IIETA International Information and Engineering Technology Association
Advancing the World of Information and Engineering

Mathematical Modelling of Engineering Problems
Vol. 9, No. 4, August, 2022, pp. 1043-1052
Journal homepage: http://iieta.org/journals/mmep

Picture Fuzzy Choquet Integral Based Geometric Aggregation Operators and Its Application to Multi Attribute Decision-Making

Krishnan Deva, Swaminathan Mohanaselvi*

Department of Mathematics, College of Engineering and Technology, SRM Institute of Science and Technology, Kattankulathur 603203, Tamil Nadu, India

Corresponding Author Email: mohanass@srmist.edu.in

https://doi.org/10.18280/mmep.090422

Received: 1 July 2022
Accepted: 16 August 2022

Keywords:
aggregation operators, Choquet integral, fuzzy measure, multi attribute decision-making problem, picture fuzzy set

ABSTRACT

Picture fuzzy set is an extension of intuitionistic fuzzy set, which allows interpreting uncertain data in decision-making problems. This study provides aggregation operators based on Choquet integral, namely Choquet integral picture fuzzy geometric aggregation (CIPFGA) operator and Choquet integral picture fuzzy hybrid geometric aggregation (CIPFHGA) operator with certain properties of these operators are established. We validate the functioning of the operators with illustrative examples. The CIPFHGA operator has an added benefit of combining the weights of positions along with capturing the comprehensive correlative relationships of the criteria. Further, the proposed operators enable us to solve a numerical problem in picture fuzzy Multi attribute decision-making problem and also allows to make a comparison study with the existing literature.

1. INTRODUCTION

Multi attribute decision-making (MADM) is the process of selecting the most suitable alternative from among a number of available alternatives. Zadeh [1] introduced the main concept of fuzzy set (FS) theory in 1965 and examined the degree of membership of an object in a set, which ranges from zero to one. Later, Intuitionistic fuzzy set (IFS) was established by Atanassov [2] in 1986 as an extension of FS by including the degree of non-membership as well as the degree of membership to each object in a set and demonstrating various properties associated with operations and relations over sets. If an expert obtains an opinion from a certain person regarding any object, that person may indicate that 0.5 is the likelihood that the statement is true, 0.3 is the possibility that the statement is false and 0.2 is that he or she is not sure. As a result, the role of neutrality is not handled by FSs or IFSs but must be addressed in real-life decision making. Cuong [3, 4] proposed a picture fuzzy set (PFS) and investigated certain essential operations and properties. The PFS is influenced by the degree of membership, the degree of neutral membership and the degree of non-membership. The only requirement is that the sum of the three degrees be less than or equal to one. Essentially, PFS based models may be applied to situations requiring human opinions involving more sorts of answers: yes, abstain, no, refuse, which cannot be adequately described in FS and IFS. Dutta et al. [5] defined (α, δ, β)-cuts under picture fuzzy (PF) environment. Many researchers introduced PFSs into the MADM field to express fuzzy evaluations due to their high performance in dealing fuzzy information.

Aggregation operators (AOs) are especially important in the decision-making (DM) process since they combine all of the provided individual assessment values into a single form. Yager [6] explored the features of an ordered weighted AOs and described a new form of AOs. Cuong et al. [7] investigated picture t-norm and picture t-conorms and its properties for PFSs. Wei [8] described various PF Weighted average (WA) and Weighted geometric (WG) operators as well as PF hybrid aggregators and their applications. Wang et al. [9] studied MADM problems utilising some PF geometric operators. Garg [10] demonstrated a series of AOs for the PFSs that were used to solve the multi criteria decision-making problem. Wei [11] used arithmetic and geometric AOs, Hamacher operations with PF environment to study the MADM problem. Jana et al. [12] developed PF Dombi arithmetic and geometric AOs are used to solve MADM problem. Zhang et al. [13] developed new PF operational laws based on Dombi t-norm and t-conorm and utilised Heronian mean (HM) information AO to combine picture fuzzy numbers (PFN) and the suggested operators not only combine individual attribute values but they are also able to model the general correlation between attributes. Wang et al. [14] used a case study regarding financial investment risk to provide various picture fuzzy aggregation operators (PFAO) that were based on the classic Muirhead mean (MM) operators and developed the MADM approach. Xu et al. [15] proposed a family of MM operators with PF environment and its properties are investigated. Jana et al. [16] developed MADM approaches for enterprise performance evaluation using PF Hamacher AOs. Khan et al. [17] suggested an AOs based on PF Einstein operations for the MADM problem using PF environment. Qiyas et al. [18] developed some AO based on the idea of yager operators with PF environment based on Archimedean t-norm and t-conorm. Qin et al. [19] proposed a novel MADM approach that uses a set of Archimedean power Maclaurin symmetric mean (MSM) operators of PFNs. Tapan Senapati [20] introduced the AOs of PFNs and a few AOs mainly PF Aczel-Alsine average AOs as well as these operators to develop a method for solving MADM in a PF environment. Table 1 is a list of the AOs of PFNs on which these approaches are based.

Table 1. Existing aggregation operators of PFNs

Authors	Aggregation operators of PFNs	Operation
Wei [8]	WA, WG, Ordered WA, Ordered WG, Hybrid average (HA) and Hybrid geometric (HG) operators.	Algebraic
Garg [10]	WA, Ordered WA and HA operators.	Archimedean
Wei [11]	HamacherWA, Hamacher Ordered WA, HamacherHA, HamacherWG, Hamacher Ordered WG, HamacherHG, Hamacher Correlated averaging (CA) and Hamacher Correlated geometric (CG), Induced Hamacher Ordered WA, Induced Hamacher: Ordered WG, Induced Hamacher CA and Induced Hamacher CG, Hamacher weighted prioritised average, prioritised geometric, power average and power geometric operators.	Hamacher
Jana et al. [12]	Dombi WA, Ordered WA, HA, WG, Ordered WG, HG operators.	Dombi
Zhang et al. [13]	Dombi weighted Heronian mean, Dombi weighted dual Heronian mean operators.	Dombi
Xu et al. [15]	Weighted Muirhead mean, Weighted dual Muirhead mean operators.	Algebraic
Jana et al. [16]	Hamacher WA and WG operators.	Hamacher
khan et al. [17]	Einstein WA and Ordered WA operators.	Einstein
Qiyas et al. [18]	Yager WA, Ordered WA, Hybrid WA, WG, Ordered WG, Hybrid WG operators.	Yager
Qin et al. [19]	Archimedean power MSM and Archimedean power weighted MSM operators.	Archimedean
Senapati [20]	Aczel–Alsina WA, Ordered WA and HA operators.	Aczel–Alsina

The above mentioned PFAOs are initiated within a presumption: The criteria are independent. In most of the MADM problems which are dependent on the contrary, they are correlative. In such cases, the important degrees of criteria are given as fuzzy measures (FM) rather than weights, because weighted AOs cannot be aggregated. FMs can well reflect the correlative relationships between criteria sets, such as redundant, independent, and complimentary. In reality, FMs are extensions of weights due to the fact that the sum of all FMs of criteria set can exceed one, whereas the FM of the entire set_is_restricted to one. Consequently, the FMs are more flexible. Choquet integral is an efficient method for aggregating ranking based on correlated criteria. To handle multi-criteria group decision making (MCGDM) problems where the criteria and expert opinions frequently involve inter-dependent or interaction phenomena among criteria. Tan et al. [21] investigated the generalised IF ordered geometric averaging operator. Zhang et al. [22] developed the Einstein interval IF Choquet geometric operator and investigated the relationship with the IF Choquet geometric operator. Singh et al. [23] described Choquet averaging and geometric mean operators for PFSs and also presented a VIKOR for MCGDM problems based on PF Choquet integrals. IF arithmetic and hybrid arithmetic AOs are proposed by Jia and Wang [24] based on Choquet integral as well as proposed operator computes the correlative criteria and combines the weight of position.

In MADM problems, the decision-making criteria may be dependent or independent. FMs were first presented by Sugeno [25] in 1974 to model interactions between decision criteria. The aforementioned AOs, which are unable to properly handle the specific conditions with correlative criteria and weighted positions, we define an AOs as the Choquet integral picture fuzzy geometric aggregation (CIPFGA) operator. After that, we combine the FMs of the criteria and the weights of the positions to construct the Choquet integral picture fuzzy hybrid geometric aggregation (CIPFHGA) operator.

The summary of the motivations of this article are outlined as follows:

- When compared to weighted AOs, the CIPFGA operator result can indicate the connections between the criteria.
- In comparison to the Choquet integral based AOs (Singh et al. [23]), the CIPFGA operator result may not only reflect the comprehensive connections of each criteria, but also reflect the unique relevance of each criteria.
- When compared to existing AOs, the CIPFHGA operator results can reflect weights of position. which are listed in Table 2.

The summary of the contributions of this article are outlined as follows:

- It has been demonstrated that the CIPFGA operator can effectively combine PFNs with correlative criteria and it has several properties that have been proved.
- The CIPFHGA operator is described to aggregate PFNs using a combination of FMs and weights position, with various properties are proved.
- To solve decision-making challenges in the PF environment, a MADM approach based on the CIPFHGA operator is provided.

Table 2. A comparison of the existing operators and the proposed operator

	Weight of the criteria	Fuzzy measure and weight of the criteria	Correlative relationships among criterion sets
Wei [8]	√	×	×
Wang et al. [9]	√	×	×
Garg [10]	√	×	×
Jana et al. [16]	√	×	×
Khan et al. [17]	√	×	×
Qiyas et al. [18]	√	×	×
Proposed operator	√	√	√

From Table 2, existing AOs [8-10, 16-18] are not dealing with FMs and correlative relationships among criteria sets, whereas the proposed CIPFHGA operator completely deals with them.

The present article is structured in a comprehensive manner into seven detailed sections: In Section 2, we briefly review the concepts with respect to the PFSs and the fuzzy measures. In Section 3, the CIPFGA and CIPFHGA operators are defined in detail, along with their properties. Section 4, a MADM approach that relies on the CIPFHGA operator is proposed. Section 5, applying the proposed MADM approach to the numerical problem to demonstrate its practicality. Section 6, Comparative analyses are conducted to demonstrate the advantages of our suggested MADM approach over existing aggregation operators. Conclusions and potential study directions are discussed in the final section.

2. PRELIMINARIES

This section provides the definitions required to support the proposed research work.

2.1 Picture fuzzy set

Cuong [3, 4] developed the PFS as an extension of the IFS. PFS is represented mathematically as follows:

Definition 1: A picture Fuzzy "sets A on universal set X is defined by:

$$\mathrm{A}=\left\{\left\langle x, \mu_{\mathrm{A}}(x), \eta_{\mathrm{A}}(x), \nu_{\mathrm{A}}(x)\right\rangle \mid x \in X\right\}$$

where, $\mu_A(x)$, $\eta_A(x)$ and $\nu_A(x) \in [0,1]$ are the degree of membership, the degree of neutral membership and the degree of non-membership of $x \in A$ respectively, with the following condition: $0 \leq \mu_A(x)+\eta_A(x)+\nu_A(x) \leq 1$ $\forall x \in X$. Then for $x \in X$, $\pi_A(x)=1-\mu_A(x)-\eta_A(x)-\nu_A(x)$ could be called the degree of refusal membership of x in A. For convenience, let $\alpha=(\mu_\alpha, \eta_\alpha, \nu_\alpha)$ is called a PFN.

2.2 Comparison for picture fuzzy numbers

According to Garg [10] the score and accuracy functions are as follows:

Definition 2: Let $\alpha=(\mu_\alpha, \eta_\alpha, \nu_\alpha)$ be a PFN and its score function $S(\alpha)$ and its accuracy function $A(\alpha)$ is defined by:

$$S(\alpha)=\mu_\alpha-\eta_\alpha-\nu_\alpha; \quad S(\alpha) \in [-1,1],$$
$$A(\alpha)=\mu_\alpha+\eta_\alpha+\nu_\alpha; \; A(\alpha) \in [0,1].$$

Based on the $S(\alpha)$ and $A(\alpha)$an order relationship between two PFNs is defined as follows.

Definition 3: Let $\alpha_1=(\mu_{\alpha_1}, \eta_{\alpha_1}, \nu_{\alpha_1})$ and $\alpha_2=(\mu_{\alpha_2}, \eta_{\alpha_2}, \nu_{\alpha_2})$ are two PFNs, the following is their order using the score and accuracy functions:

(1) If $S(\alpha_1) > S(\alpha_2)$ then $\alpha_1 > \alpha_2$;

(2) If $S(\alpha_1) = S(\alpha_2)$ then

(a) If $A(\alpha_1) > A(\alpha_2)$ then $\alpha_1 > \alpha_2$;

(b) If $A(\alpha_1) = A(\alpha_2)$ then $\alpha_1 = \alpha_2$.

2.3 Operation laws of picture fuzzy numbers

Wang et al. [9] following operational laws are defined for PFNs.

Definition 4: Le $\alpha_1=(\mu_{\alpha_1}, \eta_{\alpha_1}, \nu_{\alpha_1})$ and $\alpha_2=(\mu_{\alpha_2}, \eta_{\alpha_2}, \nu_{\alpha_2})$ are two PFNs and λ>0. Then PFN operations can be defined as follows:

$$\alpha_1 \otimes \alpha_2=\left\langle \begin{array}{c}\left(\mu_{\alpha_1}+\eta_{\alpha_1}\right)\left(\mu_{\alpha_2}+\eta_{\alpha_2}\right)-\eta_{\alpha_1}\eta_{\alpha_2}, \eta_{\alpha_1}\eta_{\alpha_2}, \\ 1-\left(1-\nu_{\alpha_1}\right)\left(1-\nu_{\alpha_2}\right)\end{array}\right\rangle$$
$$\alpha_1^{\lambda}=\left\langle\left(\mu_{\alpha_1}+\eta_{\alpha_1}\right)^{\lambda}-\eta_{\alpha_1}^{\lambda}, \eta_{\alpha_1}^{\lambda}, 1-\left(1-\nu_{\alpha_1}\right)^{\lambda}\right\rangle.$$

These two operational laws are also satisfying the following properties under condition $\lambda, \lambda_1, \lambda_2 > 0$.

(1) $\alpha_1 \otimes \alpha_2 = \alpha_2 \otimes \alpha_1$

(2) $(\alpha_1 \otimes \alpha_2)^{\lambda} = \alpha_1{}^{\lambda} \otimes \alpha_2{}^{\lambda}$

(3) $\alpha_1{}^{\lambda_1} \otimes \alpha_1{}^{\lambda_2} = \alpha_1{}^{\lambda_1+\lambda_2}$.

2.4 Fuzzy measure and existing PFCGM operator

Analyzing the relationships between a set of criteria is accomplished using FMs. Unlike weights, which only represent the importance of a single criteria, FMs properly reflect the importance of a collection of criteria. Fuzzy measure Sugeno [25] and picture fuzzy Choquet geometric mean (PFCGM) operator Singh et al. [23] are discussed in this section.

Definition 5: Sugeno et al. [25] Let $S=\{S_1, S_2, ..., S_n\}$ be an universe of discourse, a set function ϑ: $P(S) \rightarrow [0, 1]$ is a FM on S, if satisfying the following conditions:

(1) $\vartheta(\phi)=0, \vartheta(S)=1$;

(2) If $R_1, R_2 \subseteq P(S)$ and $R_1 \subseteq R_2$, then $\vartheta(R_1) \leq \vartheta(R_2)$.

Sugeno [25] suggested a λ-fuzzy measure, which may be described as follows:

$$\vartheta(R_1 \cup R_2)=\vartheta(R_1)+\vartheta(R_2)+\lambda \cdot \vartheta(R_1)\vartheta(R_2)$$
$$\forall R_1, R_2 \subseteq P(S) \text{ and } R_1 \cap R_2=\phi, \; \lambda \geq -1$$

λ diagnoses the relationship between the criteria in this case. For different values of λ, the λ- FM reduces to various measures. If $\lambda = 0$, then it becomes simply additive measure, if $\lambda > 0$ then it becomes to super-additive measure, and if $\lambda < 0$ then it becomes to sub additive measure.

$$\vartheta(S)=\left(\begin{array}{ll}\frac{1}{\lambda}\left(\prod_{i=1}^{n}\left(1+\lambda \cdot \vartheta(S_i)\right)-1\right) & if\ \lambda \neq 0 \\ \sum_{i=1}^{n}\vartheta(S_i) & if\ \lambda=0\end{array}\right) \tag{1}$$

Eq. (1), boundary condition $\vartheta(S_i)$ = 1 can be used to find parameter λ, following Eq. (2) is similar to solve Eq. (1).

$$\lambda+1=\left(\prod_{i=1}^{n}\left(1+\lambda\vartheta(S_i)\right)\right) \tag{2}$$

Definition 6: Singh et al. [23] Let $\alpha_j=(\mu_{\alpha_j}, \eta_{\alpha_j}, \nu_{\alpha_j})$ (j=1, 2, ..., n) and ϑ be a family of PFNs and fuzzy measure on S. The PFCGM operator of α_i with respects to ϑ is defined as follows:

$$PFCGM(\alpha_1, \alpha_2, ..., \alpha_n) = \left\langle \begin{array}{c}\prod_{j=1}^{n}\left(\mu_{\alpha_{(j)}}\right)^{\vartheta(R_{(j)})-\vartheta(R_{(j+1)})}, 1-\prod_{j=1}^{n}\left(1-\eta_{\alpha_{(j)}}\right)^{\vartheta(R_{(j)})-\vartheta(R_{(j+1)})}, \\ 1-\prod_{j=1}^{n}\left(1-\nu_{\alpha_{(j)}}\right)^{\vartheta(R_{(j)})-\vartheta(R_{(j+1)})}\end{array}\right\rangle$$

where, (j) denotes a permutation on C such that $\alpha_{(j)} \leq \alpha_{(j+1)}$, j = 1, 2, ⋯, n-1 and $R_{(j)} = \{S_{(j)}, S_{(j+1)}, \cdots, S_{(n)}\}$ with $R_{(n+1)} = \phi$.

3. PICTURE FUZZY CHOQUET INTEGRAL OPERATORS AND SOME OF ITS PROPERTIESS

The Choquet integral based CIPFGA operator is defined in this section and it is combined with the FMs of criteria and the weights of positions to define the CIPFHGA operator.

3.1 CIPFGA operator

Wei [8] and Garg [10] presented certain AOs to aggregate PF environment for MADM problems under the premise of independent criteria. In this part, we propose the Choquet integral operator for the PF environment, which provides for the interaction phenomena between the criteria represented by FM. The CIPFGA operator is defined as follows.

Definition 7: Let $\alpha_j = \left\langle \mu_{\alpha_j}, \eta_{\alpha_j}, \nu_{\alpha_j} \right\rangle (j = 1,2,\dots,n)$ and ϑ be a family of PFNs and FM on S. The discrete choquet integral CIPFGA operator of α_j with respect to ϑ is define as

$$CIPFGA(\alpha_1,\alpha_2,...,\alpha_n) = \alpha_1 \frac{\sum_{R_1 \subseteq P_1(C)} \vartheta(R_1 \cup \{S\}) - \vartheta(R_1)}{\sum_{j=1}^{n} \sum_{R_j \subseteq P_j(C)} \vartheta(R_j \cup \{S_j\}) - \vartheta(R_j)} \oplus \alpha_2 \frac{\sum_{R_2 \subseteq P_2(C)} \vartheta(R_2 \cup \{S_2\}) - \vartheta(R_2)}{\sum_{j=1}^{n} \sum_{R_j \subseteq P_j(C)} \vartheta(R_j \cup \{S_j\}) - \vartheta(R_j)} \oplus \cdots \oplus \alpha_n \frac{\sum_{R_n \subseteq P_n(C)} \vartheta(R_n \cup \{S_n\}) - \vartheta(R_n)}{\sum_{j=1}^{n} \sum_{R_j \subseteq P_j(C)} \vartheta(R_j \cup \{S_j\}) - \vartheta(R_j)} \quad (3)$$

where, $R_j \subseteq P_j(S), P_j(S) = P(S\{S_j\})$ Indicates the power of S. For convenience, let $\beta_j = \frac{\sum_{R_j \subseteq P_j(C)} \vartheta(R_j \cup \{S_j\}) - \vartheta(R_j)}{\sum_{j=1}^{n} \sum_{R_j \subseteq P_j(C)} \vartheta(R_j \cup \{S_j\}) - \vartheta(R_j)}$, then Eq. (3) can be simplified as $CIPFGA(\alpha_1,\alpha_2,...,\alpha_n) = \otimes_{j=1}^{n} \alpha_j^{\beta_j}$.

Theorem 1: Let $\alpha_j = \left\langle \mu_{\alpha_j}, \eta_{\alpha_j}, \nu_{\alpha_j} \right\rangle (j = 1,2,\dots,n)$ and ϑ be a family of PFNs and FM on S. Using the CIPFGA operator, their aggregated value is also a PFN and

$$CIPFGA(\alpha_1,\alpha_2,\cdots,\alpha_n) = \left\langle \begin{array}{l} \prod_{j=1}^{n} (\mu_{\alpha_j} + \eta_{\alpha_j})^{\beta_j} - \prod_{j=1}^{n} (\eta_{\alpha_j})^{\beta_j}, \\ \prod_{j=1}^{n} (\eta_{\alpha_j})^{\beta_j}, 1 - \prod_{j=1}^{n} (1 - \nu_{\alpha_j})^{\beta_j} \end{array} \right\rangle \quad (4)$$

where, $\beta_j = \frac{\sum_{B_j \subseteq P_j(C)} \vartheta(R_j \cup \{S_j\}) - \vartheta(R_j)}{\sum_{j=1}^{n} \sum_{R_j \subseteq P_j(C)} \vartheta(R_j \cup \{S_j\}) - \vartheta(R_j)}$, $R_j \subseteq P_j(S), P_j(S) = P(S\{S_j\})$.

Proof

From the definition (4), the first results come quickly Below, we use mathematical induction on n to show that Eq. (4).

When n=2, since:

$$\alpha_1^{\beta_1} = \left\langle \left(\mu_{\alpha_1} + \eta_{\alpha_1}\right)^{\beta_1} - \eta_{\alpha_1}^{\beta_1}, \eta_{\alpha_1}^{\beta_1}, 1 - \left(1 - \nu_{\alpha_1}\right)^{\beta_1} \right\rangle$$

$$\alpha_2^{\beta_2} = \left\langle \left(\mu_{\alpha_2} + \eta_{\alpha_2}\right)^{\beta_2} - \eta_{\alpha_2}^{\beta_2}, \eta_{\alpha_2}^{\beta_2}, 1 - \left(1 - \nu_{\alpha_2}\right)^{\beta_2} \right\rangle$$

Then, $CIPFGA(\alpha_1, \alpha_2, \dots, \alpha_n) = \alpha_1^{\beta_1} \otimes \alpha_2^{\beta_2} = \left\langle \begin{array}{l} (\mu_{\alpha_1} + \eta_{\alpha_1})^{\beta_1} (\mu_{\alpha_2} + \eta_{\alpha_2})^{\beta_2} - \eta_{\alpha_1}^{\beta_1} \eta_{\alpha_2}^{\beta_2}, \\ \eta_{\alpha_1}^{\beta_1} \eta_{\alpha_2}^{\beta_2}, 1 - (1 - \nu_{\alpha_1})^{\beta_1} (1 - \nu_{\alpha_2})^{\beta_2} \end{array} \right\rangle$.

If Eq. (4) holds for n=k, that is:

$$CIPFGA(\alpha_1,\alpha_2,\cdots,\alpha_k) = \left\langle \begin{array}{l} \prod_{j=1}^{k} (\mu_{\alpha_j} + \eta_{\alpha_j})^{\beta_j} - \prod_{j=1}^{k} (\eta_{\alpha_j})^{\beta_j}, \\ \prod_{j=1}^{k} (\eta_{\alpha_j})^{\beta_j}, 1 - \prod_{j=1}^{k} (1 - \nu_{\alpha_j})^{\beta_j} \end{array} \right\rangle$$

Then, for n=k+1, we have:

$$CIPFGA(\alpha_1,\alpha_2,...,\alpha_{k+1}) = \left\langle \begin{array}{l} \prod_{j=1}^{k} (\mu_{\alpha_j} + \eta_{\alpha_j})^{\beta_j} - \prod_{j=1}^{k} (\eta_{\alpha_j})^{\beta_j}, \\ \prod_{j=1}^{k} (\eta_{\alpha_j})^{\beta_j}, 1 - \prod_{j=1}^{k} (1 - \nu_{\alpha_j})^{\beta_j} \end{array} \right\rangle \bullet \left\langle \begin{array}{l} (\mu_{\alpha_{k+1}} + \eta_{\alpha_{k+1}})^{\beta_{k+1}} - (\eta_{\alpha_{k+1}})^{\beta_{k+1}}, \\ (\eta_{\alpha_{k+1}})^{\beta_{k+1}}, 1 - (1 - \nu_{\alpha_{k+1}})^{\beta_{k+1}} \end{array} \right\rangle$$

$$= \left\langle \begin{array}{l} \prod_{j=1}^{k+1} (\mu_{\alpha_j} + \eta_{\alpha_j})^{\beta_j} - \prod_{j=1}^{k+1} (\eta_{\alpha_j})^{\beta_j}, \\ \prod_{j=1}^{k+1} (\eta_{\alpha_j})^{\beta_j}, 1 - \prod_{j=1}^{k+1} (1 - \nu_{\alpha_j})^{\beta_j} \end{array} \right\rangle$$

That is to say, the condition n=k+1 is satisfied by Eq. (4).

As a result, Eq. (4) obtains for all n, concluding the proof of Theorem 1.

Theorem 2: (Idempotency) Let $\alpha_j = \left\langle \mu_{\alpha_j}, \eta_{\alpha_j}, \nu_{\alpha_j} \right\rangle$ (j = 1, 2, ..., n) and ϑ be a family of PFNs and FM on S. If all α_j are equal, i.e., $\alpha_j = \alpha$: ∀j, then CIPFGA (α_1, α_2, ..., α_n) = α.

Proof

Since $\alpha_j = \alpha = \langle \mu_\alpha, \eta_\alpha, \nu_\alpha \rangle$ according to theorem 1, we have:

$$CIPFGA(\alpha_1,\alpha_2,...,\alpha_n) = \left\langle \begin{array}{l} \prod_{j=1}^{n}(\mu_{\alpha_j}+\eta_{\alpha_j})^{\beta_j} - \prod_{j=1}^{n}(\eta_{\alpha_j})^{\beta_j}, \\ \prod_{j=1}^{n}(\eta_{\alpha_j})^{\beta_j}, 1-\prod_{j=1}^{n}(1-\nu_{\alpha_j})^{\beta_j} \end{array} \right\rangle = \langle \mu_\alpha, \eta_\alpha, \nu_\alpha \rangle = \alpha$$

where, $\beta_j = \dfrac{\sum_{B_j \subseteq P_j(C)} \vartheta\left(R_j \cup \{S_j\}\right) - \vartheta\left(R_j\right)}{\sum_{j=1}^{n}\sum_{R_j \subseteq P_j(C)} \vartheta\left(R_j \cup \{S_j\}\right) - \vartheta\left(R_j\right)}$, $R_j \subseteq P_j(S), P_j(S) = P(S\{S_j\})$.

Theorem 3: (Monotonicity) Let $\alpha_{1j} = \left(\mu_{\alpha_{1j}}, \eta_{\alpha_{1j}}, \nu_{\alpha_{1j}}\right)$ and $\alpha_{2j} = \left(\mu_{\alpha_{2j}}, \eta_{\alpha_{2j}}, \nu_{\alpha_{2j}}\right)$ (j=1, 2, ..., n) be two family of PFNs on S and ϑ be a FM on S. If $\mu_{\alpha_{1j}} \leq \mu_{\alpha_{2j}}, \eta_{\alpha_{1j}} \geq \eta_{\alpha_{2j}}$ and $\nu_{\alpha_{1j}} \geq \nu_{\alpha_{2j}}$, then $CIPFAA(\alpha_{11}, \alpha_{12}, \ldots, \alpha_{1n}) \leq CIPFAA(\alpha_{21}, \alpha_{22}, \ldots, \alpha_{2n})$.

Proof

$$CIPFGA(\alpha_{11},\alpha_{12},...,\alpha_{1n}) = \left\langle \begin{array}{l} \prod_{j=1}^{n}(\mu_{\alpha_{1j}}+\eta_{\alpha_{1j}})^{\beta_j} - \prod_{j=1}^{n}(\eta_{\alpha_{1j}})^{\beta_j}, \\ \prod_{j=1}^{n}(\eta_{\alpha_{1j}})^{\beta_j}, 1-\prod_{j=1}^{n}(1-\nu_{\alpha_{1j}})^{\beta_j} \end{array} \right\rangle$$

$$CIPFGA(\alpha_{21},\alpha_{22},...,\alpha_{2n}) = \left\langle \begin{array}{l} \prod_{j=1}^{n}(\mu_{\alpha_{2j}}+\eta_{\alpha_{2j}})^{\beta_j} - \prod_{j=1}^{n}(\eta_{\alpha_{2j}})^{\beta_j}, \\ \prod_{j=1}^{n}(\eta_{\alpha_{2j}})^{\beta_j}, 1-\prod_{j=1}^{n}(1-\nu_{\alpha_{2j}})^{\beta_j} \end{array} \right\rangle$$

Since $\mu_{\alpha_{1j}} \leq \mu_{\alpha_{2j}}, \eta_{\alpha_{1j}} \geq \eta_{\alpha_{2j}}$ and $\nu_{\alpha_{1j}} \geq \nu_{\alpha_{2j}}$ we have:

$$\prod_{j=1}^{n}(\mu_{\alpha_{1j}}+\eta_{\alpha_{1j}})^{\beta_j} - \prod_{j=1}^{n}(\eta_{\alpha_{1j}})^{\beta_j} \leq \prod_{j=1}^{n}(\mu_{\alpha_{2j}}+\eta_{\alpha_{2j}})^{\beta_j} - \prod_{j=1}^{n}(\eta_{\alpha_{2j}})^{\beta_j},$$

$$\prod_{j=1}^{n}\left(\eta_{\alpha_{1j}}\right)^{\beta_j} \geq \prod_{j=1}^{n}\left(\eta_{\alpha_{2j}}\right)^{\beta_j} \text{ and } 1-\prod_{j=1}^{n}(1-\nu_{\alpha_{1j}})^{\beta_j} \geq 1-\prod_{j=1}^{n}(1-\nu_{\alpha_{2j}})^{\beta_j}.$$

Hence

$$\prod_{j=1}^{n}(\mu_{\alpha_{1j}}+\eta_{\alpha_{1j}})^{\beta_j} - \prod_{j=1}^{n}(\eta_{\alpha_{1j}})^{\beta_j} - \prod_{j=1}^{n}(\eta_{\alpha_{1j}})^{\beta_j} - 1 - \prod_{j=1}^{n}(1-\nu_{\alpha_{1j}})^{\beta_j}$$

$$\leq \prod_{j=1}^{n}(\mu_{\alpha_{2j}}+\eta_{\alpha_{2j}})^{\beta_j} - \prod_{j=1}^{n}(\eta_{\alpha_{2j}})^{\beta_j}, \prod_{j=1}^{n}(\eta_{\alpha_{2j}})^{\beta_j}, 1-\prod_{j=1}^{n}(1-\nu_{\alpha_{2j}})^{\beta_j}$$

i.e.,

$CIPFGA(\alpha_{11}, \alpha_{12}, \ldots, \alpha_{1n}) \leq CIPFGA(\alpha_{21}, \alpha_{22}, \ldots, \alpha_{2n})$.

Theorem 4: (Boundedness) Let $\alpha_j = \left\langle \mu_{\alpha_j}, \eta_{\alpha_j}, \nu_{\alpha_j} \right\rangle$ (j = 1, 2, ..., n) and ϑ be a family of PFNs and FM on S.

If $\alpha^- = \left\langle \min_j\left(\mu_{\alpha_j}\right), \max_j\left(\eta_{\alpha_j}\right), \max_j\left(\nu_{\alpha_j}\right) \right\rangle$ and $\alpha^+ = \left\langle \max_j\left(\mu_{\alpha_j}\right), \min_j\left(\eta_{\alpha_j}\right), \min_j\left(\nu_{\alpha_j}\right) \right\rangle$ then $\alpha^- \leq CIPFGA(\alpha_1, \alpha_2, \ldots, \alpha_n) \leq \alpha^+$.

Proof

Using theorem 2, α^- and α^+ can be defined as $CIPFGA(\alpha^-, \alpha^-, ..., \alpha^-)$ and $CIPFGA$ $(\alpha^+, \alpha^+, ..., \alpha^+)$ respectively.

Since

$$\alpha^- = \left\langle \min_j\left(\mu_{\alpha_j}\right), \max_j\left(\eta_{\alpha_j}\right), \max_j\left(\nu_{\alpha_j}\right) \right\rangle = \left\langle \mu_{\alpha^-}, \eta_{\alpha^-}, \nu_{\alpha^-} \right\rangle$$

$$\alpha^+ = \left\langle \max_j\left(\mu_{\alpha_j}\right), \min_j\left(\eta_{\alpha_j}\right), \min_j\left(\nu_{\alpha_j}\right) \right\rangle = \left\langle \mu_{\alpha^+}, \eta_{\alpha^+}, \nu_{\alpha^+} \right\rangle$$

Then $\mu_{\alpha^-} \leq \mu_{\alpha_j}, \eta_{\alpha^-} \geq \eta_{\alpha_j}, \nu_{\alpha^-} \geq \nu_{\alpha_j}, \mu_{\alpha^+} \geq \mu_{\alpha_j}, \eta_{\alpha^+} \leq \eta_{\alpha_j}$ and $\nu_{\alpha^+} \leq \nu_{\alpha_j}$ $\forall j$. According to Theorem 3, we have:

$$CIPFGA\left(\alpha^-, \alpha^-, ..., \alpha^-\right) \leq CIPFGA(\alpha_1, \alpha_2, ..., \alpha_n) \leq CIPFGA\left(\alpha^+, \alpha^+, ..., \alpha^+\right).$$

Theorem 5: (Permutation) Let $\alpha_j = \left(\mu_{\alpha_j}, \eta_{\alpha_j}, \nu_{\alpha_j}\right)$ and $\alpha_{(j)} = \left(\mu_{\alpha_{(j)}}, \eta_{\alpha_{(j)}}, \nu_{\alpha_{(j)}}\right)$ (j=1, 2, ..., n) be two family of PFNs on S and ϑ be a FM on S, where (j) indicates an arbitrary permutation of j. then $CIPFGA(\alpha_1, \alpha_2, \ldots, \alpha_n) = CIPFGA\left(\alpha_{(1)}, \alpha_{(2)}, \ldots, \alpha_{(n)}\right)$.

Proof

$$CIPFGA(\alpha_1,\alpha_2,...,\alpha_n) = \left\langle \begin{array}{l} \prod_{j=1}^{n}(\mu_{\alpha_j}+\eta_{\alpha_j})^{\beta_j} - \prod_{j=1}^{n}(\eta_{\alpha_j})^{\beta_j}, \\ \prod_{j=1}^{n}(\eta_{\alpha_j})^{\beta_j}, 1-\prod_{j=1}^{n}(1-\nu_{\alpha_j})^{\beta_j} \end{array} \right\rangle$$

$$= \left\langle \begin{array}{l} \prod_{(j)=1}^{n}(\mu_{\alpha_{(j)}}+\eta_{\alpha_{(j)}})^{\beta_{(j)}} - \prod_{(j)=1}^{n}(\eta_{\alpha_{(j)}})^{\beta_{(j)}}, \\ \prod_{(j)=1}^{n}(\eta_{\alpha_{(j)}})^{\beta_{(j)}}, 1-\prod_{(j)=1}^{n}(1-\nu_{\alpha_{(j)}})^{\beta_{(j)}} \end{array} \right\rangle = CIPFGA\left(\alpha_{(1)}, \alpha_{(2)}, ..., \alpha_{(n)}\right).$$

The proof of Theorem 5 is complete.

Example 1: Let $\alpha_1 = \langle 0.4, 0.2, 0.3 \rangle$, $\alpha_2 = \langle 0.3, 0.1, 0.5 \rangle$, $\alpha_3 = \langle 0.6, 0.2, 0.1 \rangle$ be three PFNs on S, $S=\{S_1, S_2, S_3\}$ and ϑ be a fuzzy measure on S. $\vartheta(S_1) = 0.2$, $\vartheta(S_2) = 0.3$, $\vartheta(S_3) = 0.4$, $\vartheta(S_1, S_2) = 0.5$, $\vartheta(S_1, S_3) = 0.6$, $\vartheta(S_2, S_3) = 0.7$, $\vartheta(S_1, S_2, S_3) = 1$. Then:

$$CIPFGA(\alpha_1,\alpha_2,\alpha_3)$$
$$=\langle 0.4,0.2,0.3\rangle\frac{0.71}{3.09}+\langle 0.3,0.1,0.5\rangle\frac{1.03}{3.09}+\langle 0.6,0.2,0.1\rangle\frac{1.35}{3.09}$$
$$=\left\langle \begin{array}{c} \left(\begin{array}{l}(0.4+0.2)^{0.2298}-0.2^{0.2298}\cdot(0.3+0.1)^{0.3333}\\ -0.1^{0.3333}\cdot(0.6+0.2)^{0.4369}-0.2^{0.4369}\end{array}\right), \\ 0.2^{0.2298}\cdot 0.1^{0.3333}\cdot 0.2^{0.4369}, \\ 1-\left((1-0.3)^{0.2298}\cdot(1-0.5)^{0.3333}\cdot(1-0.1)^{0.4369}\right) \end{array}\right\rangle$$
$$=\langle 0.4356,0.1587,0.3016\rangle.$$

3.2 CIPFHGA operator

We propose the CIPFHGA operator, which is motivated by the PFHA operator (Wang et al. [9]).

Definition 8: Let $\alpha_j=\left\langle\mu_{\alpha_j},\eta_{\alpha_j},\nu_{\alpha_j}\right\rangle$ $(j=1,2,\ldots,n)$ and ϑ be a family of PFNs and FM on S and has an weight vector $w=(w_1,w_2,\ldots,w_n)^T$ with $0\le w_j\le 1$ and $\sum_{j=1}^n w_j=1$. The (discrete) CIPFHGA operator of α_j with respect to w and ϑ is defined as:

$$CIPFHGA(\alpha_1,\alpha_2,\ldots,\alpha_n)=\overset{n}{\underset{j=1}{\otimes}}\tilde{\alpha}_{(j)}^{w_j}$$

where, $\tilde{\alpha}_{(j)}$ is the jth largest of the $\tilde{\alpha}_j$,

$$\tilde{\alpha}_j=n\cdot\alpha_j\cdot\frac{\sum_{R_j\subseteq P_j(C)}\vartheta(R_j\cup\{S_j\})-\vartheta(R_j)}{\sum_{j=1}^n\sum_{R_j\subseteq P_j(C)}\vartheta(R_j\cup\{S_j\})-\vartheta(R_j)},$$

If FM ϑ is additive and the FMs of S_j(j=1, 2, ⋯, n) are equivalent, then $R_j\subseteq P_j(S)$, $P_j(S)=P$ (S{S_j}) and n is the balancing coefficient, which functions as a balancing factor in this situation, i.e., reduces to weight $\left(\frac{1}{n},\frac{1}{n},\cdots,\frac{1}{n}\right)^T$ then the vector reduces to $(\alpha_1,\alpha_2,\ldots,\alpha_n)^T$.

$$\left(\begin{array}{l} n\cdot\alpha_1\frac{\sum_{R_1\subseteq P_1(C)}\vartheta(R_1\cup\{S_1\})-\vartheta(R_1)}{\sum_{j=1}^n\sum_{R_j\subseteq P_j(C)}\vartheta(R_j\cup\{S_j\})-\vartheta(R_j)}, \\ n\cdot\alpha_2\frac{\sum_{R_2\subseteq P_2(C)}\vartheta(R_2\cup\{S_2\})-\vartheta(R_2)}{\sum_{j=1}^n\sum_{R_j\subseteq P_j(C)}\vartheta(R_j\cup\{S_j\})-\vartheta(R_j)}, \\ \cdots,n\cdot\alpha_n\frac{\sum_{R_n\subseteq P_n(C)}\vartheta(R_n\cup\{S_n\})-\vartheta(R_n)}{\sum_{j=1}^n\sum_{R_j\subseteq P_j(C)}\vartheta(R_j\cup\{S_j\})-\vartheta(R_j)} \end{array}\right)^T$$

For convenience,

let $\beta_j=\frac{\sum_{R_j\subseteq P_j(C)}\vartheta(R_j\cup\{S_j\})-\vartheta(R_j)}{\sum_{j=1}^n\sum_{R_j\subseteq P_j(C)}\vartheta(R_j\cup\{S_j\})-\vartheta(R_j)}$, it can be found that the CIPFHGA operator is three phases:

- To obtain the $n\cdot\alpha_j\cdot\beta_j$, it multiplies the importance degrees of the PFNs α_j by the appropriate FM ϑ as well as multiply these PFNs by a balancing coefficient n.
- It orders every $n\cdot\alpha_j\cdot\beta_j$ $(j=1,2,\ldots,n)$ in descending order $\left(\tilde{\alpha}_{(1)},\tilde{\alpha}_{(2)},\cdots,\tilde{\alpha}_{(n)}\right)$, where $\tilde{\alpha}_{(j)}$ is the j^{th} greatest of $n\cdot\alpha_j\cdot\beta_j$.
- It gives each of these ordered PFNs $\tilde{\alpha}_{(j)}$ a weight w_j $(j=1,2,\ldots,n)$ and then adds aggregated of the weighted PFNs $\tilde{\alpha}_{(j)}$ into a single one.

Theorem 6: Let $\alpha_j=\left\langle\mu_{\alpha_j},\eta_{\alpha_j},\nu_{\alpha_j}\right\rangle$ $(j=1,2,\ldots,n)$ and ϑ be a family of PFNs and FM on S and has an weight vector $w=(w_1,w_2,\ldots,w_n)^T$ with $0\le w_j\le 1$ and $\sum_{j=1}^n w_j=1$, $\tilde{\alpha}_{(j)}=\left\langle\mu_{\tilde{\alpha}_{(j)}},\eta_{\tilde{\alpha}_{(j)}},\nu_{\tilde{\alpha}_{(j)}}\right\rangle$ be the j^{th} largest of $\tilde{\alpha}_j$,

$$\tilde{\alpha}_j=n\cdot\alpha_j\cdot\frac{\sum_{R_j\subseteq P_j(C)}\vartheta(R_j\cup\{S_j\})-\vartheta(R_j)}{\sum_{j=1}^n\sum_{R_j\subseteq P_j(C)}\vartheta(R_j\cup\{S_j\})-\vartheta(R_j)},$$

Then, using the CIPFHGA operator, their aggregated value is also a PFN

$$CIPFHGA(\alpha_1,\alpha_2,\ldots,\alpha_n)=\left\langle\begin{array}{c}\prod_{j=1}^n(\mu_{\tilde{\alpha}_{(j)}}+\eta_{\tilde{\alpha}_{(j)}})^{w_j}-\prod_{j=1}^n(\eta_{\tilde{\alpha}_{(j)}})^{w_j}, \\ \prod_{j=1}^n(\eta_{\tilde{\alpha}_{(j)}})^{w_j},1-\prod_{j=1}^n(1-\nu_{\tilde{\alpha}_{(j)}})^{w_j}\end{array}\right\rangle$$

Theorem 7: (Idempotency) Let $\alpha_j=\left\langle\mu_{\alpha_j},\eta_{\alpha_j},\nu_{\alpha_j}\right\rangle$ (j=1, 2, ..., n) and ϑ be a family of PFNs and FM on S and has an weight vector $w=(w_1,w_2,\ldots,w_n)^T$ with $0\le w_j\le 1$ and $\sum_{j=1}^n w_j=1$. If all $\alpha_j(j=1,2,\ldots,n)$ are equal, i.e., $\alpha_j=\alpha:\forall j$, then $CIPFHGA(\alpha_1,\alpha_2,\cdots,\alpha_n)=\alpha$.

Theorem 8: (Monotonicity) Let $\alpha_{1j}=\left(\mu_{\alpha_{1j}},\eta_{\alpha_{1j}},\nu_{\alpha_{1j}}\right)$ and $\alpha_{2j}=\left(\mu_{\alpha_{2j}},\eta_{\alpha_{2j}},\nu_{\alpha_{2j}}\right)$ (j=1, 2, ..., n) be a family of PFNs and ϑ be a FM on S and has an weight vector $w=(w_1,w_2,\ldots,w_n)^T$ with $0\le w_j\le 1$ and $\sum_{j=1}^n w_j=1$. if $\mu_{\alpha_{1j}}\le\mu_{\alpha_{2j}},\eta_{\alpha_{1j}}\ge\eta_{\alpha_{2j}}$ and $\nu_{\alpha_{1j}}\ge\nu_{\alpha_{2j}}$, then $CIPFHGA(\alpha_{11},\alpha_{12},\cdots,\alpha_{1n})\le CIPFHGA(\alpha_{21},\alpha_{22},\cdots,\alpha_{2n})$.

Theorem 9: (Boundedness) Let $\alpha_j=\left\langle\mu_{\alpha_j},\eta_{\alpha_j},\nu_{\alpha_j}\right\rangle$ $(j=1,2,\ldots,n)$ and ϑ be a family of PFNs and FM on S and has a weight" vector $w=(w_1,w_2,\ldots,w_n)^T$ with $0\le w_j\le 1$ and $\sum_{j=1}^n w_j=1$.

If $\alpha^{-}=\left\langle \min_{j}\left(\mu_{\alpha_j}\right), \max_{j}\left(\eta_{\alpha_j}\right), \max_{j}\left(\nu_{\alpha_j}\right)\right\rangle$ and $\alpha^{+}=\left\langle \max_{j}\left(\mu_{\alpha_j}\right), \min_{j}\left(\eta_{\alpha_j}\right), \min_{j}\left(\nu_{\alpha_j}\right)\right\rangle$, then $\alpha^{-} \leq CIPFHAA(\alpha_1, \alpha_2, \cdots, \alpha_n) \leq \alpha^{+}$.

Theorem 10: (Permutation) let $\alpha_j=\left(\mu_{\alpha_j}, \eta_{\alpha_j}, \nu_{\alpha_j}\right)$ $\alpha_{(j)}=\left(\mu_{\alpha_{(j)}}, \eta_{\alpha_{(j)}}, \nu_{\alpha_{(j)}}\right)$ (j=1, 2, ..., n) be two familys of PFNs on S, ϑ be a fuzzy measure on S and has an weight vector $w=(w_1, w_2, ..., w_n)^T$ with $0 \leq w_j \leq 1$ and $\sum_{j=1}^{n} w_j = 1$. where (j) denotes an arbitrary permutation of j. Then

$CIPFHGA(\alpha_1, \alpha_2, \ldots, \alpha_n) = CIPFHGA(\alpha_{(1)}, \alpha_{(2)}, \ldots, \alpha_{(n)})$.

Theorem 2 - Theorem 5 proofs are similar to Theorem 7 - Theorem 10 proofs, which are ignored here.

Example 2: Suppose $w=(0.2429, 0.5142, 0.2429)^T$ by referring to (Xu [26]) then, the example 1, by using the CIPFHGA operator we have:

$$\alpha_1^{\beta_1} = 0.2298 \cdot \langle 0.4, 0.2, 0.3 \rangle = \langle 0.1984, 0.6908, 0.0787 \rangle$$
$$\alpha_2^{\beta_2} = 0.3333 \cdot \langle 0.3, 0.1, 0.5 \rangle = \langle 0.2726, 0.4642, 0.2063 \rangle$$
$$\alpha_3^{\beta_3} = 0.4369 \cdot \langle 0.6, 0.2, 0.1 \rangle = \langle 0.4121, 0.4950, 0.0449 \rangle.$$

Then, $S(\alpha_1^{\beta_1}) = -0.5711$, $S(\alpha_2^{\beta_2}) = -0.3989$ and $S(\alpha_3^{\beta_3}) = -0.1278$.

Following the rules of section 2.2, we may conclude that based on these comparative laws,

$$\tilde{\alpha}_1 = \alpha_3^{3\beta_3} = \langle 0.3734, 0.3297, 0.2180 \rangle$$
$$\tilde{\alpha}_2 = \alpha_2^{3\beta_2} = \langle 0.2999, 0.1207, 0.5000 \rangle$$
$$\tilde{\alpha}_3 = \alpha_1^{3\beta_1} = \langle 0.6251, 0.1213, 0.1287 \rangle.$$

then,

$$\begin{aligned} & CIPFHGA(\alpha_1, \alpha_2, \cdots, \alpha_n) \\ & 0.2429 \cdot \langle 0.3734, 0.3297, 0.2180 \rangle \otimes \\ & 0.5142 \cdot \langle 0.2999, 0.1207, 0.5000 \rangle \otimes \\ & 0.2429 \cdot \langle 0.6251, 0.1213, 0.1287 \rangle \\ & = \langle 0.1556, 0.1543, 0.3621 \rangle. \end{aligned}$$

4. MADM PROBLEM

To begin, we discuss MADM problem with correlative criteria in a PF environment. Then, we provide a MADM problem that is using CIPFHGA operator.

4.1 Description of MADM problems

Let $A_i=\{A_1, A_2, \cdots A_m\}$ (i=1, 2,$\cdots$, m) is the finite set of alternative, $S_j=\{S_1, S_2, \cdots S_n\}$ (j=1, 2, $\cdots$ n) is the finite set of criteria and the importance degree of S_j is denote as FM ϑ. The evaluation of the alternatives under each criteria is provide by the decision making and is denoted by the PFNs where $\mu_{\alpha_{ij}}$, $\eta_{\alpha_{ij}}$, $\nu_{\alpha_{ij}}$ denotes the extent to which the decision-making analyses whether alternative A_i should satisfy, neutral satisfy and non satisfy of the criteria S_j. The evaluation of matrix is constructed as followes.

$$E = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{n1} & \alpha_{n2} & \cdots & \alpha_{mn} \end{bmatrix}$$

A flowchart for the proposed MADM problem is shown in Figure 1 and the detailed steps are presented as follows.

Steps of MADM problem:

Following are the steps aimed at solving the MADM problem with PF environment using the proposed operator.

Step 1: The normalized PF decision matrix is generated by defining the attribute set S_j into S_1 and S_2, with S_1 representing benefit attributes and S_2 representing cost attributes. Eq. (5) is used to convert the benefit type to the cost type:

$$\alpha_{ij} = \begin{cases} \left\langle \mu_{\alpha_{ij}}, \eta_{\alpha_{ij}}, \nu_{\alpha_{ij}} \right\rangle; & \text{for cost type attribute} \\ \left\langle \nu_{\alpha_{ij}}, \mu_{\alpha_{ij}}, \mu_{\alpha_{ij}} \right\rangle; & \text{for benefit type attribute} \end{cases} \quad (5)$$

Step 2: Calculate the FM on S_j using definition 5.

Step 3: Confirm the weighting vector $w=(w_1, w_2, ..., w_n)^T$ with respect to the orders $0 \leq w_j \leq 1$ and $\sum_{j=1}^{n} w_j = 1$.

Step 4: Use the CIPFHGA operator to combine the evaluation for each alternative under each criteria, such that $\alpha_i = CIPFHGA(\alpha_{i1}, \alpha_{i2}, \cdots, \alpha_{in})$.

Step 5: Select the best alternative by ranking the alternatives based on the score and accuracy values of α_j.

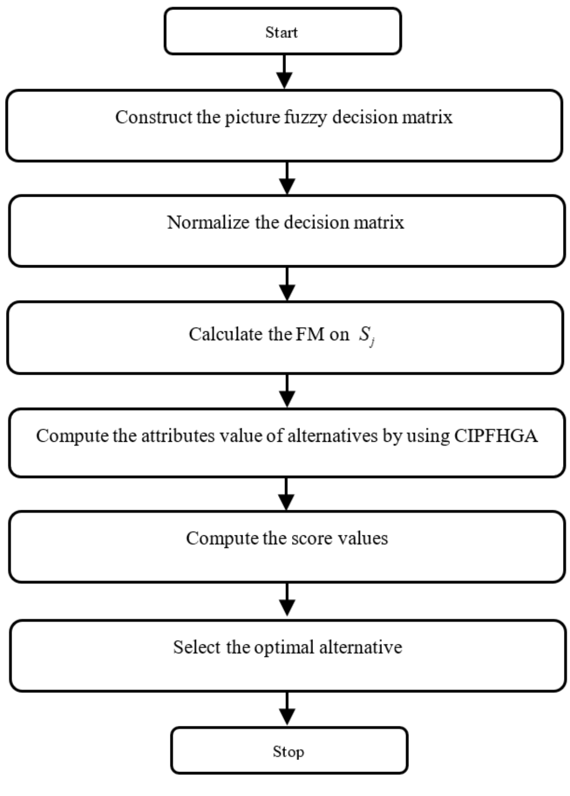

Figure 1. Flowchart for the selection of the best alternative

5. NUMERICAL EXAMPLE

In this part, we will use the numerical example below to demonstrate the proposed approach. Assume that decision makers must select between four alternatives A_1, A_2, A_3 and A_4. Four attributes S_1, S_2, S_3 and S_4 where the attributes are cost attributes. The PF ratings of alternatives $\{A_i\}i$=1, 2, ..., m according to attributes$\{S_j\}j$=1, 2, ..., n are evaluated by decision makers and form the decision matrix is given below and the weight information of attribute w=(0.1550, 0.3450, 0.3450, 0.1550)^T by referring to (Xu [26]).

$$\begin{array}{cc} & \begin{array}{cccc} A_1 & A_2 & A_3 & A_4 \end{array} \\ \begin{array}{c} S_1 \\ S_2 \\ S_3 \\ S_4 \end{array} & \begin{bmatrix} \langle 0.5,0.2,0.3\rangle & \langle 0.2,0.3,0.1\rangle & \langle 0.2,0.3,0.3\rangle & \langle 0.7,0.2,0.1\rangle \\ \langle 0.4,0.3,0.2\rangle & \langle 0.6,0.2,0.1\rangle & \langle 0.4,0.3,0.3\rangle & \langle 0.5,0.2,0.3\rangle \\ \langle 0.6,0.1,0.1\rangle & \langle 0.4,0.2,0.3\rangle & \langle 0.5,0.2,0.3\rangle & \langle 0.3,0.4,0.1\rangle \\ \langle 0.5,0.3,0.1\rangle & \langle 0.7,0.1,0.1\rangle & \langle 0.4,0.3,0.2\rangle & \langle 0.6,0.1,0.2\rangle \end{bmatrix} \end{array}$$

Step 1: There is no need to convert the criteria because they all cost.

Step 2: Evaluate the FM on S_j. The FM are sets as follows:

$$\begin{gathered} \vartheta(S_1)=0.23, \vartheta(S_2)=0.31, \vartheta(S_3)=0.18, \\ \vartheta(S_4)=0.24, \vartheta(S_1,S_2)=0.5482, \\ \vartheta(S_1,S_3)=0.4148, \vartheta(S_1,S_4)=0.4764, \\ \vartheta(S_2,S_3)=0.4964, \vartheta(S_2,S_4)=0.5586, \\ \vartheta(S_3,S_4)=0.4249, \vartheta(S_1,S_2,S_3)=0.7396, \\ \vartheta(S_1,S_2,S_4)=0.8033, \vartheta(S_1,S_3,S_4)=0.6663, \\ \vartheta(S_2,S_3,S_4)=0.7501, \vartheta(S_1,S_2,S_3,S_4)=1. \end{gathered}$$

Step 3: Confirm the weighting vector w=(0.1550, 0.3450, 0.3450, 0.1550)^T by referring to (Xu [26]).

Step 4: Calculate the evaluations. As an example, consider the calculation of A_1.

$$\begin{gathered} \text{Since } \langle 0.5,0.2,0.3\rangle^{0.2468}=\langle 0.2435,0.6722,0.0843\rangle, \\ \langle 0.2,0.3,0.1\rangle^{0.3114}=\langle 0.1185,0.6873,0.0823\rangle, \\ \langle 0.2,0.3,0.3\rangle^{0.1938}=\langle 0.0824,0.7919,0.0663\rangle \text{ and} \\ \langle 0.7,0.2,0.1\rangle^{0.2480}=\langle 0.3033,0.6709,0.0258\rangle \end{gathered}$$

Their respective score values are -0.523, -0.6511, -0.7763, and -0.3934. We get their order as:

$$\langle 0.3033,0.6709,0.0258\rangle > \langle 0.2435,0.6722,0.0843\rangle > \langle 0.1185,0.6873,0.0823\rangle > \langle 0.0824,0.7919,0.0668\rangle.$$

Hence, we have:

$$\begin{aligned} & CIPFHGA(\alpha_{11},\alpha_{12},\alpha_{13},\alpha_{14}) = \\ & \langle 0.3033,0.6709,0.0258\rangle^{4\cdot 0.1550} \otimes \langle 0.2435,0.6722,0.0843\rangle^{4\cdot 0.3450} \\ & \otimes \langle 0.1185,0.6873,0.0823\rangle^{4\cdot 0.3450} \otimes \langle 0.0824,0.7919,0.0668\rangle^{4\cdot 0.1550} \\ & = \left\langle \begin{array}{c} (0.3033+0.6709)^{0.62}-0.6709^{0.62} \\ \cdot(0.2435+0.6722)^{1.38}-0.6722^{1.38} \\ \cdot(0.1185+0.6873)^{1.38}-0.6722^{1.38} \\ \cdot(0.0824+0.7919)^{0.62}-0.7919^{0.62}, \\ 0.6709^{0.62}\cdot 0.6722^{1.38}\cdot 0.6722^{1.38}\cdot 0.7919^{0.62}, \\ 1-\begin{bmatrix} (1-0.0258)^{0.62}\cdot(1-0.0843)^{1.38} \\ \cdot(1-0.0823)^{1.38}\cdot(1-0.0668)^{0.62} \end{bmatrix} \end{array} \right\rangle \\ & = \langle 0.0134,0.0252,0.6839\rangle \end{aligned}$$

At the same reason, A_2 = ⟨0.0016, 0.2426, 0.2300⟩, A_3 = ⟨0.0015, 0.1988, 0.2449⟩ and A_4=⟨0.0033, 0.1831, 0.1473⟩.

Step 5: Rank the alternatives. The A_i(i =1, 2, 3, 4) score values are -0.6957, -0.4710, -0.4422 and -0.3271 respectively. Hence alternative can be ordered as A_4>A_3>A_2>A_1 the best alternative A_4.

6. COMPARATIVE ANALYSIS

We evaluated the proposed MADM problem to the existing AOs to demonstrate the utility and effectiveness of the obtained results.

6.1 Comparison with existing approaches

The following Table 3 compares the proposed CIPFHGA operator with several existing aggregation operators.

Table 3. A comparison of various existing approaches and the proposed approach

Authors	Aggregate Operators	Ranking Order
Wei [8]	PFWG	$A_4 > A_2 > A_3 > A_1$
Wang et al. [9]	PFWG	$A_4 > A_2 > A_3 > A_1$
Garg [10]	PFWA	$A_4 > A_2 > A_3 > A_1$
Wei [11]	PFHWA	$A_4 > A_2 > A_3 > A_1$
Jana et al. [16]	PFHWG	$A_4 > A_2 > A_3 > A_1$
Khan et al. [17]	PFEWA	$A_4 > A_2 > A_3 > A_1$
Proposed operator	CIPFHGA	$A_4 > A_3 > A_2 > A_1$

We compare the proposed CIPFHGA operator with six existing AOs [8-11, 16, 17] in Table 3. It can be noted that all the existing AOs obtain the same ranking results and the same optimal alternative, while the proposed operator also optimal alternative is same, which is A_4 but obtain different ranking results from the existing AOs.

The suggested approach select option A_4 as the best choice, which is consistent with the findings produced by previous AOs. This validates the suggested operators practicality. Combining the Choquet integral with the OWG operator, the CIPFHGA operator ensures that all criteria establish correlative relationships and position weights are taken into consideration, providing a robust solution for dealing with MADM problems.

6.2 Advantages of the proposed operators

The benefits of our proposed approach can be described based on the aforementioned comparisons [8-11, 16, 17].

- The computation for fuzzy measure is easier and may accurately deliberate the importance degree of each distinct criteria.
- To maintain consistency during the computation process, the types of criteria are evaluated and the benefit criteria are converted into their cost types.
- There are several MADM problems that could be solved using the proposed operators.

7. CONCLUSIONS

A PFS is a more effective way of dealing with uncertainty in given information and it has a various of applications in DM. AOs are especially important in the DM process since they combine all of the provided individual assessment values into a single form. The correlative relationships between criteria sets in fuzzy information systems are well defined by FMs, which may be further dealt with using the Choquet integral. We introduced the CIPFGA operator as an aggregation operator inspired by the Choquet integral and also proposed the CIPFHGA operator further by combining weights position. Some main properties of these proposed operators have been demonstrated and the functioning of the operators have been validated with illustrations. Further, the proposed operators enable us to solve a numerical example in MADM problem with PF environments. Finally, we have compared the proposed operators with some existing AOs. The viability of the proposed technique is demonstrated by different results of the ranking alternatives.

The specified AOs can be enhanced in future to handle more complex environments, such as interval valued PF environments, triangular PF environments and trapezoidal PF environments. In addition, we will continue to work on extending and applying the proposed operators to other domains.

REFERENCES

[1] Zadeh, L.A. (1965). Fuzzy sets. Information and Control, 8(3): 338-353. https://doi.org/10.1016/S0019-9958(65)90241-X

[2] Atanassov, K.T. (1986). Intuitionistic fuzzy sets. Fuzzy Sets Syst, 20(1): 87-96. https://doi.org/10.1016/S0165-0114(86)80034-3

[3] Cuong, B.C., Kreinovich, V. (2013). Picture fuzzy sets: A new concept for computational intelligence problems. In 2013 Third World Congress on Information and Communication Technologies(WIICT-2013), pp. 1-6. https://doi.org/10.1109/WICT.2013.7113099

[4] Cuong, B.C. (2014). Picture fuzzy sets. Journal of Computer Science and Cybernetics, 30(4): 409-420. http://dx.doi.org/10.15625/1813-9663/30/4/5032

[5] Dutta, P., Ganju, S. (2018). Some aspects of picture fuzzy set. Transactions of A. Razmadze Mathematical Institute, 172(2): 164-175. https://doi.org/10.1016/j.trmi.2017.10.006

[6] Yager, R.R. (1988). On ordered weighted averaging aggregation operators in multicriteria decisionmaking. IEEE Transactions on Systems, Man, and Cybernetics, 18(1): 183-190. https://doi.org/10.1109/21.87068

[7] Cuong, B.C., Kreinovitch, V., Ngan, R.T. (2016). A classification of representable t-norm operators for picture fuzzy sets. In 2016 Eighth International Conference on Knowledge and Systems Engineering (KSE), pp. 19-24. https://doi.org/10.1109/KSE.2016.7758023

[8] Wei, G. (2017). Picture fuzzy aggregation operators and their application to multiple attribute decision making. Journal of Intelligent & Fuzzy Systems, 33(2): 713-724. https://doi.org/10.3233/JIFS-161798

[9] Wang, C., Zhou, X., Tu, H., Tao, S. (2017). Some geometric aggregation operators based on picture fuzzy sets and their application in multiple attribute decision making. Ital. J. Pure Appl. Math, 37: 477-492.

[10] Garg, H. (2017). Some picture fuzzy aggregation operators and their applications to multicriteria decision-making. Arabian Journal for Science and Engineering, 42(12): 5275-5290. https://doi.org/10.1007/s13369-017-2625-9

[11] Wei, G. (2018). Picture fuzzy Hamacher aggregation operators and their application to multiple attribute decision making. Fundamenta Informaticae, 157(3): 271-320. https://doi.org/10.3233/FI-2018-1628

[12] Jana, C., Senapati, T., Pal, M., Yager, R.R. (2019). Picture fuzzy Dombi aggregation operators: Application to MADM process. Applied Soft Computing, 74: 99-109. https://doi.org/10.1016/j.asoc.2018.10.021.

[13] Zhang, H., Zhang, R., Huang, H., Wang, J. (2018). Some picture fuzzy Dombi Heronian mean operators with their application to multi-attribute decision-making. Symmetry, 10(11): 593. https://doi.org/10.3390/sym10110593

[14] Wang, R., Wang, J., Gao, H., Wei, G. (2018). Methods for MADM with picture fuzzy muirhead mean operators and their application for evaluating the financial investment risk. Symmetry, 11(1): 6. https://doi.org/10.3390/sym11010006

[15] Xu, Y., Shang, X., Wang, J., Zhang, R., Li, W., Xing, Y. (2019). A method to multi-attribute decision making with picture fuzzy information based on Muirhead mean. Journal of Intelligent & Fuzzy Systems, 36(4): 3833-3849. https://doi.org/10.3233/JIFS-172130

[16] Jana, C., Pal, M. (2019). Assessment of enterprise performance based on picture fuzzy Hamacher aggregation operators. Symmetry, 11(1): 75. https://doi.org/10.3390/sym11010075

[17] Khan, S., Abdullah, S., Ashraf, S. (2019). Picture fuzzy aggregation information based on Einstein operations and their application in decision making. Mathematical Sciences, 13(3): 213-229. https://doi.org/10.1007/s40096-019-0291-7

[18] Qiyas, M., Khan, M. A., Khan, S., Abdullah, S. (2020). Concept of Yager operators with the picture fuzzy set

environment and its application to emergency program selection. International Journal of Intelligent Computing and Cybernetics. https://doi.org/10.1108/IJICC-06-2020-0064

[19] Qin, Y., Cui, X., Huang, M., Zhong, Y., Tang, Z., Shi, P. (2021). Multiple-attribute decision-making based on picture fuzzy Archimedean power Maclaurin symmetric mean operators. Granular Computing, 6(3): 737-761. https://doi.org/10.1007/s41066-020-00228-0

[20] Senapati, T. (2022). Approaches to multi-attribute decision-making based on picture fuzzy Aczel–Alsina average aggregation operators. Computational and Applied Mathematics, 41(1): 1-19. https://doi.org/10.1007/s40314-021-01742-w

[21] Tan, C., Chen, X. (2010). Intuitionistic fuzzy Choquet integral operator for multi-criteria decision making. Expert Systems with Applications, 37(1): 149-157. https://doi.org/10.1016/j.eswa.2009.05.005

[22] Zhang, S., Yu, D. (2014). Some geometric Choquet aggregation operators using Einstein operations under intuitionistic fuzzy environment. Journal of Intelligent & Fuzzy Systems, 26(1): 491-500. https://doi.org/10.3233/IFS-130924

[23] Singh, A., Kumar, S. (2021). Picture fuzzy Choquet integral-based VIKOR for multicriteria group decision-making problems. Granular Computing, 6(3): 587-601. https://doi.org/10.1007/s41066-020-00218-2

[24] Jia, X., Wang, Y. (2022). Choquet integral-based intuitionistic fuzzy arithmetic aggregation operators in multi-criteria decision-making. Expert Systems with Applications, 191: 116242. https://doi.org/10.1016/j.eswa.2021.116242

[25] Sugeno M. (1974). Theory of fuzzy integral and its applications. Doct. Theses, Tokyo.

[26] Xu, Z. (2005). An overview of methods for determining OWA weights. International Journal of Intelligent Systems, 20(8): 843-865. https://doi.org/10.1002/int.20097