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Analysis of an N-policy $M^X/M/1$ Two-phase Queueing System with State-dependent Arrival Rates and Unreliable Server

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ABSTRACT

The aim of this paper is to analyze the performance of the two-phase MX/M/1 queueing mechanism. The other conditions of the queueing system under study are state-dependent arrival rates, N-policy, unreliable server and delayed repair. A single server provide service in two stages. The first stage is batch service and the second one is individual to each customer in the batch. The client's arrival rate depends upon the state of the server. We developed the steady state equations. Probability generating functions were used to solve the equations. The expected size of the queue while the server is at different states are derived. Cost function has been developed to determine the optimal threshold of N. Sensitivity analysis is presented to study the effect of the system parameters at the threshold of N for the geometric batch size distribution. The findings of this research help in designing two-phase queueing systems that occur in telecommunication networks, production etc. at a low cost.

1. INTRODUCTION

In many queueing systems that are observed in telecommunication networks, production systems etc. the arrival rates are not constant, but depends on the state of the server. Various authors have studied queueing problems with state dependent arrival rates under different queue control policies.

One of the critical works done towards this path was performed by Yechiali and Naor [1]. Haris and Marchal [2] studied the M/G/1 queue with server vacations whose distributions can be state dependent. Yijun and Quanlin [3] examined the two-stage queueing system with state dependent vacation policy. They derived the stationary distributions of the queue length and the cycle time for the closed state-dependent vacation model.

Kumar and Chandan [4], analyzed the performance of two-phase $M^X/E_K/1$ queuing system with server startups and N-policy. They derived optimal threshold value of N by computing average queue size of the system and average cost considering three batch size distributions. Al Hanbali, & Boxma [5] studied Busy period analysis of the state dependent M/M/1/K queue. Vasanta Kumar et al. [8, 9] examined the performance of two-phase $M^X/M/1$ and $M^X/E_k/1$ queueing systems with N-policy and unreliable server, respectively.

Madhu Jain and Agarval [6] dealt with a state dependent batch arrival queueing system with modified Bernoulli vacation under N-policy. They proposed a method to find the optimal value of the threshold parameter to minimize the total expected cost. Singh et al. [10] considered a single server, state dependent Poisson arrival system. They used supplementary variable technique to obtain probability generating function of number of units in the system. In addition, some special cases are also provided. Charan Jeet Singh and Binay Kumar [11] investigated a batch arrival queueing system with unreliable server, state dependent arrival rates and two stages of heterogeneous service. They studied the transient and steady-state behaviour of the queue length distribution.

Banik [7] assessed state-dependent arrival in GI/BMSP/1/∞ queue. Rashmita [12] evaluated $M^X/G/1$ queueing model. They considered the state-dependent server vacation and derived the explicit expressions for the system size. Charan Jeet Singh et al. [15] investigated a single repairable server queueing system with bulk input and state dependent rates considering the general distributions for the repair, delay to repair and service processes.

Hanumantha Rao et al. [13] studied two-phase M/M/1 queueing system with server breakdowns, delayed repair, and impatient customers. They found expected loss due to balking and reneging. Numerical illustrations are presented to support the model. Recently, Hanumantha Rao et al. [14] examined the M/M/1 two -phase queueing system with state dependent arrival rates under N policy.

As observed from the review of literature two-phase $M^X/M/1$ queueing system with state dependent arrival rates, N-policy and unreliable server has not been studied so far. Thus the present study is aimed at the analysis of this queueing system. The remainder of this paper is organized as follows: Section 2 describes the model and its assumptions. Section 3 describes the analysis under steady state, Section 4 presents the performance analysis of the system, Section 5 describes the cost function and the optimal operating policy, Section 6 describes the sensitivity analysis and summary is presented in Section 7.

2. DESCRIPTION OF PROPOSED MODEL AND UNDERLYING ASSUMPTIONS

In the present research we examine the performance analysis of $M^X/M/1$ queueing system. The queueing system is considered with two phases of service, state-dependent arrival rates, server breakdowns and delayed repair under N-policy.

Notation symbols used in this paper are presented below: Notation symbols used in the present paper are presented below.

λ_1: Arrival rate while idle or startup states

λ_2: Arrival rate while batch and individual services

λ_3: Arrival rate while breakdown and delay states

θ: Startup rate

β: Batch service rate

μ: Individual service rate

α_1: Breakdown rate while batch service

α_2: Breakdown rate while individual service

δ: Delay rate while batch and individual services

γ: Repair rate while batch and individual services

W_V, Ws, W_b, W_{bb},, W_{db}, W_i, W_{bi}, W_{db}: Average length of vacation time, startup time, first phase service time, delay period during first phase service, waiting time for repair during first phase service, second phase service time, during second phase service, waiting time for repair during second phase service, and the cost parameter notations are taken from our previous publication [14].

Assumptions

The first assumption of present queueing model is that a single server provides with two-phases of service. The first stage of service is batch service while the second one is individual.

The customers arrive into the queue in batches of size 'X' according to Poisson process. The queueing model has state-dependent arrival rates, as given in the notation. The queue discipline is FCFS.

After providing batch service, the server proceeds to the second phase. In the second phase individual customers in the entire batch are served.

The service times for the batch stage and individual stage are exponentially distributed with parameters β and μ.

After offering individual service to the customers, the server returns to the batch service queue and serve the newly joined customers to the queue. After finishing batch service for the waiting customers, the server proceeds to individual service.

At the moment of system being unoccupied, the server turns off. The server will turn on as and when the number of arrivals in the queue meet a preset threshold 'N'. However, on return the server is not available momentarily to restart the service to the clients in waiting. At this moment it calls for a startup time. The startup time of the server follows a negative exponential distribution with average $1/\theta$. On fulfillment of startup time requirement, the server starts serving the clients in batch.

During the preservice and batch service, the new customers are permitted to be included in the ongoing batch of service.

The server in the system which may face breakdown at whatsoever moment follows Poisson breakdown with parameters α_1 for initial phase of batch service and α_2 for the second phase of individual serve. At whatever point the server breaks down, the server is sent for repairing and it cannot accomplish the service till it gets repaired. The delayed time and repair time are considered to be negative exponentially distributed with means $1/\delta$ and$1/\gamma$.

When there is server failure during service, the customers in process and in queue need to wait till the server available to complete the service. The customers are allowed to join the queue even during the delay time and repair time.

3. ANALYSIS

In the current paper, the following notations have been utilized as below:

$\Pi_{0,m,0}$=Steady state probability when m customers are in the batch queue and the server is on vacation state, m= 0,1, 2, ...(N-1).

$\Pi_{1,m,0}$=Steady state probability when m customers are in the batch queue while the server is in startup state, where m = N, N+1, N+2, …

$\Pi_{2,m,0}$=Steady state probability when m customers are in the batch which is in batch service state, m = 1, 2, 3…

$\Pi_{3,m,0}$= Steady state probability when m customers are in the batch which is in batch service, while the server is found to be broken down and waiting for repair state, m = 1, 2, 3, …

$\Pi_{4,m,0}$= Steady state probability when m customers are in the batch which is in batch service, while the server is undergoing repair, m = 1, 2, 3, …

$\Pi_{5,m,n}$= Steady state probability when m customers are in the batch queue service and n customers in the individual service while the server is in individual service state, m = 0, 1, 2…, and n = 1, 2, 3, …

$\Pi_{6,m,n}$= Steady state probability when m customers are in the batch service and n customers in the individual service state, while the server is in individual service but found to be broken down and waiting for repair, m = 0,1,2 …, and n = 1,2,3….

$\Pi_{7,m,n}$ = Steady state probability when m customers are in the batch service and n customers in the individual service queue while the server is in individual service, but undergoing repair, m =0, 1, 2…, and n =1, 2, 3….

The steady state equations for the queue length distribution

$$\lambda_1 \Pi_{0,0,0} = \mu \Pi_{5,0,1}. \quad (1)$$

$$\lambda_1 \Pi_{0,m,0} = \lambda_1 \textstyle\sum_{l=1}^{m} a_l \Pi_{0,m-l,0}, 1 \le m \le (N-1). \quad (2)$$

$$(\lambda_1 + \theta)\Pi_{1,N,0} = \lambda_1 \textstyle\sum_{l=1}^{N} a_l \Pi_{0,N-l,0}. \quad (3)$$

$$(\lambda_1 + \theta)\Pi_{1,m,0} = \lambda_1 \textstyle\sum_{l=1}^{m-N} a_l \Pi_{1,m-l,0} + \lambda_1 \textstyle\sum_{l=m-(N-1)}^{m} a_l \Pi_{0,m-l,0}, \ m \ge N+1. \quad (4)$$

$$(\lambda_2 + \beta + \alpha_1)\Pi_{2,m,0} = \lambda_2 \textstyle\sum_{l=1}^{m} a_l \Pi_{2,m-l,0} + \mu\Pi_{5,m,1} + \gamma\Pi_{4,m,0}, 1 \le m \le (N-1). \quad (5)$$

$$(\lambda_2 + \beta + \alpha_1)\Pi_{2,m,0} = \lambda_2 \textstyle\sum_{l=1}^{m} a_l \Pi_{2,m-l,0} + \mu\Pi_{5,m,1} + \gamma\Pi_{4,m,0} + \theta\Pi_{1,m,0}, m \ge N \quad (6)$$

$$(\lambda_3 + \delta)\Pi_{3,1,0} = \alpha_1 \Pi_{2,1,0}. \quad (7)$$

$$(\lambda_3 + \delta)\Pi_{3,m,0} = \alpha_1 \Pi_{2,m,0} + \lambda_3 \textstyle\sum_{l=1}^{m} a_l \Pi_{3,m-l,0}, \quad m > 1 \quad (8)$$

$$(\lambda_3 + \gamma)\Pi_{4,1,0} = \delta\Pi_{3,1,0}. \quad (9)$$

$$(\lambda_3 + \gamma)\Pi_{4,m,0} = \delta\Pi_{3,m,0} + \lambda_3 \textstyle\sum_{l=1}^{m} a_l \Pi_{3,m-l,0}, \ m > 1 \quad (10)$$

$$(\lambda_2 + \alpha_2 + \mu)\Pi_{5,0,n} = \mu\, \Pi_{5,0,n+1} + \beta\Pi_{2,n,0} + \gamma\Pi_{7,0,n}, n \ge 1 \quad (11)$$

$$(\lambda_2+\alpha_2+\mu)\Pi_{5,m,n} = \lambda_2\sum_{l=1}^{m} a_l\,\Pi_{5,m-l,n} + \mu\,\Pi_{5,m,n+1} + \gamma\Pi_{7,m,n},\ \ m\geq 1, \& n\geq 1. \quad (12)$$

$$(\lambda_3+\delta)\Pi_{6,0,n} = \alpha_2\Pi_{5,0,n}, n\geq 1. \quad (13)$$

$$(\lambda_3+\delta)\Pi_{6,m,n} = \alpha_2\Pi_{5,m,n} + \lambda_3\sum_{l=1}^{m} a_l\,\Pi_{6,m-l,n},\ \ m\geq 1, \& n\geq 1 \quad (14)$$

$$(\lambda_3+\gamma)\Pi_{7,0,n} = \delta\Pi_{6,0,n}, n\geq 1 \quad (15)$$

$$(\lambda_3+\gamma)\Pi_{7,m,n} = \delta\Pi_{6,m,n} + \lambda_3\sum_{l=1}^{m} a_l\,\Pi_{7,m-l,n}, m\geq \& n\geq 1 \quad (16)$$

In the next step PGF's (Probability generating functions) of queue size at an arbitrary time epoch are derived for different states of the system. Prior to that PGF's are defined below:

$$F_0(s) = \sum_{m=0}^{\infty}\Pi_{0,m,0}s^m,\ F_1(s) = \sum_{m=0}^{\infty}\Pi_{1,m,0}s^m,\ F_2(s) = \sum_{m=0}^{\infty}\Pi_{2,m,0}s^m,$$

$$F_3(s) = \sum_{m=0}^{\infty}\Pi_{3,m,0}s^m\ F_4(s) = \sum_{m=0}^{\infty}\Pi_{4,m,0}s^m, F_5(s,y) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\Pi_{5,m,n}s^m y^n,$$

$$F_6(s,y) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\Pi_{6,m,n}s^m y^n, F_7(s,y) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\Pi_{7,m,n}s^m y^n,$$

$$S_n(s) = \sum_{m=0}^{\infty}\Pi_{6,m,n}s^m, T_n(s) = \sum_{m=0}^{\infty}\Pi_{7,m,n}s^m, \text{and} R_n(s) = \sum_{m=0}^{\infty}\Pi_{5,m,n}s^m, |s|\leq 1, |y|\leq 1.$$

Let $B(s) = \sum_{m=1}^{\infty} a_m s^m$ be the probability generating function of the arrival batch size random variable X and $B'(s), B''(s)$ represents the first and second order derivatives of $B(s)$ respectively.

From equation (1) to (16), using the PGFs, we will get $F_0(s) = \Pi_{0,0,0}Y_N(s)$,
where

$$Y_N(s) = \sum_{m=0}^{N-1} y_m s^m, Y_N(1) = \sum_{m=0}^{N-1} y_m \& Y_N'(1) = \sum_{m=1}^{N-1} m\, y_m. \quad (17)$$

$$[\lambda_1(1-B(s)+\theta]F_1(s) = \lambda_1\Pi_{0,0,0} + \lambda_1(B(s)-1)F_0(s) \quad (18)$$

$$[\lambda_2(1-B(s)+\beta+\alpha_1]F_2(s) = \mu R_1(s) + \theta F_1(s) - \lambda_1\Pi_{0,0,0} + \gamma F_4(s). \quad (19)$$

$$[\lambda_3(1-B(s)+\delta]F_3(s) = \alpha_1 F_2(s). \quad (20)$$

$$[\lambda_3(1-B(s)+\gamma]F_4(s) = \delta F_3(s). \quad (21)$$

$$[\lambda_2(1-B(s)+\alpha_2+\mu]R_n(s) = \mu R_{n+1}(s) + \gamma T_n(s) + \beta\Pi_{2,n,0,} \quad (22)$$

$$[\lambda_2 y(1-B(s)+\alpha_2 y+\mu(y-1)]F_5(s,y) = -\mu y R_1(s) + \gamma y F_7(s,y) + \beta y F_2(y). \quad (23)$$

$$[\lambda_3(1-B(s)+\delta]S_n(s) = \alpha_2 R_n(s), \quad (24)$$

$$[\lambda_3(1-B(s)+\delta]F_6(s,y) = \alpha_2 F_5(s,y). \quad (25)$$

$$[\lambda_3(1-B(s)+\gamma]T_n(s) = \delta\, S_n(s), \quad (26)$$

$$[\lambda_3(1-B(s)+\gamma]F_7(s,y) = \delta F_6(s,y). \quad (27)$$

Put the value of $F_6(s,y)$ in equation (27) we obtain

$$[\lambda_3(1-B(s)+\gamma]F_7(s,y) = \frac{\delta\alpha_2 F_5(s,y)}{[\lambda_3(1-A(s)+\delta]} \quad (28)$$

Put y = s in Eq. (23), we get

$$[\lambda_2 s(1-B(s)) + \alpha_2 s + \mu(s-1)]F_5(s,s) = -\mu s R_1(s) + \gamma s F_7(s,s) + \beta s F_2(s).$$

Put the values of $F_2(s)$ and $F_7(s,s)$ obtained through equations (19) and (28) respectively, we get

$$[(\lambda_2 s(1-B(s)+\alpha_2 s+\mu(s-1)(\lambda_3(1-B(s)+\delta)(\lambda_3(1-B(s)+\gamma) - \gamma\delta\alpha_2 s]F_5(s,s) = \left(\beta s F_2(s) - \mu s R_1(s)\right)[\lambda_3(1-B(s)+\gamma][\lambda_3(1-B(s)+\delta]. \quad (29)$$

Put s=1 and y=1 in Eqns. (17), (18), (19), (20), (21), (25), (28), and (29), we get

$$F_0(1) = Y_N(1)\Pi_{0,0,0}. \quad (30)$$

$$F_1(1) = \frac{\lambda_1\Pi_{0,0,0}}{\theta} \quad (31)$$

$$F_2(1) = \frac{\mu}{\beta}R_1(1) \quad (32)$$

$$F_3(1) = \frac{\alpha_1}{\delta}F_2(1) \quad (33)$$

$$F_4(1) = \frac{\alpha_1}{\gamma}F_2(1) \quad (34)$$

$$F_5(1,1) = \frac{\frac{\lambda_2 A'(1)}{\mu}\left(1+\frac{\lambda_3}{\lambda_2}\left(\frac{\alpha_1}{\delta}+\frac{\alpha_1}{\gamma}\right)\right)F_2(1) + \frac{\theta}{\mu}F_1'(1)}{1-\frac{\lambda_2 A'(1)}{\mu}\left(1+\frac{\lambda_3}{\lambda_2}\left(\frac{\alpha_2}{\delta}+\frac{\alpha_2}{\gamma}\right)\right)} \quad (35)$$

$$F_6(1,1) = \frac{\alpha_2}{\delta}F_5(1,1) \quad (36)$$

$$F_7(1,1) = \frac{\alpha_2}{\gamma}F_5(1,1) \quad (37)$$

Thequeue length distribution is given by

$$F(s,s) = F_0(s) + F_1(s) + F_2(s) + F_3(s) + F_4(s) + F_5(s,s) + F_6(s,s) + F_7(s,s).$$

Probability that the server is not doing any service is given by

$$F_0(1) + F_1(1) = 1 - \frac{\lambda_2 B'(1)}{\beta} - \frac{\lambda_3 B'(1)}{\beta}\left(\frac{\alpha_1}{\gamma}+\frac{\alpha_1}{\delta}\right) - \frac{\lambda_2 B'(1)}{\mu} - \frac{\lambda_3 B'(1)}{\mu}\left(\frac{\alpha_1}{\gamma}+\frac{\alpha_1}{\delta}\right).$$

This gives

$$\left(\frac{\lambda_1+\theta Y_N(1)}{\theta}\right)\Pi_{0,0,0} = 1-\rho_1-\rho_2 \qquad (38)$$

where
$$\rho_1 = \frac{\lambda_2 B'(1)}{\beta}\left(1+\frac{\lambda_3}{\lambda_2}\left(\frac{\alpha_1}{\gamma}+\frac{\alpha_1}{\delta}\right)\right),$$
$$\rho_2 = \frac{\lambda_2 B'(1)}{\mu}\left(1+\frac{\lambda_3}{\lambda_2}\left(\frac{\alpha_2}{\gamma}+\frac{\alpha_2}{\delta}\right)\right),$$

and $\rho=\rho_1+\rho_2$ is the utilizing factor of the system.

Hence, $\Pi_{0,0,0}=(1-\rho)\frac{\theta}{(\lambda_1+\theta Y_N(1))}$.

The Normalizing condition is

$$F(1,1) = F_0(1)+F_1(1)+F_2(1)+F_3(1)+F_4(1)+F_5(1,1)+F_6(1,1)+F_7(1,1)=1. \qquad (39)$$

Using the normalizing condition, we get

$$R_1(1) = \frac{\beta\left[(\rho_1+\rho_2)(1-\rho_2)\mu-\left(1+\frac{\alpha_2}{\gamma}+\frac{\alpha_2}{\delta}\right)\theta F_1'(1)\right]}{\mu\left[\left(1+\frac{\alpha_1}{\gamma}+\frac{\alpha_1}{\delta}\right)\mu+(\alpha_2-\alpha_1)\left(\frac{1}{\gamma}+\frac{1}{\delta}\right)(\lambda_2-\lambda_3)B'(1)\right]}. \qquad (40)$$

Let $F(1,1)=1$ in the forms $\lim_{y\to 1}F(1,y)=1$ and $\lim_{s\to 1}F(s,1)=1$we find

$$R_1'(1) = \frac{\rho_2}{(1-\rho_2)}\left[\left(\frac{\lambda_2 B'(1)}{\mu}+\frac{\lambda_3 B'(1)}{\beta}\left(\frac{\alpha_1}{\delta}+\frac{\alpha_1}{\gamma}\right)\right)F_2(1)+\frac{\lambda_1 B'(1)}{\mu}(1-\rho_1-\rho_2)\right]. \qquad (41)$$

4. PERFORMANCE ANALYSIS OF THE SYSTEM

In the present segment, authors evaluated the mathematical expressions for average size of the system at distinct states of the server:

- The average size of the system while the server is on vacation state is

$$L_v = \sum_{m=1}^{N-1} m\Pi_{0,m,0} = F_0'(1) = Y_N'(1)\Pi_{0,0,0}. \qquad (42)$$

- The average size of the system while the server is in startup state is

$$L_s = \sum_{m=N}^{\infty} m\Pi_{0,m,0} = F_1'(1) = \frac{\lambda_1 B'(1)(\lambda_1+\theta Y_N(1))}{\theta^2}\Pi_{0,0,0} \qquad (43)$$

- The average size of the system while the server is offering in first of batch service

$$L_b = \sum_{m=1}^{N-1} m\Pi_{2,m,0} = F_2'(1) = \frac{{\lambda_2}^2\left(B'(1)\right)^2}{\beta^2}\left(1+\frac{\lambda_3}{\lambda_2}\left(\frac{\alpha_1}{\delta}+\frac{\alpha_1}{\gamma}\right)\right)+\frac{\mu}{\beta}R_1'(1)+\frac{\theta}{\beta}F_1'(1), \qquad (44)$$

- The average size of the system while the server is waiting for repair during service

$$L_{bb} = \sum_{m=1}^{\infty} m\Pi_{3,m,0} = F_3'(1) = \frac{\alpha_1\lambda_3 B'(1)}{\delta^2}F_2(1)+\frac{\alpha_1}{\delta}F_2'(1) \qquad (45)$$

- The average size of the system while the server is in under repair during batch service,

$$L_{db} = \sum_{m=1}^{N-1} m\Pi_{4,m,0} = F_4'(1) = \frac{\lambda_3 B'(1)\alpha_1}{\gamma^2}F_2(1)+\frac{\alpha_1\lambda_3 B'(1)}{\gamma\delta}F_2(1)+\frac{\alpha_1}{\gamma}F_2'(1), \qquad (46)$$

- The average size of the system while the server is offering individual service state

$$\begin{aligned}L_m &= \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}(m+n)\Pi_{5,m,n} = F_5'(1,1)\\ &= \frac{\lambda_3 B'(1)}{\mu(1-\rho_2)}\left[\frac{\lambda_2}{\lambda_3}+\left(\mu+\alpha_2-\lambda_2 B'(1)\right)\left(\frac{1}{\delta}+\frac{1}{\gamma}\right)+\frac{A''(1)}{2A'(1)}\left(\frac{\lambda_2}{\lambda_3}+\frac{\alpha_2}{\delta}+\frac{\alpha_2}{\gamma}\right)-\frac{\lambda_3 A'(1)\alpha_2}{\gamma\delta}\right]F_5(1,1)\\ &+\frac{\lambda_3 B'(1)}{\mu(1-\rho_2)}\left[\left(\frac{\lambda_2}{\lambda_3}+\frac{\alpha_1}{\delta}+\frac{\alpha_2}{\gamma}\right)\left(F_2(1)+F_2'(1)\right)\lambda_3 B'(1)\left(\frac{\alpha_1}{\gamma^2}+\frac{\alpha_1}{\gamma\delta}+\frac{\alpha_2}{\delta^2}\right)F_2(1)+\right.\\ &\frac{B''(1)}{2A'(1)}\left(\frac{\lambda_2}{\lambda_3}+\left(\frac{\alpha_1}{\delta}+\frac{\alpha_1}{\gamma}\right)\right)F_2(1)\frac{\lambda_1^2 B''(1)}{2\lambda_3}(1-\rho_1-\rho_2)\frac{\lambda_1^2 B'(1)}{\lambda_3}(1-\rho_1-\rho_2)\\ &\left.+\frac{\lambda_1^2 B'(1)}{\lambda_3}(1-\rho_1-\rho_2)+\frac{\lambda_1}{\lambda_3}(1-\rho_1-\rho_2)+\frac{\lambda_1}{\lambda_3}Y_N'(1)\Pi_{0,0,0}\right]\\ &-\frac{\lambda_3 B'(1)}{\mu(1-\rho_2)}\left(\frac{1}{\delta}+\frac{1}{\gamma}\right)\left[\lambda_3 B'(1)\left(\frac{\lambda_2}{\lambda_3}+\frac{\alpha_1}{\delta}+\frac{\alpha_1}{\gamma}\right)F_2(1)+\lambda_1 B'(1)(1-\rho_1-\rho_2)\right] \qquad (47)\end{aligned}$$

- The average size of the system while the server is in waiting for repair in second phase of service

$$L_{bi} = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}(m+n)\Pi_{6,m,n} = F_6'(1,1) = \frac{\lambda_3 B'(1)\alpha_2}{\delta^2}F_5(1,1)+\frac{\alpha_2}{\delta}F_5'(1,1) \qquad (48)$$

- The average size of the system while the server is under repair in second phase service

$$L_{di} = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}(m+n)\Pi_{7,m,n} = F_7'(1,1) = \frac{\lambda_3 B'(1)\alpha_2}{\gamma}\left(\frac{1}{\gamma}+\frac{1}{\delta}\right)+\frac{\alpha_2}{\gamma}F_5'(1,1) \qquad (49)$$

The average quantity of clients in the system is given by

$$\begin{aligned}L(N) &= F_0'(1)+F_1'(1)+F_2'(1)+F_3'(1)+F_4'(1)+F_5'(1,1)+F_6'(1,1)+F_7'(1,1).\\ &= Y_N'(1)\Pi_{0,0,0}+\lambda_1 B'(1)\frac{\left(\lambda_1+\theta Y_N(1)\right)}{\theta^2}\Pi_{0,0,0}\\ &+\left(1+\frac{\alpha_1}{\gamma}+\frac{\alpha_1}{\delta}\right)F_2'(1)\\ &+\lambda_3 B'(1)\left(\frac{\alpha_1}{\delta^2}+\frac{\alpha_1}{\delta\gamma}+\frac{\alpha_1}{\gamma^2}\right)F_2(1)\\ &+\lambda_3 B'(1)\left(\frac{\alpha_2}{\delta^2}+\frac{\alpha_2}{\delta\gamma}+\frac{\alpha_2}{\gamma^2}\right)F_5(1,1)\\ &+\left(1+\frac{\alpha_2}{\delta}+\frac{\alpha_2}{\gamma}\right)F_5'(1,1). \qquad (50)\end{aligned}$$

Then the average length of a busy cycle is given by

$$W_c = W_v+W_s+W_b+W_{bb}+W_{db}+W_i+W_{bi}+W_{di}. \qquad (51)$$

The long run fractions of time, the server is in distant states are given below:

- The fraction of time the server is in vacation state

$$\frac{W_v}{W_c} = \Pi_v = Y_N(1)\Pi_{0,0,0} \quad (52)$$

- The fraction of time the server is in startup state

$$\frac{W_s}{W_c} = \Pi_s = \frac{\lambda_1 \Pi_{0,0,0}}{\theta}, \quad (53)$$

- The fraction of time the server is in first phase service state

$$\frac{W_b}{W_c} = \Pi_b = \frac{\lambda_2 B'(1)}{\beta}, \quad (54)$$

- The fraction of time the server is in delay period during first phase service state

$$\frac{W_{bb}}{W_c} = \Pi_{bb} = \frac{\alpha_1}{\delta} F_2(1) \quad (55)$$

- The fraction of time the server is in waiting time for repair during first phase service state

$$\frac{W_{db}}{W_c} = \Pi_{db} = \frac{\alpha_1}{\gamma} F_2(1) \quad (56)$$

- The fraction of time the server is in second phase service state

$$\frac{W_i}{W_c} = \Pi_i = \frac{\frac{\lambda_2 B'(1)}{\mu}\left(1+\frac{\lambda_3}{\lambda_2}\left(\frac{\alpha_1}{\delta}+\frac{\alpha_1}{\gamma}\right)\right)F_2(1)+\frac{\theta}{\mu}F_1'(1)}{1-\frac{\lambda_2 B'(1)}{\mu}\left(1+\frac{\lambda_3}{\lambda_2}\left(\frac{\alpha_2}{\delta}+\frac{\alpha_2}{\gamma}\right)\right)} \quad (57)$$

- The fraction of time the server is in delay state during second phase service and

$$\frac{W_{bi}}{W_c} = \Pi_{bi} = \frac{\alpha_2}{\delta} F_5(1,1), \text{ and} \quad (58)$$

- The fraction of time the server is in waiting time for repair state during second phase service respectively.

$$\frac{W_{di}}{W_c} = \Pi_{di} = \frac{\alpha_2}{\gamma} F_5(1,1) \quad (59)$$

- The expected length of vacation period $W_V = \frac{Y_N(1)}{\lambda_1 B'(1)}$. Substituting this in equation (52), we get $W_C = \frac{1}{\lambda_1 B'(1)\Pi_{0,0,0}}$.

5. COST FUNCTION AND OPTIMAL OPERATING POLICY

System managers are keen on limiting the total expenditure. For this reason, we establish the cost function in terms of adequate performance measures and interrelated cost components to ascertain the optimal threshold parameters.

Let C_A (N) be the average cost per unit of time. Then

$$C_A(N) = C_h L(N) + C_o\left(\frac{W_b}{W_c}+\frac{W_i}{W_c}\right) + C_m\left(\frac{W_s}{W_c}\right) + C_b\left(\frac{W_{bb}+W_{db}+W_{bi}+W_{di}}{W_c}\right) + C_s\left(\frac{1}{W_c}\right) - C_r\left(\frac{W_v}{W_c}\right) \quad (60)$$

Neglecting the terms independent of N in C_A(N), from (61) we get the new cost function

$$T_A(N) = \left(\frac{\lambda_1 B'(1)}{\mu(1-\rho_2)}+1\right)Y_N'(1)\Pi_{0,0,0}\frac{C_h}{k} + C_m\frac{\lambda_1}{\theta}\Pi_{0,0,0} + \lambda_1\Pi_{0,0,0}C_s - C_r Y_N(1)\Pi_{0,0,0}$$

$$= \frac{1}{(1-\rho_2)}\left[(\lambda_1 B'(1)+\mu(1-\rho_2))Y_N'(1)\frac{C_h}{k} + \mu(1-\rho_2)\left(C_m\frac{\lambda_1}{\theta}+\lambda_1 C_s - C_r Y_N(1)\right)\right]\Pi_{0,0,0}$$

$$= \left[(\lambda_1 B'(1)+\mu(1-\rho_2))Y_N'(1)\frac{C_h}{k} + \mu(1-\rho_2)\left(C_m\frac{\lambda_1}{\theta}+\lambda_1 C_s - Y_N(1)C_r\right)\right]\left(\frac{\theta}{\lambda_1+\theta Y_N(1)}\right)(1-\rho_1-\rho_2). \quad (61)$$

Subsequently for determination of the superior operating N-policy, minimizing C_A(N) in (60) is equivale to minimizing $T_A(N)$ in (61).

It is tight to prove that TA(N) is convex but now we presented a technique to determine the optimal threshold N*.

Result

Utilizing the long run expected average cost criterion, the optimal threshold N* is given by

$$N^* = \min\left\{k \ge 1 / \left(\sum_{n=0}^{k-1}(k-n)y_n + \frac{k\lambda_1}{\theta}\right) > \frac{\lambda_1}{C_h}\left(\frac{C_m+C_r}{\theta}+C_s\right)(1-\rho_2)\right\}. \quad (62)$$

Proof: Let $J(k) = \sum_{m=1}^{k} y_m + \frac{\lambda_1}{\theta}$ and $I(k)=\sum_{m=1}^{k} m y_m$.

Consider the following difference

$$\Delta T_A(k) = T_A(k+1) - T_A(k) = \frac{MY_k}{(1-\rho_2)}\frac{H(k)}{J(k)J(k-1)}.$$

where Δ is the difference operator, and

$$H(k) = C_h(k\,J(k) - I(k)) - \lambda_1\left(\frac{C_m+C_r}{\theta}+C_s\right)(1-\rho_2)$$

$$= C_h\left[\sum_{n=0}^{k-1}(k-n)\,y_k + \frac{k\lambda_1}{\theta}\right] - \lambda_1\left(\frac{C_m+C_r}{\theta}+C_s\right)(1-\rho_2) \quad (63)$$

By definition, $C_h\left[\sum_{n=0}^{k-1}(k-n)\,y_k + \frac{k\lambda_1}{\theta}\right] > 0$ and

$$\frac{MY_k}{(1-\rho_2)J(k)J(k-1)} > 0$$

Then it follows that $\Delta T_A(k) > 0$.

Thus, the sign of H(k) determines whether T_A(N) increases or decreases,

Let m be the first k such that $H(k) > 0$, then we have

$$H(m+1) = C_h[(m+1)\,J(m+1) - P(m+1)] - \lambda_1\left(\frac{C_m+C_r}{\theta}+C_s\right)(1-\rho_2) = H(m) + C_h P(m).$$

It follows that H(m+1)> H(m).

Hence for some n>m, we have $T_A(n) > T_A(m)$.

Let N^* be the optimal value of N, which minimizes $T_A(N)$, Then from equation (63) we have

$$N^{*}=\min\left\{k \geq 1 / \left(\sum_{n=0}^{k-1}(k-n)y_n + \frac{k\lambda_1}{\theta}\right) > \frac{\lambda_1}{C_h}\left(\frac{C_m+C_r}{\theta} + C_s\right)(1-\rho_2)\right\} \quad (64)$$

therefore, the premier threshold of N may be evaluated from equation (64), through selecting the pleasant value of k, that is one of the integers surrounding 'N'.

Also, note that if $\frac{C_h}{\left(\frac{C_m+C_r}{\theta}+C_s\right)} > \lambda_1(1-\rho_2)$, the optimal threshold N^*should be 1.

To carry out the sensitivity analysis, we assume that the arrival batch size follows geometric distribution. Then $b_k = P(X=k) = p(1-p)^{k-1},\ 0 < p < 1, k = 1,2,\ldots$ with probability generating function $B(s) = \frac{1-p}{(1-ps)}$. $E(X) = B'(1)= \frac{1}{p}$ and $E(X(X-1)) = B''(1)= \frac{2(1-p)}{p^2}$. Then

$$L(N) = Y_N'(1)\Pi_{0,0,0} + \frac{\lambda_1}{p}\frac{(\lambda_1+\theta Y_N(1))}{\theta^2}\Pi_{0,0,0} + \left(1+\frac{\alpha_1}{\gamma}+\frac{\alpha_1}{\delta}\right)F_2'(1) + \frac{\lambda_3}{p}\left(\frac{\alpha_1}{\delta^2}+\frac{\alpha_1}{\delta\gamma}+\frac{\alpha_1}{\gamma^2}\right)F_2(1) + \frac{\lambda_3}{p}\left(\frac{\alpha_2}{\delta^2}+\frac{\alpha_2}{\delta\gamma}+\frac{\alpha_2}{\gamma^2}\right)F_5(1,1) + \left(1+\frac{\alpha_2}{\delta}+\frac{\alpha_2}{\gamma}\right)F_5'(1,1).$$

where, $\Pi_{0,0,0} = (1-\rho_1-\rho_2)\left(\frac{\theta}{\lambda_1+\theta Y_N(1)}\right)$, $,\rho_1 = \frac{\lambda_2}{p\beta}\left(1+\frac{\lambda_3}{\lambda_2}\left(\frac{\alpha_1}{\gamma}+\frac{\alpha_1}{\delta}\right)\right)$, and $\rho_2 = \frac{\lambda_2}{p\mu}\left(1+\frac{\lambda_3}{\lambda_2}\left(\frac{\alpha_2}{\delta}+\frac{\alpha_2}{\gamma}\right)\right)$.

6. SENSITIVITY ANALYSIS

In this section, authorsdemonstrated thenumerical illustrations of the model to study the variations in various performance measures with respect to some parameters. The illustrations presented in ensuing section show the analytical outcomes acquired and display a way to reach to a decision.

The most suitable threshold N*, mean number of jobs inside the system and minimal expected cost arediscovered for a targeted range of values of λ_1, λ_2, λ_3, μ, β, θ,γ, δ, α_1, α_2, C_h, C_o, C_m, C_b, C_r, and C_s. Let us we ussume the λ_1=0.1, λ_2=0.9, λ_3=0.5, μ=15, β=25, θ=2,γ=2, δ=1, α_1=0.2, α_2=0.5, C_h=20, C_o=50, C_m=35, C_b=40, C_r=30, and C_s=30, and m=3.

From Table 1, it can be observed that

(i) N^* shows enhancing trend with rise in the values of λ_1, diminishing trend for increase in λ_2, and is no significant with increase in the values of λ_3,

(ii) both $L(N^*)$ and $T(N^*)$ increases with rise in the values of λ_1,λ_2, andλ_3.

From Table 2, we can concludeas

(i) N^* shows increasing trend with increase in the values of μ, and is insensitive with increase in the values of β and θ.

(ii) (N^*) and $T(N^*)$ decrease for increase in the values of μ and β, and decreases slightly with increase in the values of θ.

Table 1. Influence of ($\lambda_1, \lambda_2, \lambda_3$) on N^*, $L(N^*)$, and $T(N^*)$

λ_1	N^*	$L(N^*)$	$T(N^*)$
0.1	4	18.62	481.35
0.2	6	18.68	485.71
0.3	7	18.71	489.29
0.4	8	18.77	492.65
0.5	9	18.87	496.04
0.6	9	18.89	499.62
λ_2	N^*	$L(N^*)$	$T(N^*)$
0.9	10	7.61	273.31
1	10	8.42	291.08
1.1	10	9.46	312.64
1.3	9	12.47	375.26
1.4	9	15.07	424.42
1.6	8	24.77	609.85
λ_3	N^*	$L(N^*)$	$T(N^*)$
0.5	9	8.22	282.65
0.6	9	9.45	307.38
0.7	9	11.01	338.72
0.8	9	13	378.64
0.9	9	15.56	429.81
1	9	18.87	496.04

Table 2. Influence of (μ, β, θ) on N^*, $L(N^*)$, and $T(N^*)$

μ	N^*	$L(N^*)$	$T(N^*)$
13	8	29.7	708.24
15	9	18.97	496.04
17	9	14.18	406.75
20	9	10.92	342.41
25	10	8.39	295.54
30	10	7.21	273.78
β	N^*	$L(N^*)$	$T(N^*)$
20	9	20.29	526.75
24	9	19.1	501.09
28	9	18.39	483.19
32	9	17.66	469.99
36	9	17.19	459.87
40	9	16.82	451.86
θ	N^*	$L(N^*)$	$T(N^*)$
1.3	9	18.93	496.93
1.4	9	18.92	496.74
1.5	9	18.91	496.58
1.6	9	18.9	496.44
1.7	9	18.89	496.32
1.8	9	18.88	496.22

Table 3. Influence of (α_1, α_2) on N^*, $L(N^*)$, and $T(N^*)$

α_1	N*	$L(N^*)$	$T(N^*)$
0.1	9	23.48	621.08
0.2	9	18.87	496.04
0.3	9	16.55	432.25
0.4	9	15.16	393.32
0.5	9	14.24	366.92
0.6	9	13.6	347.73
α_2	N*	$L(N^*)$	$T(N^*)$
0.2	10	9.47	320.97
0.3	10	11.65	360.1
0.4	9	14.52	415.18
0.5	9	18.87	496.04
0.6	8	25.43	622.1
0.7	7	36.54	837.68

From Table 3, one can conclude as

(i) N^* is insensitive with enhance in the values of α_1,and shows decreasing trend with increase in the values of α_2,

(ii) Both $L(N^*)$ and $T(N^*)$ decrease for rise in the values of α_1, and

(iii) with enhance in the values of α_2, both $L(N^*)$ and $T(N^*)$ increase.

Table 4. Influence of (C_b, C_s, C_m) on N^*, $L(N^*)$, and $T(N^*)$

C_b	N*	L(N*)	T(N*)
25	9	18.87	484.78
30	9	18.87	498.54
35	9	18.87	492.29
40	9	18.87	496.04
45	9	18.87	499.79
50	9	18.87	503.55
C_s	N*	L(N*)	T(N*)
250	7	18.67	486.68
300	8	18.77	490.06
400	9	18.87	496.04
500	10	18.97	501.42
600	11	19.07	506.34
700	12	19.17	510.88
C_m	N*	L(N*)	T(N*)
25	9	18.87	495.95
30	9	18.87	495.99
35	9	18.87	496.04
40	9	18.87	496.09
45	9	18.87	496.13
50	9	18.87	496.17

From Table 4, it is observed that,

(i) N^* and $L(N^*)$ are insensitive with enhance in the values of C_b and C_m, and increases with rise in the values of C_s, and

(ii) $T(N^*)$rises for enhance in the values of C_b and C_s, and $T(N^*)$ increases slightly with rise in the values of C_m.

Table 5. Influence of (C_h, C_o, C_r) on N^*, $L(N^*)$, and $T(N^*)$

C_h	N*	L(N*)	T(N*)
15	11	19.07	401.4
17	10	18.97	439.35
21	9	18.87	514.91
23	8	18.77	552.5
31	7	18.67	702.19
33	7	18.67	739.53
C_o	N*	L(N*)	T(N*)
25	9	18.67	460.69
30	9	18.67	467.76
35	9	18.67	474.83
40	9	18.67	481.98
45	9	18.67	488.97
50	9	18.67	496.04
C_r	N*	L(N*)	T(N*)
15	9	18.87	498.78
20	9	18.87	498.1
25	9	18.87	497.41
30	9	18.87	496.73
35	9	18.87	496.04
40	9	18.87	495.36

From Table 5, it observed that,

(i) $N^*$and $L(N^*)$ show diminishing trend with rise in the values of C_h, and are no significant with enhance in the values of C_o and C_r, and

(ii) $T(N^*)$rises with enhance in the values of C_h and C_o, and diminishes with increase in the values of C_r.

7. SUMMARY

In the present paper, we investigated some important performance measures of the two-phase $M^X/M/1$ queueing system with state-dependent arrival rates, server startup and unreliable server. The average cost function per unit time is formulated to conclude the best threshold of N. Impact of the system parameters on N, mean system size and minimum cost are studied via numerical values. This work can be generalized, considering the general distributions for the batch size, service time, startup time and repair times.

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