpinski carpets from the data found in Table 2, suggest that the site filtration thresholds of standard Sierpinski folders can be adjusted by generalized Eq. (1) if the dimension.

Table 1. Site percolation thresholds and critical exponents reported in the literature for three infinitely ramified Sierpinski carpets (Hausdorff carpets)

Property	Reference	Sierpinski carpet		
		S_3^1	S_4^2	S_5^3
D_H	Eq. (3)	Ln8/ln3	Ln12/ln4	Ln16/ln5
pc	[39]	0.85	0.92	0.95
	[40]	--	--	0.815
	[41]	0.6898	--	0.8273
	[42]	0.759	<0.864	--
	This work	0.758± 0.002	0.858± 0.004	0.914± 0.004
γ	[39]	2.13	2.43	2.69
	[40]	1.718	--	1.937
	[41]	2.194	--	--
	[42]	1.786	<2.5	--
	This work	1.790 ± 0.007	2.65± 0.004	2.40 +0.005
β	[39]	0.27	--	--
	[43]	0.234	--	--
	[42]	0.115	>0.066	--
	This work	0.114± 0.004	0.067± 0.003	0.001± 0.003
D_c	[43]	1.828	1.766	--
	This work	1.829	1.767	1.718

Table 2. Values obtained from the scaling behavior for a non-consensus network

	Square Lattice	WPLS	Sierpinski carpet				
			S_5^1	S_3^1	S_3^1 NC	S_4^2	S_5^3
D_H	2	2	1.9746…	1.8927…	1.8927…	1.7924…	1.7227…
D_{tH}	2	2	1.8613…	1.6309…	1.6309…	1.5	1.4306…
$Z\infty$	4	5.33	3.7894…	3.2	3.2	3	2.9090…
q_{tH}	3	4.33	2.402…	1.388	1.388	1	0.822
P_c	0.592745	0.527	0.634± 0.004	0.758± 0.002	0.758± 0.002	0.858± 0.004	0.914 ± 0.004
D_s	2	<2	1.951	1.806	1.806	1.659	1.572
D_H - D_{tH}	0	0	0.1133	0.2619	0.2619	0.2925	0.2920
γ	4/3	1.635	1.50 ± 0.01	1.790± 0.007	1.825 +0.007	2.65± 0.04	2.40 ± 0.05
β	5/36	0.222	0.134± 0.004	0.114± 0.004	0.195± 0.004	0.67± 0.003	0.011± 0.003
ϒ	43/18	2.825	2.69	3.16	3.062	4.6	4.1
α	-2/3	-1.27	-0.97	-1.4	-1.4536	-2.75	-2.1
τ	187/91	2.072	2.05 ± 0.05	2.03 ± 0.05	2.06± 0.05	2.01± 0.04	2.00± 0.02
D_c	91/48	1.864	1.885	1.829	1.7854	1.767	1.718

We also found that the data shown in Table 2. found in the simulations confirm that there are minimal differences between the p_C values and their critical exponents of the sierpisky S_n^m=folders. =0.758 and the folder S_n^mNO=0.858 obtained in numerical simulations and calculated with Eqns. (1), (24), and (25) and these are in the commonly accepted range for the application of approximate threshold by universal formulas (see Refs. [32, 33]). Indicating that the dimension number relevant to the percolation threshold is the Topological Hausdorff dimension.

This confirms our initial hypothesis that starting from a random distribution of agents with certain initial conditions, and using a stable opinion dynamic by means of numerical simulation to calculate the percolation threshold and its critical exponents, the universality class to which the model belongs is checked and how the fractal characteristics in an infinitely branched network affect its percolation properties.

6. CONCLUSIONS

We found that filtration thresholds and critical exponents were determined for standard Sierpinski folders S_n^m infinitely branched from a random distribution of agents given the initial conditions, and using stable opinion dynamics by numerical simulation to calculate the filtration threshold and its critical exponents meet the kind of universality for which the non-consensual opinion model belongs and validate the fractal characteristics in an infinitely branched network derived from its percolation properties.

It was also found that data from two-dimensional simulations and networks reported in the literature show that percolation in infinitely branched Sierpinski carpets suggest that Sierpinski folder site filtration thresholds infer that a universality exists. This was based on the fact that the percolation values of the component critical exponents immersed in a random percolation universality class are determined by the established dimensions (DH, DtH, ds), rather than solely by the spatial dimension (d). on the other hand it was inferred that the hyper scalability relationship between the critical exponents is governed by the Hausdorff dimension of the network. Therefore, we discuss that the percolation site in infinitely branched Sierpinski carpets (DtH, ds, DH) concerns the universality class of random percolation, and that the values of the critical exponents of percolation are functions of DtH, DH, and ds.

In summary it was confirmed that the hyperscalability relationship between the critical exponents is governed by the Hausdorff dimension of the network, thus validating the non-consensual opinion model using a percolation site in Sierpinski carpets in infinitely branched networks belonging to the universality class of random percolation comparing it with previously reported results, while the values of the critical exponents of percolation are functions of the aforementioned dimensions. So, in the context of the site's filtering threshold, these networks belong to the same kind of universality.

REFERENCES

[1] Galam, S. (1999). Application of statistical physics to politics. Physica A: Statistical Mechanics and its Applications, 274(1-2): 132-139. https://doi.org/10.1016/S0378-4371(99)00320-9

[2] Galam, S. (2004). The dynamics of minority opinions in democratic debate. Physica A: Statistical Mechanics and its Applications, 336(1-2): 56-62. https://doi.org/10.1016/j.physa.2004.01.010

[3] Broadbent, S.R., Hammersley, J.M. (1957). Percolation processes: I. Crystals and mazes. In Mathematical Proceedings of the Cambridge Philosophical Society, 53(3): 629-641. https://doi.org/10.1017/S0305004100032680

[4] Domb, C. (1959). Fluctuation phenomena and stochastic processes. Nature, 184(4685): 509-512.

[5] Essam, J.W. (1980). Percolation theory. Reports on Progress in Physics, 43(7): 833. https://doi.org/10.1088/0034-4885/43/7/001

[6] Stauffer, D., Aharony, A. (2018). Introduction to Percolation Theory. Taylor & Francis.

[7] Vogel, E.E., Lebrecht, W., Valdés, J.F. (2010). Bond percolation for homogeneous two-dimensional lattices. Physica A: Statistical Mechanics and its Applications, 389(8): 1512-1520. https://doi.org/10.1016/j.physa.2009.12.049

[8] Ziff, R.M., Newman, M.E.J. (2002). Convergence of threshold estimates for two-dimensional percolation. Physical Review E, 66(1): 016129. https://doi.org/10.1103/PhysRevE.66.016129

[9] Lebrecht, W. (2010). Umbrales de percolación exactos en redes duales. Revista Mexicana de fÍsica E, 56(2): 190-196.

[10] Galam, S., Mauger, A. (1996). Universal formulas for percolation thresholds. Phys. Rev. E, 53(1996): 2177-

2181. https://doi.org/10.1103/PhysRevE.53.2177
[11] van der Marck, S.C. (1997). Comment on "Universal formulas for percolation thresholds". Physical Review E, 55(1): 1228-1229. https://doi.org/10.1103/PhysRevE.55.1228
[12] Suding, P.N., Ziff, R.M. (1999). Site percolation thresholds for Archimedean lattices. Physical Review E, 60(1): 275-283. https://doi.org/10.1103/PhysRevE.60.275
[13] van der Marck, S.C. (1997). Percolation thresholds and universal formulas. Physical Review E, 55(2): 1514-1517. https://doi.org/10.1103/PhysRevE.55.1514
[14] Van der Marck, S.C. (1998). Calculation of percolation thresholds in high dimensions for FCC, BCC and diamond lattices. International Journal of Modern Physics C, 9(04): 529-540. https://doi.org/10.1142/S0129183198000431
[15] Kawamoto, H., Takayasu, H., Takayasu, M. (2017). Network anatomy controlling abrupt-like percolation transition. Scientific Reports, 7(1): 1-8. https://doi.org/10.1038/s41598-017-00242-4
[16] Gefen, Y., Aharony, A., Mandelbrot, B.B. (1984). Phase transitions on fractals. III. Infinitely ramified lattices. Journal of Physics A: Mathematical and General, 17(6): 1277-1289. https://doi.org/10.1088/0305-4470/17/6/024
[17] Balka, R., Buczolich, Z., Elekes, M. (2015). A new fractal dimension: The topological Hausdorff dimension. Advances in Mathematics, 274: 881-927. https://doi.org/10.1016/j.aim.2015.02.00
[18] Torquato, S., Jiao, Y. (2013). Effect of dimensionality on the percolation threshold of overlapping nonspherical hyperparticles. Physical Review E, 87(2): 022111. https://doi.org/10.1103/PhysRevE.87.032149
[19] McMullen, C. (1984). The Hausdorff dimension of general Sierpiński carpets. Nagoya Mathematical Journal, 96: 1-9. https://doi.org/10.1017/S0027763000021085
[20] Hsiao, P.Y., Monceau, P., Perreau, M. (2000). Magnetic critical behavior of fractals in dimensions between 2 and 3. Phys. Rev. E, 62(2000): 13856. https://doi.org/10.1103/PhysRevB.62.13856
[21] Balankin, A.S. (2017). The topological Hausdorff dimension and transport properties of Sierpiński carpets. Physics Letters A, 381(34): 2801-2808. https://doi.org/10.1016/j.physleta.2017.06.049
[22] Galam, S., Mauger, A. (1997). Universal formulas for percolation thresholds. II. Extension to anisotropic and aperiodic lattices. Physical Review E, 56(1): 322-325. https://doi.org/10.1103/PhysRevE.56.322
[23] Wierman, J.C. (2002). Accuracy of universal formulas for percolation thresholds based on dimension and coordination number. Physical Review E, 66(2): 027105. https://doi.org/10.1103/PhysRevE.66.027105
[24] Galam, S., Mauger, A. (2005). Possible crossover of a nonuniversal quantity at the upper critical dimension. Physical Review E, 71(3): 036136. https://doi.org/10.1103/PhysRevE.71.036136
[25] Wierman, J.C., Naor, D.P., Smalletz, J. (2007). Incorporating variability into an approximation formula for bond percolation thresholds of planar periodic lattices. Physical Review E, 75(1): 011114. https://doi.org/10.1103/PhysRevE.75.011114
[26] Neher, R.A., Mecke, K., Wagner, H. (2008). Topological estimation of percolation thresholds. Journal of Statistical Mechanics: Theory and Experiment, 2008(01): P01011. https://doi.org/10.1088/1742-5468/2008/01/P01011
[27] Rahman, M.M., Hassan, M.K. (2017). Explosive percolation on a scale-free multifractal weighted planar stochastic lattice. Physical Review E, 95(4): 042133. https://doi.org/10.1103/PhysRevE.95.042133
[28] Newman, M.E.J., Ziff, R.M. (2000). Efficient Monte Carlo algorithm and high-precision results for percolation. Physical Review Letters, 85(19): 4104. https://doi.org/10.1103/PhysRevLett.85.4104
[29] Kosior, A., Sacha, K. (2017). Localization in random fractal lattices. Physical Review B, 95(10): 104206. https://doi.org/10.1103/PhysRevB.62.13856
[30] Monceau, P., Hsiao, P.Y. (2004). Percolation transition in fractal dimensions. Physics Letters A, 332(3-4): 310-319. https://doi.org/10.1016/j.physleta.2004.09.068
[31] Zhu, Y., Yang, Z.Q., Zhang, X., Chen, X.S. (2015). Critical behaviors and universality classes of percolation phase transitions on two-dimensional square lattice. Communications in Theoretical Physics, 64(2): 231-236. https://doi.org/10.1088/0253-6102/64/2/231
[32] Galam, S., Mauger, A. (1998). Topology invariance in percolation thresholds. The European Physical Journal B-Condensed Matter and Complex Systems, 1(2): 255-258. https://doi.org/10.1007/s100510050
[33] Torquato, S., Jiao, Y. (2013). Effect of dimensionality on the percolation thresholds of various d-dimensional lattices. Physical Review E, 87: 032149. https://doi.org/10.1103/PhysRevE.87.032149
[34] Hassan, M.K., Rahman, M.M. (2015). Percolation on a multifractal scale-free planar stochastic lattice and its universality class. Physical Review E, 92(4): 040101. https://doi.org/10.1103/PhysRevE.92.040101
[35] Ziff, R.M., Newman, M.E.J. (2002). Convergence of threshold estimates for two-dimensional percolation. Physical Review E, 66(1): 016129. https://doi.org/10.1103/PhysRevE.66.016129
[36] Yang, H. (2012). Alternative criterion for two-dimensional wrapping percolation. Physical Review E, 85(4): 042106. https://doi.org/10.1103/PhysRevE.85.042106
[37] Lee, M.J. (2008). Pseudo-random-number generators and the square site percolation threshold. Physical Review E, 78(3): 031131. https://doi.org/10.1103/PhysRevE.78.031131
[38] Monseau Hsiao, P.Y. (2004). Percolation transition in fractal dimensions. Physics Letters A, 332(3-4): 310-319. https://doi.org/10.1016/j.physleta.2004.09.068
[39] Ben-Avraham, D., Havlin, S., Movshovitz, D. (1984). Infinitely ramified fractal lattices and percolation. Philosophical Magazine B, 50(2): 297-306. https://doi.org/10.108013642818408238847
[40] Yu, B.M., Yao, K.L. (1988). Critical percolation probabilities for site problems on Sierpinski carpets. Zeitschrift für Physik B Condensed Matter, 70(2): 209-212. https://doi.org/10.1007/BF01318301
[41] Lin, Z.Q., Yang, Z.R. (1997). Thresholds and Universality of the Site Percolation on the Sierpinski Carpets1. Communications in Theoretical Physics, 27(2): 145. https://doi.org/10.1088/0253-6102/27/2/145
[42] Shinoda, M. (2003). Non-existence of phase transition of oriented percolation on Sierpinski carpet lattices. Probability Theory and Related Fields, 125(3): 447-456. https://doi.org/10.1007/s00440-002-0247-x

[43] Herega, A.N., Drik, N.G., Ugol'nikov, A.P. (2012). Hybrid ramified Sierpinski carpet: percolation transition, critical exponents, and force field. Physics-Uspekhi, 55(5): 519. https://doi.org/10.3367/UFNr.0182.201205f.0555

NOMENCLATURE

m	Number of copies of similarity
n	Magnification factor
DH	Dimensions of Hausdorff
DtH	Dimensions topological Hausdorff
ds	Spectral dimension
Pc	Percolation threshold
Z_∞	Average coordination number
Q	Topological constraint
Z_K	Number of iteration steps k
$\alpha, \beta, \gamma, \tau$	Critical exponent