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Vibration Analysis of a Symmetric Double-Beam with an Elastic Middle Layer at Arbitrary Boundary Conditions

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ABSTRACT

Vibration of double beams with an elastic connected layer has been studied in this paper by assuming that the beam is a Bernoulli-Euler beam. The natural frequencies equations of the symmetric double beam have been computed at arbitrary boundary conditions. The behavior of those frequencies has been investigated with a change in the stiffness of connected layer, modulus of elasticity of beam, length of beam, mass density of beam, and thickness of beam. The high effect of the elastic connected layer on the higher natural frequencies of a cantilever double beam is less than that in the clamped and free double beams. The increase in the thickness of upper and lower beams made a high increase in the values of lower natural frequencies in all types of beams. The change in the modulus of elasticity values of double beam becomes high on the lower natural frequencies but without enlarging the influence on the higher frequencies, especially in the cantilever double beam. The similar effect of change in the mass density of the beam resulted in the same influence on the higher and lower natural frequencies in all types of beams. The length of the beam enlarges the influence on the higher natural frequencies of clamped and free.

1. INTRODUCTION

In many current engineering applications, double beam systems are commonly employed such as aircraft structures and civil buildings. As a result, researchers continue to be interested in the dynamic behavior of double beam structures. Li and Sun [1] developed a numerical approach for analyzing the mode shapes and the natural frequencies of a double beam structure with a general boundary and any beam mass, made up of double beams bonded by an elastic layer that is uniformly distributed between them. Hao et al. [2] enhanced an analytical method for investigating the vibration characteristics of a double beam under different boundaries. The current framework provides the impact of the connected layer stiffness on the vibration characteristics of double beam. Lai et al. [3] used a mix of finite sin–Fourier transforms and numerical Laplace transforms depending on Durbin transform to investigate the displacement response in the time domain of a double simply supported Euler–Bernoulli beam system with elastic connection. The Bernoulli–Euler beam theory was used by Zhang et al. [4] to study the characteristics of buckling and vibration of a double beam structure under a compression force. The results indicated that the system's critical buckling load is linked to the compression ratio of elastic connected layer and beams, and that the axial compressions have a significant impact on the parameters of the system's free transverse vibration. The vibrational properties of double beams under compressive stress were studied by Kozic et al. [5]. The system's two parallel beams are easily and regularly connected by a Kerr-type three-parameter. The impact of non-linear elasticity on the frequencies of sandwich beams under varied boundary conditions was demonstrated by Abdulsahib and Atiyah [6]. The energy balance technique based on Galerkin-Petrov (EGP) and the Homotopy Perturbation Method were used to study the influence of the inner layer's non-linearity stiffness on those frequencies (HPM). By distinguishing between the synchronous and asynchronous movements of beams, Mirzabeigy and Madoliat [7] explored the influence of nonlinearity in connected layer on the vibration of double beam studied and, concluded the high frequencies are more accurate if ignoring the effect of the nonlinearity of the elastic layer. Mao [8] analyzed the frequencies behavior of double beams using the AMDM technique, The suggested technique was applied to systems containing any number of beams to compute vibration characteristics with varied parameters. Oniszczuk [9] demonstrated the continuous vibration characteristics of double beams. A uniform set of dynamic equations was solved using analytical technique to characterize the system's motion. The analytical technique was used to determine the ultimate shape of the vibrations. The vibration properties of a double beam were examined by Rezaiee-Pajand and Hozhabrossadati [10]. This structure consists of two beams, one end is elastic and the other is free, as well as the two beams are connected by a mass-spring mechanism. The impact of four geometric and material parameters on the vibration of twin beams was examined by Atiyah and Abdulsahib [11]. Those parameters of two beams were mass density, thickness, modulus of elasticity, and the properties of the intermediate layer. The Bernoulli-Euler beam was used to compute the frequencies of the double beams.

In this paper, a number of variables of the elastic connecting layer that are believed to affect the vibration behavior of the double beams, which were not fully studied previously are investigated. Those parameters of two beams were mass density, thickness, modulus of elasticity, and the properties of the intermediate layer. The Bernoulli-Euler beam was used to compute the frequencies of the double beams. the equations of motion are derived to calculate symmetric and asymmetric frequencies at different boundary conditions, which are the most common in various engineering applications, with calculating the effect of a number of connecting layers variables on those frequencies.

2. THEORETICAL WORK

Two beams are connected by an elastic layer with arbitrary boundary conditions. The two beams are symmetric and have the same length, as shown in Figure 1. The Bernoulli-Euler beam theory for free vibrations is used to describe the equations of motion [1]:

$$\frac{\partial^2}{\partial x^2}\left(E_1 I_1 \frac{\partial^2 W_1}{\partial x^2}\right) + K(W_1 - W_2) + \rho_1 A_1 \frac{\partial^2 W_1}{\partial t^2} = 0 \tag{1}$$

$$\frac{\partial^2}{\partial x^2}\left(E_2 I_2 \frac{\partial^2 W_2}{\partial x^2}\right) - K(W_1 - W_2) + \rho_2 A_2 \frac{\partial^2 W_2}{\partial t^2} = 0 \tag{2}$$

where, A_1, A_2, ρ_1, ρ_2, E_1, E_2, I_1, and I_2 are the cross-sectional area, mass density, modulus of elasticity, and moment of area for the upper and lower beam, respectively, k is the stiffness of elastic layer, and W_1, W_2 are the deflection of the upper and lower beam, respectively.

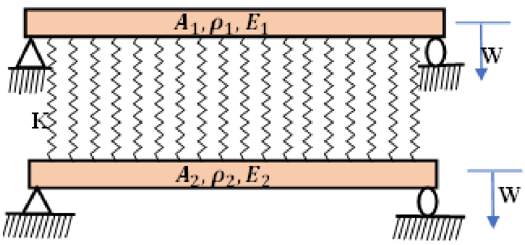

Figure 1. Double-beam with elastic connected layer

The boundary conditions in general form for clamped beams are assumed as follows: $W_i(0,t) = W'_i(0,t) = W_i(l,t) = W'_i(l,t) = 0, i = 1,2$.

The boundary conditions for simply supported beams are: $W_i(0,t) = W''_i(0,t) = W_i(l,t) = W''_i(l,t) = 0, i = 1,2$.

In addition, the boundary conditions for free beams are: $W''_i(0,t) = W'''_i(0,t) = W''_i(l,t) = W'''_i(l,t) = 0, i = 1,2$.

And, the boundary conditions for cantilever beam are: $W_i(0,t) = W'_i(0,t) = W''_i(l,t) = W'''_i(l,t) = 0, i = 1,2$.

The natural frequencies of the system will be got by solving Eqns. (1) and Eq. (2). Assume the time-harmonic motion with the above boundary conditions and by the separation of variables, the solutions of Eqns. (1) and (2) can be written as follow:

$$W_i(x,t) = \sum_{n=1}^{\infty} x_n(x) \cdot T_{ni}(t)\, , i = 1,2 \tag{3}$$

where,

$$X_n(x) = cosh(k_n x) - cos(k_n x) - \sigma_n[sinh(k_n x) - sin(k_n x)],\ k_n = \frac{\pi(2n+1)}{2l},\ n = 1,2,3, \ldots \ldots \sigma_n \cong 1 \ldots \tag{4}$$

For clamped beams,

$$X_n(x) = sin(k_n x),\ k_n = \frac{n\pi}{l},\ n = 1,2,3, \ldots \tag{5}$$

For simply-supported beams,

$$X_n(x) = cosh(k_n x) + cos(k_n x) - \sigma_n[sinh(k_n x) + sin(k_n x)],\ k_n = \frac{\pi(2n+1)}{2l},\ n = 1,2,3, \ldots \ldots \sigma_n \cong 1 \tag{6}$$

For free beams,

$$X_n(x) = cosh(k_n x) - cos(k_n x) - \sigma_n[sinh(k_n x) - sin(k_n x)],\ k_n = \frac{\pi(2n-1)}{2l},\ n = 1,2,3, \ldots \ldots \sigma_n 1 \ldots \tag{7}$$

For cantilever beam, the assumed general forms for time functions are:

$$T_{ni} = C_i e^{jw_n t}, i = 1,2 \tag{8}$$

Substituting the above expression into Eqns. (1) and (2) will get:

$$(E_1 I_1 k_n^4 + K - \rho_1 A_1 \omega_n^2)C_1 - KC_2 = 0 \tag{9}$$

$$(E_2 I_2 k_n^4 + K - \rho_2 A_2 \omega_n^2)C_2 - KC_1 = 0 \tag{10}$$

These equations can be solved when the two beams are symmetric; the lower and higher frequencies will be obtained as follows:

$$\omega_{1n} = \sqrt{\frac{EIk_n^4}{\rho AL^4}} \tag{11}$$

$$\omega_{2n} = \sqrt{\frac{EIk_n^4 + 2KL^4}{\rho AL^4}} \tag{12}$$

3. RESULTS AND DISCUSSION

A convergence test is utilized to compare the accuracy of Eqns. (11) & (12) with the results in reference [1]. The numerical values are used as in reference [11], such as EI=4×10^6 N.m^2, L=10 m, ρA=1×10^2 kg.m^{-1}, K=1~5×10^5 N.m^{-2}, and ω_n (Hz). Table 1 shows the comparison results between the present results and the Ref. [1] when the boundary conditions of double beams are simply supported. This table shows a good convergence of results between the present work and the literature by Li and Sun [1]. The maximum difference between the present and results of reference [1] is less than 1%. this convergence confirms the validity of the derived equations in this work.

The natural frequencies for the clamped double beam are same for the free double beam, because it has the same dimensionless natural frequency function but it has another mode shape. Therefore, the behavior is same for the both cases in the all figures of this paper. Figure 2 manifests the behavior of higher frequencies with the change in stiffness of the elastic connected layer (k), in all cases of boundary conditions. The change in the values of k has not affected the lower frequencies (synchronous), but it caused an increasing in the higher natural frequencies (asynchronous) in all modes when increasing the k values.

In Table 2, it is seen that when k is increased from 100000 to 1800000 N/m^2, the higher natural frequencies (asynchronous) increased about 367% in the simply supported beam, about 265% in the clamped and free double beams, and about 415% in the cantilever double beam. As a result, there is a high effect of elastic connected layer on the cantilever double beam, and this effect is less in the clamped and free double beams.

Figure 3 and Table 3 elucidated that for the clamped and free double beams, the higher natural frequencies decrease when the ratio (h/b) increases from 1 to 12 times. However, if the ratio is more than 12, the higher natural frequencies start increasing. The same behavior for the simply supported beam can be seen for the ratio (h/b) between 1 to 22 times, but the higher frequencies decrease with the increase in this ratio for more than 22 times. In the cantilever double beam, the higher frequencies also decrease with the (h/b) ratio increase. The behavior of lower natural frequencies (synchronous) with the changes in the values of thickness in the upper and lower beams (h_1 & h_2) is depicted in Figure 4 and Table 3. When the thickness increases from 0.02 to 0.38 m (h_1=h_2), the lower natural frequencies increase about 3000% in the simply supported beam and increase in the same ratio approximately 3000% in the clamped, free, and the cantilever beams. Consequently, the effect of the change in thickness on the higher natural frequencies (asynchronous) is higher in the simply supported, and the clamped, free double beams, but it generally causes an increase in those frequencies with the thickness increase in a cantilever beam. The increase in thickness of the upper and lower beams made a great increase in the values of the lower natural frequencies in all types of beams.

The effects of changing the modulus of elasticity of the upper and lower beams (E_1 & E_2) on the higher natural frequencies are demonstrated in Figure 5 and Table 4. When the modulus of elasticity changes from 10 GPa to 140 GPa (E_1=E_2), the frequencies of simply supported beams increase about 28%, the frequencies of clamped and free beams increase about 93%, and the frequencies of cantilever beam increase just 4%. Figure 6 and Table 4 portrays the behavior of lower natural frequencies when the modulus of elasticity of upper and lower beams changes. When the elasticity modulus increases from 10 GPa to 140 GPa, the lower natural frequencies increase about 275% in all types of beams. The change in the values of the elasticity modulus of double beam has a great effect on the lower natural frequencies but not as much as that effect on the higher frequencies, especially in the cantilever double beam.

Figures 7-8 and Table 5 reveal the effect of changing the mass density of the upper and lower beams (ρ_1 & ρ_2) on the higher and lower natural frequencies, respectively. The natural frequencies (higher & lower) of the simply supported, clamped, free, and cantilever beams decrease about 45% when the mass density of beam (ρ_1=ρ_2) changes from 1500 kg/m^3 to 5000 kg/m^3. Accordingly, the same effect of the change in the mass density of the beam results in the same effect on the higher and lower natural frequencies in all types of beams.

Figure 9 and Table 6 exhibit the effect of changing the length of the upper and lower beams (L_1 & L_2) on the higher natural frequencies. When the length of beam (L_1 =L_2) changes from 5 m to 14 m, the higher frequencies of simply supported beam decrease about 58%, the frequencies of clamped and free beams decrease about 75%, and the higher frequencies of cantilever beam decrease about 23%. The behavior of lower natural frequencies with change in length of the beam is shown in Figure 10 and Table 6. If the length of beam changes from 5 m to 14 m, the lower natural frequencies for the all types of beams decrease about 83%. Consequently, the length of the beam enlarges the effect on the higher natural frequencies of the clamped and free beams and makes the same effect on the lower natural frequencies of the all types of double beams.

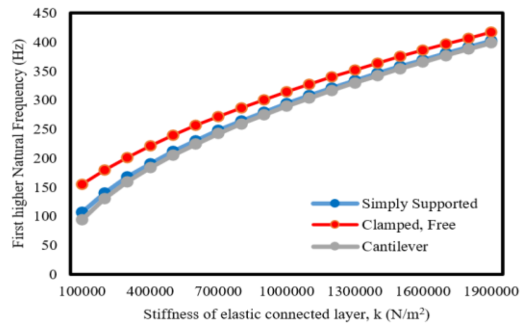

Figure 2. Higher natural frequencies versus stiffness of elastic layer

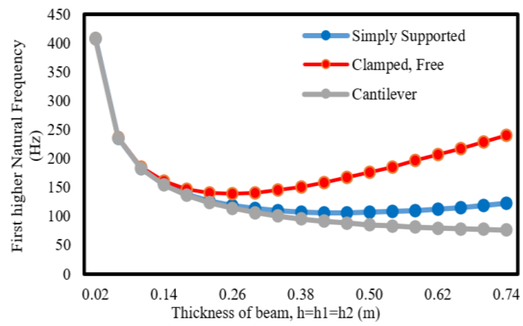

Figure 3. Higher natural frequencies versus thickness of double beam

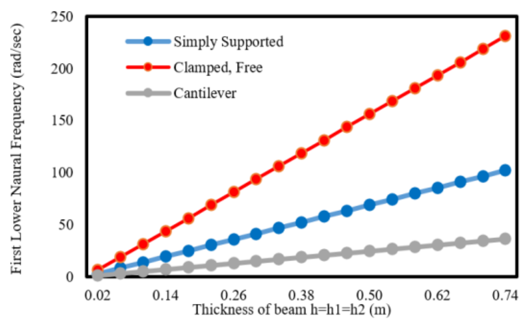

Figure 4. Lower frequencies versus thickness of beam

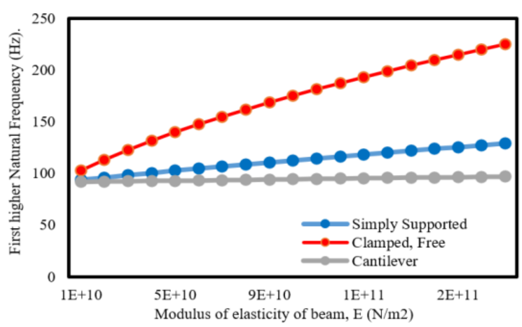

Figure 5. Higher natural frequencies versus modulus of elasticity of double beam

Table 1. A comparison test for present work with reference [1]

No. of mode	k=1×10⁵ N/m²		k=2×10⁵ N/m²		k=3×10⁵ N/m²		k=4×10⁵ N/m²		k=5×10⁵ N/m²	
	Present	Li & Sun [1]	Present	Li & Sun [1]	Present	Li & Sun [1]	Present	Li & Sun [1]	Present	Li & Sun [1]
1	19.739	19.74	19.739	19.74	19.739	19.74	19.739	19.74	19.739	19.74
2	48.884	48.88	66.254	66.25	78.957	78.94	78.957	78.96	78.957	78.96
3	78.957	78.96	78.957	78.96	79.935	79.96	91.595	91.59	101.930	101.93
4	90.742	90.74	101.164	101.16	110.608	110.61	119.307	119.31	127.413	127.41
5	177.653	177.65	177.653	177.65	177.653	177.65	177.653	177.65	177.653	177.65
6	183.195	183.20	188.575	188.58	193.805	193.81	198.898	198.90	203.864	203.86

Table 2. Higher natural frequencies (Hz) versus stiffness of elastic connected layer $E = 70 \times \frac{10^9 N}{m^2}, L = 10\ m, b = 0.02\ m, \rho = 3000 \frac{kg}{m^3}, h = 0.02\ m$

K *10⁵(N/m²)	Simply supported	Clamped, Free	Cantilever	K*10⁵(N/m²)	Simply supported	Clamped, Free	Cantilever
1	106.601	154.617	93.369	10	293.877	314.493	289.340
2	140.346	179.554	130.580	11	307.729	327.474	303.399
3	167.423	201.427	159.325	12	320.983	339.960	316.835
4	190.693	221.147	183.624	13	333.712	352.003	329.723
5	211.417	239.248	205.063	14	345.972	363.647	342.127
6	230.283	256.072	224.465	15	357.813	374.930	354.096
7	247.717	271.857	242.317	16	369.274	385.883	365.674
8	264.002	286.774	258.942	17	380.390	396.534	376.896
9	279.339	300.953	274.562	18	391.191	406.906	387.794

Table 3. Natural frequencies (Hz) versus thickness of beam $E = 70 \times 10^9 N/m^2, L = 10\ m, b = 0.02\ m, \rho = 3000\ kg/m^3, K = 10^5$

Thickness of beam, h (m)	Simply supported N.F.		Clamped, Free N.F.		Cantilever N.F.	
	Lower	Higher	Lower	Higher	Lower	Higher
0.020	2.753	408.258	6.240	408.296	0.980	408.249
0.060	8.258	235.847	18.719	236.444	2.941	235.721
0.100	13.763	183.092	31.198	185.221	4.902	182.640
0.140	19.268	155.502	43.677	160.366	6.863	154.456
0.180	24.773	138.319	56.156	147.214	8.824	136.369
0.220	30.278	126.761	68.636	140.934	10.785	123.563
0.260	35.783	118.747	81.115	139.284	12.746	113.943
0.300	41.288	113.207	93.594	140.964	14.707	106.430
0.340	46.793	109.515	106.073	145.105	16.668	100.408
0.380	52.298	107.271	118.553	151.085	18.629	95.493
0.420	57.803	106.196	131.032	158.448	20.590	91.435
0.460	63.308	106.086	143.511	166.859	22.551	88.062
0.500	68.813	106.779	155.990	176.067	24.512	85.250
0.540	74.318	108.148	168.469	185.889	26.472	82.907
0.580	79.823	110.085	180.949	196.187	28.433	80.967
0.620	85.328	112.504	193.428	206.859	30.394	79.374
0.660	90.833	115.330	205.907	217.826	32.355	78.086
0.700	96.338	118.502	218.386	229.029	34.316	77.068
0.740	101.843	121.969	230.865	240.423	36.277	76.292

Table 4. Natural frequencies (Hz) versus modulus of elasticity of double beam $L = 10\ m, b = 0.02\ m, h = 0.4\ m, \rho = 3000 \frac{kg}{m^3}, K = 10^5\ N/m^2$

Modulus of elasticity E N/m²	Simply supported N.F.		Clamped, Free		Cantilever	
	Lower	Higher	Lower	Higher	Lower	Higher
1E+10	20.806	93.628	47.166	102.752	7.411	91.587
2E+10	29.425	95.912	66.704	113.061	10.481	91.886
3E+10	36.038	98.143	81.695	122.505	12.837	92.185
4E+10	41.613	100.324	94.333	131.271	14.823	92.482
5E+10	46.525	102.459	105.468	139.488	16.572	92.779
6E+10	50.966	104.551	115.535	147.247	18.154	93.074
7E+10	55.050	106.601	124.792	154.617	19.609	93.369
8E+10	58.850	108.612	133.408	161.651	20.963	93.663
9E+10	62.420	110.588	141.501	168.392	22.234	93.955
1E+11	65.797	112.528	149.155	174.873	23.437	94.247
1.1E+11	69.008	114.435	156.435	181.122	24.581	94.538
1.2E+11	72.077	116.312	163.391	187.163	25.674	94.828
1.3E+11	75.020	118.158	170.063	193.014	26.722	95.118
1.4E+11	77.852	119.976	176.482	198.694	27.731	95.406

1.5E+11	80.585	121.7673	182.677	204.216128.7049695.69382
1.6E+11	83.22783	123.5322	188.668	209.592329.6463595.98041
1.7E+11	85.78928	125.2722	194.4745	214.834 30.5587696.26615
1.8E+11	88.27644	126.9884	200.1126219.950931.4447196.55104	
1.9E+11	90.69543	128.6818	205.5962224.951432.3063696.83509	

Table 5. Natural frequencies (Hz) versus mass density of double beam $E = 70 \times 10^9 \frac{N}{m^2}, L = 10\ m, b = 0.02\ m, h = 0.02\ m, K = 10^5 \frac{N}{m^2}$

Mass density of beam (ρ) kg/m³	Simply supported N.F.		Clamped, Free N.F.		Cantilever N.F.	
	Lower	Higher	Lower	Higher	Lower	Higher
1500	77.853	150.757	176.483	218.661	27.732	132.044
1750	72.077	139.574	163.391	202.441	25.674	122.249
2000	67.422	130.559	152.839	189.366	24.016	114.354
2250	63.566	123.093	144.098	178.536	22.643	107.814
2500	60.304	116.776	136.703	169.374	21.481	102.281
2750	57.498	111.341	130.341	161.492	20.481	97.521
3000	55.050	106.601	124.792	154.617	19.609	93.369
3250	52.890	102.419	119.896	148.551	18.840	89.706
3500	50.966	98.694	115.535	143.148	18.155	86.443
3750	49.238	95.347	111.617	138.294	17.539	83.512
4000	47.675	92.319	108.073	133.902	16.982	80.860
4250	46.251	89.563	104.846	129.904	16.475	78.446
4500	44.948	87.040	101.892	126.244	16.011	76.236
4750	43.749	84.718	99.175	122.877	15.584	74.203
5000	42.642	82.573	96.664	119.766	15.189	72.324
5250	41.614	80.583	94.334	116.879	14.823	70.581
5500	40.657	78.730	92.165	114.192	14.482	68.958
5750	39.763	77.000	90.139	111.682	14.164	67.442
6000	38.926	75.379	88.241	109.331	13.866	66.022

Table 6. Natural frequencies (Hz) versus length of double beam $E = 70 \times 10^9 \frac{N}{m^2}, h = 0.02\ m, b = 0.02\ m, \rho = 3000 \frac{kg}{m^3}, K = 10^5 \frac{N}{m^2}$

Length of beam, L (m)	Simply supported N.F.		Clamped, Free N.F.		Cantilever N.F.	
	Lower	Higher	Lower	Higher	Lower	Higher
5	220.200	238.372	499.169	507.447	78.437	120.356
5.5	181.984	203.596	412.536	422.515	64.824	111.962
6	152.917	178.092	346.645	358.463	54.470	106.303
6.5	130.296	159.092	295.366	309.151	46.412	102.408
7	112.347	144.759	254.678	270.544	40.019	99.674
7.5	97.867	133.833	221.853	239.900	34.861	97.717
8	86.016	125.427	194.988	215.299	30.639	96.292
8.5	76.194	118.907	172.723	195.362	27.141	95.236
9	67.963	113.808	154.064	179.079	24.209	94.443
9.5	60.997	109.791	138.274	165.689	21.728	93.837
10	55.050	106.601	124.792	154.617	19.609	93.369
10.5	49.932	104.051	113.190	145.414	17.786	93.004
11	45.496	101.996	103.134	137.731	16.206	92.714
11.5	41.626	100.330	94.361	131.291	14.827	92.483
12	38.229	98.969	86.661	125.871	13.618	92.297
12.5	35.232	97.850	79.867	121.293	12.550	92.146
13	32.574	96.925	73.841	117.413	11.603	92.022
13.5	30.206	96.155	68.473	114.114	10.760	91.919
14	28.087	95.510	63.669	111.297	10.005	91.834

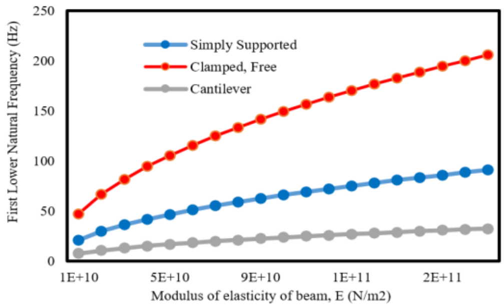

Figure 6. Lower natural frequencies versus modulus of elasticity of double beam

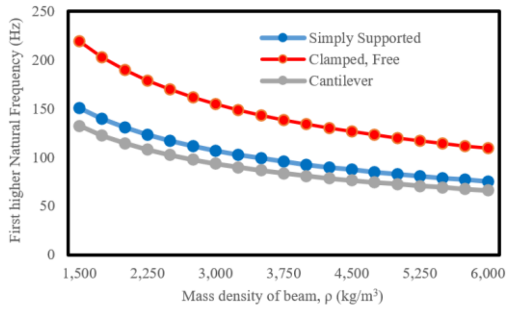

Figure 7. Higher natural frequencies versus mass density of double beam

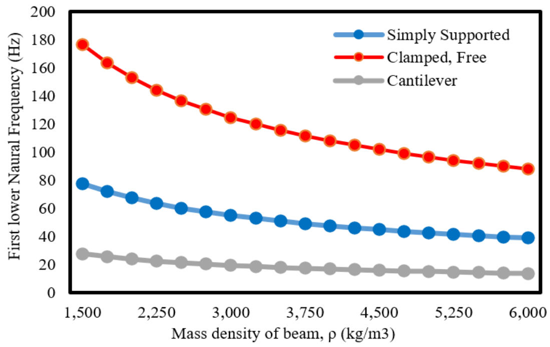

Figure 8. Lower natural frequencies versus mass density of double beam

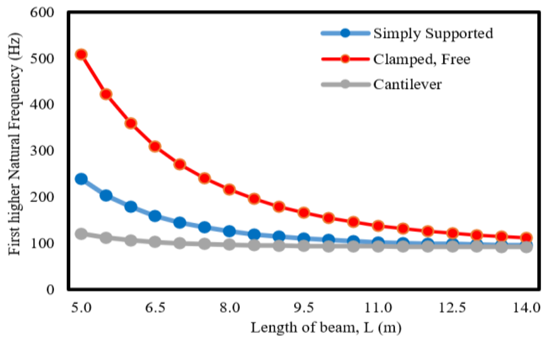

Figure 9. Higher natural frequencies versus length of double beam

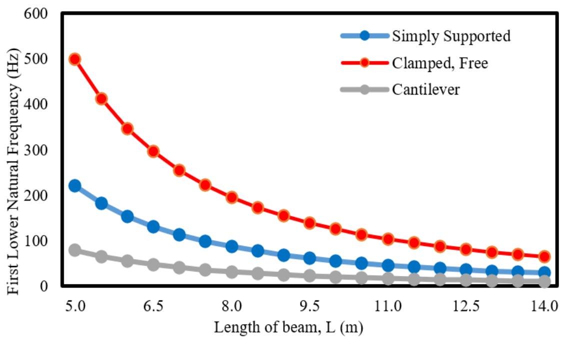

Figure 10. Lower natural frequencies versus length of double beam

4. CONCLUSIONS

In this paper, a good convergence in the results between the present work and the reference Li and Sun [1] is evinced. The great effect of elastic connected layer on the cantilever double beam, and this effect is less than that in the clamped and free double beams. The effect of the change in thickness on the higher natural frequencies (asynchronous) is higher in the simply supported, and the clamped, free double beams, but it generally causes an increase in those frequencies with the thickness increase in a cantilever beam. The increase in the thickness of upper and lower beams made a great increase in the values of lower natural frequencies in all types of beams. The change in the values of the elasticity modulus of double beam has a great effect on the lower natural frequencies but not as much as that effect on the higher frequencies, especially in the cantilever double beam. The higher and lower frequencies of the all types of beams decrease when the mass density of beam increases**.** The lower frequencies decrease with the increase in the beam length. The length of the beam enlarges the effect on the higher natural frequencies of clamped and free beams.

In all cases, the same behavior in all modes is recognized; therefore, only the first mode has been studied in this work, because there is no large difference in behavior occurring in 2nd, 3rd or nth mode.

In the future, it is possible to study the effect of the properties of the elastic connecting layer of asymmetric double beams.

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NOMENCLATURE

B	dimensionless heat source length
CP	specific heat, J. kg^{-1}. K^{-1}
g	gravitational acceleration, m.s^{-2}
k	thermal conductivity, W.m^{-1}. K^{-1}
Nu	local Nusselt number along the heat source

Greek symbols

α	thermal diffusivity, m^2. s^{-1}
β	thermal expansion coefficient, K^{-1}
ϕ	solid volume fraction
Θ	dimensionless temperature
μ	dynamic viscosity, kg. m^{-1}.s^{-1}

Subscripts

p	nanoparticle
f	fluid (pure water)
nf	nanofluid