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Optimality conditions and duality for nondifferentiable multiobjective programming

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ABSTRACT

In this paper, we study a class of nonsmooth multiobjective optimization problem including inequality constraints. To the aim, some new functions named (pseudo, quasi) invex of order σ $(B,\varphi)-V-$type II and strongly (quasi, pseudo) invex of order σ $(B,\varphi)-V-$type II are introduced by using the tools of Clarke subdifferential. These new functions are used to derive and prove the sufficient optimality condition for a strict minimizer of the multiobjective programming problems. Moreover, the corresponding duality theorems are formulated for general Mond-Weir type dual program.

1. INTRODUCTION

The multiobjective programming is an extension of mathematical programming where a scalar value objective function is replaced by a vector function. In 1970, Rockafellar [1] wrote in his book that practical applications are not necessarily differentiable in applied mathematics. So, considering the nondifferentiable mathematical programming problems was very important. The Clarke subdifferential [2] (also called the Clarke generalized gradient) is an important tool to derive sufficient conditions for nonsmooth optimization problems. The optimality conditions for the weak efficient solution, efficient solution and properly efficient solution and the duality results in multiobjective programming problems have attracted many researchers in recent years. For example, we can see in [3-10]. Recently, many researchers have been interested in other types of solution concepts, one of them is higher order strict minimizer. We can see in [11-13]. In particular, Kwan Deok Bae and Do Sang Kim [14] obtained necessary and sufficient optimality conditions for a nonsmooth multiobjective optimization problems and dual results are given for a strict minimizer of order m. Izhar Ahmad and Suliman Al-Homidan [15] obtained several sufficient optimality conditions and duality theorems for a strict minimizer of a nondifferentiable multiobjective programming problem under strong invexity and its generalizations of order σ.

In this paper, first, we consider the nonsmooth multiobjective programming and introduce the new functions named (pseudo, quasi) invex of order σ $(B,\varphi)-V-$type II and strongly (quasi, pseudo) invex of order σ $(B,\varphi)-V-$type II. Then, a sufficient optimality condition is obtained for the nondifferentiable multiobjective programming problem under the new fuctions. Finally, we formulate a Mond-Weir type dual problem and obtain weak and strong duality theorems.

2. NOTATIONS AND PRELIMINARIES

Let R^n be the $n-$dimensional Euclidean space and let X be a nonempty open subset of R^n. For $x=(x_1,x_2,\cdots,x_n)^T$, $y=(y_1,y_2,\cdots,y_n)^T\in R^n$, we denote:

$x=y \iff x_i=y_i,\ i=1,2,\cdots,n;$

$x\leqq y \iff x_i\le y_i,\ i=1,2,\cdots,n;$

$x\le y \iff x_i\leqq y_i,\ i=1,2,\cdots,n$ and $x\ne y$;

$x<y \iff x_i<y_i,\ i=1,2,\cdots,n;$

$x\in R_+^n \iff x\geqq 0.$

Definition2.1 [1]. The function $f:X\to R$ is said to be locally Lipschitz at $x\in X$, if there exist scalars $k>0$ and $\varepsilon>0$, such that

$$\left|f(y)-f(z)\right|\leqq k\left\|y-z\right\|,\ \text{ for all } y,z\in B(x,\varepsilon). \tag{1}$$

where $B(x,\varepsilon)$ is the open ball of radius ε about x.

Definition2.2 [1]. The generalized directional derivative of a locally Lipschitz function f at x in the direction d, denoted by $f^0(x;d)$, is as follows:

$$f^0(x;d)=\limsup_{\substack{y\to x\\ \lambda\to 0^+}}\frac{f(y+\lambda d)-f(y)}{\lambda}. \tag{2}$$

Definition2.3 [1]. The generalized gradient of $f:X\to R$ at $x\in X$, denoted by $\partial f(x)$, is defined as follows:

$$\partial f(x)=\{\xi\in R^n: f^0(x;d)\ge\langle\xi,d\rangle,\forall d\in R^n\}. \tag{3}$$

where $\langle\cdot,\cdot\rangle$ is the inner product in R^n.

Consider the following nonsmooth multiobjective programming problem:

(MP) $\quad$ Minimize $f(x)=(f_1(x), f_2(x),\cdots, f_k(x)),$
$\quad$ $s.t.\ g_j(x)\leqq 0, j=1,2,\cdots,m,$
$\quad$ $x\in X.$

where $f_i : X\to R\ (i\in K=\{1,2,\cdots,k\})$ and $g_j : X\to R$ $(j\in M=\{1,2,\cdots,m\})$ are locally Lipschitz functions and X is a convex set in R^n.

Let $X_0=\{x \mid g_j(x)\leqq 0, j\in M\}$ be the set of feasible solutions of (MP).

Definition2.4. A point $\bar{x}\in X_0$ is a local strict minimizer of order σ for (MP) with respect to a nonlinear function $\psi : X\times X\to R^n$, if for a constant $\rho\in \operatorname{int} R_+^k$, there exists no $x\in B(\bar{x},\varepsilon)\cap X_0$, such that

$$f(x)<f(\bar{x})+\rho\|\psi(x,\bar{x})\|^{\sigma} \quad . \tag{4}$$

Definition2.5. A point $\bar{x}\in X_0$ is a strict minimizer of order σ for (MP) with respect to a nonlinear function $\psi : X\times X\to R^n$, if for a constant $\rho\in \operatorname{int} R_+^k$, there exists no $x\in X_0$, such that

$$f(x)<f(\bar{x})+\rho\|\psi(x,\bar{x})\|^{\sigma} \quad . \tag{5}$$

Throughout the paper, we suppose that $\eta : X\times X\to R^n$; $b_0,b_1 : X\times X\to R_+$ $\varphi_0,\varphi_1 : R\to R$; $\alpha,\beta : X\times X \to R_+\setminus\{0\}$; $\rho_i,\tau\in R_+, i\in K$.

Definition2.6. (f,g) is said to be (pseudo, quasi) invex of order σ $(B,\varphi)-V-$type II at $\bar{x}\in X$, if there exist $\eta,\ b_0,\ b_1,\ \varphi_0,\ \varphi_1,\ \alpha,\ \beta,\ \rho_i(i\in K),\ \tau$ and some vectors $\lambda\in R_+^k$ and $\mu\in R_+^m$ such that for all $x\in X$ the following inequalities hold:

$$\begin{aligned}&\alpha_i(x,\bar{x})\left\langle \sum_{i=1}^k \lambda_i\xi_i, \eta(x,\bar{x})\right\rangle \geqq 0, \forall \xi_i\in\partial f_i(\bar{x}), i\in K\\ &\Rightarrow b_0(x,\bar{x})\varphi_0[\sum_{i=1}^k \lambda_i(f_i(x)-f_i(\bar{x})-\rho_i\|\psi(x,\bar{x})\|^{\sigma})]\geqq 0,\end{aligned} \tag{6}$$

$$\begin{aligned}&-b_1(x,\bar{x})\varphi_1[\sum_{j=1}^m \mu_j g_j(\bar{x})]\leqq 0\\ &\Rightarrow \beta(x,\bar{x})\left\langle \sum_{j=1}^m \mu_j\zeta_j, \eta(x,\bar{x})\right\rangle + \tau\|\psi(x,\bar{x})\|^{\sigma}\leqq 0,\\ &\forall \zeta_j\in\partial g_j(\bar{x}), j\in M.\end{aligned} \tag{7}$$

Definition2.7. (f,g) is said to be strongly (quasi, pseudo) invex of order σ $(B,\varphi)-V-$type II at $\bar{x}\in X$, if there exist $\eta,\ b_0,\ b_1,\ \varphi_0,\ \varphi_1,\ \alpha,\ \beta,\ \rho_i(i\in K),\ \tau$ and some vectors $\lambda\in R_+^k$ and $\mu\in R_+^m$ such that for all $x\in X$ the following inequalities hold

$$\begin{aligned}&b_0(x,\bar{x})\varphi_0[\sum_{i=1}^k \lambda_i(f_i(x)-f_i(\bar{x})-\rho_i\|\psi(x,\bar{x})\|^{\sigma})]\leqq 0\\ &\Rightarrow \alpha(x,\bar{x})\left\langle \sum_{i=1}^k \lambda_i\xi_i, \eta(x,\bar{x})\right\rangle \leqq 0, \forall \xi_i\in\partial f_i(\bar{x}), i\in K,\end{aligned} \tag{8}$$

$$\begin{aligned}&\beta(x,\bar{x})\left\langle \sum_{j=1}^m \mu_j\zeta_j, \eta(x,\bar{x})\right\rangle + \tau\|\psi(x,\bar{x})\|^{\sigma}\geqq 0\\ &\Rightarrow -b_1(x,\bar{x})\varphi_1[\sum_{j=1}^m \mu_j g_j(\bar{x})]\geqq 0, \forall \zeta_j\in\partial g_j(\bar{x}), j\in M.\end{aligned} \tag{9}$$

Remark2.1. If the second inequality in Eq (6) is strict (wherever $x\neq\bar{x}$), we say that (f,g) is (strictly pseudo, quasi) invex of order σ $(B,\varphi)-V-$type II at $\bar{x}\in X$; If the second inequality in Eq (7) is strict (wherever $x\neq\bar{x}$), we say that (f,g) is (pseudo, strictly quasi) invex of order σ $(B,\varphi)-V-$type II at $\bar{x}\in X$.

Remark2.2. If the second inequality in Eq (8) is strict (wherever $x\neq\bar{x}$), we say that (f,g) is (strictly quasi, pseudo) invex of order σ $(B,\varphi)-V-$type II at $\bar{x}\in X$; If the second inequality in Eq (9) is strict (wherever $x\neq\bar{x}$), we say that (f,g) is (quasi, strictly pseudo) invex of order σ $(B,\varphi)-V-$type II at $\bar{x}\in X$.

3. OPTIMALITY CONDITION

In this section, we establish sufficient optimality conditions for a strict minimizer of (MP).

Theorem 3.1. Let $\bar{x}\in X_0$. Suppose that

(i) There exists $\bar{\lambda}_i\geq 0$, $\sum_{i=1}^k\bar{\lambda}_i=1, \bar{\mu}_j\geqq 0, i\in K, j\in M$, such that for $\sum_{i=1}^k\bar{\lambda}_i=1, i\in K$, $\bar{\mu}_j\geqq 0, j\in M$ such that for $\bar{x}\in X_0$

$$0\in\sum_{i=1}^k\bar{\lambda}_i\partial f_i(\bar{x})+\sum_{j=1}^m\bar{\mu}_j\partial g_j(\bar{x}),$$
$$\bar{\mu}_j g_j(\bar{x})=0,\ j\in M,$$

(ii) (f,g) is (pseudo, quasi) invex of order σ $(B,\varphi)-V-$type II at $\bar{x}\in X$,

(iii) $\begin{aligned}&b_0(x,\bar{x})\geqq 0,\ b_1(x,\bar{x})\geqq 0;\\ &\varphi_0(a)\geqq 0\Rightarrow a\geqq 0, a=0\Rightarrow\varphi_1(a)\geqq 0\end{aligned}$.

Then $\bar{x}$ is a strict minimizer of order σ for (MP).

Proof: Suppose that $\bar{x}$ is not a strict minimizer of order σ for (MP). Then there exists $x\in X_0$ and $\rho_i\in R_+,\ i\in K$ such that

$$f_i(x)<f_i(\bar{x})+\rho_i\|\psi(x,\bar{x})\|^{\sigma}, i\in K. \tag{10}$$

Using $\bar{\lambda}_i\geq 0,\ \sum_{i=1}^k\bar{\lambda}_i=1$, which follows

$$\sum_{i=1}^{k}\bar{\lambda}_i(f_i(x)-f_i(\bar{x})-\rho_i\|\psi(x,\bar{x})\|^{\sigma})<0. \quad (11)$$

By hypothesis (i), we have

$$\sum_{j=1}^{m}\bar{\mu}_j g_j(\bar{x})=0. \quad (12)$$

Using hypothesis (iii), we get

$$-b_1(x,\bar{x})\varphi_1(\sum_{j=1}^{m}\bar{\mu}_j g_j(\bar{x}))\leqq 0. \quad (13)$$

From hypothesis (ii), the above inequality implies

$$\beta(x,\bar{x})\left\langle\sum_{j=1}^{m}\bar{\mu}_j\zeta_j,\eta(x,\bar{x})\right\rangle+\tau\|\psi(x,\bar{x})\|^{\sigma}\leqq 0, \quad \forall\zeta_j\in\partial g_j(\bar{x}),\ j\in M \quad (14)$$

That is

$$\left\langle\sum_{j=1}^{m}\bar{\mu}_j\zeta_j,\eta(x,\bar{x})\right\rangle\leqq-\frac{\tau}{\beta(x,\bar{x})}\|\psi(x,\bar{x})\|^{\sigma}\leqq 0. \quad (15)$$

The hypothesis (i) follows that there exist $\xi_i\in\partial f_i(\bar{x}), i\in K$ and $\zeta_j\in\partial g_j(\bar{x}), j\in M$

$$\sum_{i=1}^{k}\bar{\lambda}_i\xi_i+\sum_{j=1}^{m}\bar{\mu}_j\zeta_j=0. \quad (16)$$

Which together with the inequality (15), we obtain

$$\left\langle\sum_{i=1}^{k}\bar{\lambda}_i\xi_i,\ \eta(x,\bar{x})\right\rangle\geqq 0. \quad (17)$$

That is

$$\alpha(x,\bar{x})\left\langle\sum_{i=1}^{k}\bar{\lambda}_i\xi_i,\ \eta(x,\bar{x})\right\rangle\geqq 0. \quad (18)$$

By the hypothesis (ii), the above inequality yields

$$b_0(x,\bar{x})\varphi_0[\sum_{i=1}^{k}\bar{\lambda}_i(f_i(x)-f_i(\bar{x})-\rho_i\|\psi(x,\bar{x})\|^{\sigma})]\geqq 0 \quad (19)$$

For $b_0(x,\bar{x})\geqq 0,\ \ \varphi_0(a)\geqq 0\Rightarrow a\geqq 0$, which implies

$$\sum_{i=1}^{k}\bar{\lambda}_i(f_i(x)-f_i(\bar{x})-\rho_i\|\psi(x,\bar{x})\|^{\sigma})\geqq 0. \quad (20)$$

Which contradicts (11). Hence the result is true.

4. MOND-WEIR DUALITY

For the primal problem (MP), we formulate the following Mond-Weir type dual problem:

(MD) Maximize $f(u)=(f_1(u), f_2(u),\cdots f_k(u))$

subject to $0\in\sum_{i=1}^{k}\lambda_i\partial f_i(u)+\sum_{j=1}^{m}\mu_j\partial g_j(u),$

$\sum_{j=1}^{m}\mu_j g_j(u)\geqq 0$

$\lambda_i\geqq 0,\ \sum_{i=1}^{k}\lambda_i=1,\quad i\in K,$

$\mu_j\geqq 0,\ j\in M.$

Let

$U=\{(u,\ \lambda,\ \mu)\in X\times R_+^{\ k}\times R_+^{\ m}\ \Big|\ 0\in\sum_{i=1}^{k}\lambda_i\partial f_i(u)+\sum_{j=1}^{m}\mu_j\partial g_j(u),$ $\sum_{j=1}^{m}\mu_j g_j(u)\geqq 0,\ \lambda\geq 0,\ \sum_{i=1}^{k}\lambda_i=1, \mu\geqq 0\}$ be the set of all feasible solutions in problem (MD).

Theorem 4.1.(weak duality) Let x and (u,λ,μ) be feasible solutions for (MP) and (MD), respectively. Moreover, assume that

(i) (f,g) is (pseudo, quasi) invex of order σ $(B,\varphi)-V-$type II at u,

(ii) $b_0(x,u)>0,\ b_1(x,u)\geqq 0;$ $a<0\Rightarrow\varphi_0(a)<0, a\geqq 0\Rightarrow\varphi_1(a)\geqq 0$.

Then the following can not hold:

$$f_i(x)<f_i(u), i\in K\ . \quad (21)$$

Proof: Suppose contrary to the result that $f_i(x)<f_i(u), i\in K$ hold.

For $\rho_i\geqq 0, i\in K$, which implies

$$f_i(x)<f_i(u)+\rho_i\|\psi(x,u)\|^{\sigma},\ i\in K. \quad (22)$$

Using $\lambda_i\geq 0,\ \sum_{i=1}^{k}\lambda_i=1,\quad i\in K$ together with $a<0\Rightarrow\varphi_0(a)<0$ and $b_0(x,u)>0$, the above inequality follows

$$b_0(x,u)\varphi_0[\sum_{i=1}^{k}\lambda_i(f_i(x)-f_i(u)-\rho_i\|\psi(x,u)\|^{\sigma})]<0. \quad (23)$$

By the constraint condition of (MD) and the assumption (ii), we have

$$-b_1(x,u)\varphi_1(\sum_{j=1}^{m}\mu_j g_j(u))\leqq 0\ . \quad (24)$$

With the assumption (i), the Eq (24) yield

$$\beta(x,u)\left\langle \sum_{j=1}^{m}\mu_j\zeta_j,\eta(x,u)\right\rangle+\tau\|\psi(x,u)\|^{\sigma}\leqq 0, \quad (25)$$
$$\forall\zeta_j\in\partial g_j(u),\ j\in M.$$

That is

$$\left\langle \sum_{j=1}^{m}\mu_j\zeta_j,\eta(x,u)\right\rangle\leqq-\frac{\tau}{\beta(x,u)}\|\psi(x,u)\|^{\sigma} \quad (26)$$
$$\leqq 0,\forall\zeta_j\in\partial g_j(u),\ j\in M.$$

Using the feasibility of (u,λ,μ) in (MD), we get

$$\sum_{i=1}^{k}\lambda_i\xi_i+\sum_{j=1}^{m}\mu_j\zeta_j=0,\ \xi_i\in\partial f_i(u),i\in K, \quad (27)$$
$$\zeta_j\in\partial g_j(u),\ j\in M$$

Also we have

$$\left\langle \sum_{i=1}^{k}\lambda_i\xi_i+\sum_{j=1}^{m}\mu_j\zeta_j,\ \eta(x,u)\right\rangle=0\,. \quad (28)$$

For Eq (26) and $\alpha(x,u)>0$, the above follows

$$\alpha(x,u)\left\langle \sum_{i=1}^{k}\lambda_i\xi_i,\ \eta(x,u)\right\rangle\geqq 0\,. \quad (29)$$

From the assumption (i), Eq (29) yields

$$b_0(x,u)\varphi_0[\sum_{i=1}^{k}\lambda_i(f_i(x)-f_i(u)-\rho_i\|\psi(x,u)\|^{\sigma})]\geqq 0\,. \quad (30)$$

which contradicts Eq (23). Then the result is true.

Theorem 4.2.(weak duality) Let x and (u,λ,μ) be feasible solutions for (MP) and (MD), respectively. Moreover, assume that

(i) (f,g) is (strictly pseudo, quasi) invex of order σ $(B,\varphi)-V-$type II at u,

(ii) $b_0(x,u)\geqq 0,\ b_1(x,u)\geqq 0;$ $a<0\Rightarrow\varphi_0(a)<0, a\geqq 0\Rightarrow\varphi_1(a)\geqq 0$.

Then the following can not hold: $f_i(x)<f_i(u), i\in K$.

Proof: the proof is similar to the proof of Theorem 4.1.

Theorem 4.3.(weak duality) Let x and (u,λ,μ) be feasible solutions for (MP) and (MD), respectively. Moreover, assume that

(i) (f,g) is (pseudo, strictly quasi) invex of order σ $(B,\varphi)-V-$type II at u,

(ii) $b_0(x,u)>0,\ b_1(x,u)\geqq 0;$ $a<0\Rightarrow\varphi_0(a)<0, a\geqq 0\Rightarrow\varphi_1(a)\geqq 0$.

Then the following can not hold:

$$f_i(x)<f_i(u), i\in K\,. \quad (31)$$

Proof: Suppose contrary to the result that $f_i(x)<f_i(u), i\in K$ hold.

For $\rho_i\geqq 0, i\in K$, which implies

$$f_i(x)<f_i(u)+\rho_i\|\psi(x,u)\|^{\sigma},\ i\in K\,. \quad (32)$$

For $\lambda_i\geq 0,\ \sum_{i=1}^{k}\lambda_i=1,\quad i\in K$, with $a<0\Rightarrow\varphi_0(a)<0$ and $b_0(x,u)>0$, the above inequality implies

$$b_0(x,u)\varphi_0[\sum_{i=1}^{k}\lambda_i(f_i(x)-f_i(u)-\rho_i\|\psi(x,u)\|^{\sigma})]<0\,. \quad (33)$$

From the feasibility of (u,λ,μ) in (MD) with $b_1(x,u)\geqq 0$ and $a\geqq 0\Rightarrow\varphi_1(a)\geqq 0$, we obtain

$$b_1(x,u)\varphi_1(\sum_{j=1}^{m}\mu_j g_j(u))\geqq 0\,. \quad (34)$$

By the assumption (i), the Eq (33) and Eq (34) yield

$$\alpha(x,u)\left\langle \sum_{i=1}^{k}\lambda_i\xi_i,\ \eta(x,u)\right\rangle<0,\ \xi_i\in\partial f_i(u),i\in K\,. \quad (35)$$

$$\beta(x,u)\left\langle \sum_{j=1}^{m}\mu_j\zeta_j,\eta(x,u)\right\rangle+\tau\|\psi(x,u)\|^{\sigma}<0, \quad (36)$$
$$\forall\zeta_j\in\partial g_j(u),\ j\in M.$$

That is

$$\left\langle \sum_{i=1}^{k}\lambda_i\xi_i+\sum_{j=1}^{m}\mu_j\zeta_j,\ \eta(x,u)\right\rangle<0\,. \quad (37)$$

On the other hand, by using the constraint condition of (MD), there exist $\xi_i\in\partial f_i(u), i\in K$ and $\zeta_j\in\partial g_j(u), j\in M$ such that

$$\sum_{i=1}^{k}\lambda_i\xi_i+\sum_{j=1}^{m}\mu_j\zeta_j=0\,. \quad (38)$$

Also,

$$\left\langle \sum_{i=1}^{k}\lambda_i\xi_i+\sum_{j=1}^{m}\mu_j\zeta_j,\ \eta(x,u)\right\rangle=0\,. \quad (39)$$

which contradicts Eq (37). Then the result is true.

Theorem 4.4.(weak duality) Let x and (u,λ,μ) be feasible solutions for (MP) and (MD), respectively. Moreover, assume that

(i) (f,g) is (quasi, pseudo) invex of order σ $(B,\varphi)-V-$type II at u,

(ii) $b_0(x,u)>0,\ b_1(x,u)>0;$ $\varphi_0(a)>0\Rightarrow a>0, a\geqq 0\Rightarrow\varphi_1(a)>0$.

Then the following can not hold:

$$f_i(x)<f_i(u), i\in K\,. \quad (40)$$

Proof: Suppose contrary to the result that $f_i(x) < f_i(u), i \in K$ hold.

For $\rho_i \geqq 0, i \in K$, which implies

$$f_i(x) < f_i(u) + \rho_i \|\psi(x,u)\|^\sigma, \ i \in K. \tag{41}$$

For $\lambda_i \geq 0, \ \sum_{i=1}^{k} \lambda_i = 1, \ i \in K$, the above inequality implies

$$\sum_{i=1}^{k} \lambda_i (f_i(x) - f_i(u) - \rho_i \|\psi(x,u)\|^\sigma) < 0. \tag{42}$$

Since (u, λ, μ) is a feasible solution for (MD) with $b_1(x,u) > 0$ and $a \geqq 0 \Rightarrow \varphi_1(a) > 0$, we have

$$-b_1(x,u)\varphi_1(\sum_{j=1}^{m} \mu_j g_j(u)) < 0. \tag{43}$$

By the assumption (i), the above inequality yields

$$\beta(x,u)\left\langle \sum_{j=1}^{m} \mu_j \zeta_j, \eta(x,u) \right\rangle + \tau \|\psi(x,u)\|^\sigma < 0, \quad \forall \zeta_j \in \partial g_j(u), j \in M. \tag{44}$$

Also

$$\left\langle \sum_{j=1}^{m} \mu_j \zeta_j, \ \eta(x,u) \right\rangle < 0, \forall \zeta_j \in \partial g_j(u), j \in M. \tag{45}$$

From the constraint condition of (MD), there exist $\xi_i \in \partial f_i(u), i \in K$ and $\zeta_j \in \partial g_j(u), \ j \in M$ such that

$$\sum_{i=1}^{k} \lambda_i \xi_i + \sum_{j=1}^{m} \mu_j \zeta_j = 0. \tag{46}$$

That is

$$\left\langle \sum_{i=1}^{k} \lambda_i \xi_i + \sum_{j=1}^{m} \mu_j \zeta_j, \ \eta(x,u) \right\rangle = 0. \tag{47}$$

The above together with Eq (45), we get

$$\left\langle \sum_{i=1}^{k} \lambda_i \xi_i, \ \eta(x,u) \right\rangle > 0, \ \xi_i \in \partial f_i(u), i \in K. \tag{48}$$

Also,

$$\alpha(x,u)\left\langle \sum_{i=1}^{k} \lambda_i \xi_i, \ \eta(x,u) \right\rangle > 0, \ \xi_i \in \partial f_i(u), i \in K. \tag{49}$$

From the hypothesis (i), the above inequality follows

$$b_0(x,u)\varphi_0[\sum_{i=1}^{k} \lambda_i (f_i(x) - f_i(u) - \rho_i \|\psi(x,u)\|^\sigma)] > 0. \tag{50}$$

which contradicts Eq (42). Then the result is true.

Definition 4.1. A point $u \in X$ is a strict minimizer of order σ for (MD) with respect to a nonlinear function $\psi : X \times X \to R^n$, if there exists a constant $\rho \in \operatorname{int} R_+^k$ such that no $x \in X_0$, such that

$$f_i(u) + \rho_i \|\psi(x,\bar{x})\|^\sigma \geq f_i(x), \ \forall x \in X, i \in K. \tag{51}$$

Theorem 4.5. (strong duality) Assume that $\bar{x}$ is a strict minimizer of order σ with respect to ψ for (MP), Suppose that there exist $\bar{\lambda} \geq 0, \sum_{i=1}^{k} \bar{\lambda}_i = 1$ and $\bar{\mu} \geqq 0$, such that $(\bar{x}, \bar{\lambda}, \bar{\mu})$ is feasible solutions of the problems (MP) and (MD). Furthermore, if all the hypothesis of Theorem 4.1 (or Theorem 4.4) are satisfied for all feasible solutions of the problems (MP) and (MD), then $(\bar{x}, \bar{\lambda}, \bar{\mu})$ is a strict minimizer of order σ for (MD) with respect to ψ.

Proof: Suppose $(\bar{x}, \bar{\lambda}, \bar{\mu})$ is not a strict minimizer of order σ for (MD) with respect to ψ, there exists another feasible solution (x, λ, μ) of (MD) such that

$$f_i(\bar{x}) + \rho_i \|\psi(x,u)\|^\sigma < f_i(u), \ i \in K. \tag{52}$$

For $\rho_i \geqq 0, i \in K$, which implies

$$f_i(\bar{x}) < f_i(u), \ i \in K. \tag{53}$$

which is a contradiction to Theorem 3.1 (or Theorem 3.4).

Hence $(\bar{x}, \bar{\lambda}, \bar{\mu})$ is a strict minimizer of order σ for (MD) with respect to ψ.

5. CONCLUSIONS

In this paper, we study the multiobjective programming problems and two kinds of dual models. Then the sufficient optimality conditions, weak dual, strong dual and strict converse dual results are obtained and proved under a class of new generalized invex functions assumptions for the multiobjective programming.

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