



## A Finite Difference-Spectral Method for Solving the European Call Option Black–Scholes Equation

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### ABSTRACT

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In this paper, we present a novel technique based on backward-difference method and Galerkin spectral method for solving Black–Scholes equation. The main propose of this method is to reduce the solution of this problem to the solution of a system of algebraic equations. The convergence order of the proposed method is investigated. Also, we provide numerical experiment to show the validity of proposed method.

## 1. INTRODUCTION

The pricing of options is a main problem in financial investment. Because of both theoretical and practical importance then using options thrive in the financial market. In option pricing theory, the Black-Scholes equation is one of the most effective models for pricing options. Black-Scholes option pricing model (also called Black-Scholes-Merton Model) values a European-style call or put option based on the current price of the underlying (asset), the option's exercise price, the underlying's volatility, the option's time to expiration and the annual risk-free rate of return [1]. Consider the Black-Scholes equation based on the following options:

$$P_t(S, t) + \frac{1}{2}\sigma^2 S^2 P_{SS}(S, t) + rSP_S(S, t) - rP(S, t) = 0, \quad (1)$$

with the conditions:

$$P(S, T) = \max(E - S, 0), \quad S \in \Omega = [0, \infty),$$

$$P(0, t) = Ee^{-r(T-t)}, \quad P(S, t) = 0, \quad \text{as } S \rightarrow \infty.$$

where,  $P(S, t)$  is the European call option price at asset price  $S$  and at time  $t$ ,  $E$  is the exercise price,  $T$  is the maturity,  $r$  is the risk free interest rate, and  $\sigma$  represents the volatility function of underlying asset. During the past few decades, many researchers studied the existence of solutions of the Black-Scholes model using many methods in papers [2, 3]. In general, closed-form analytical solutions of some of these Black-Scholes PDEs do not exist and therefore one has to resort to numerical methods in order to solve them. There is an enriched literature regarding the numerical solution of Black-Scholes PDE in finance using different strategies such as Ref. [4-8] and the references cited.

The spectral method plays a significant role in various fields of applied science, especially in fluid dynamics where a large spectral hydrodynamics codes are now regularly used to study turbulence, transition, numerical weather prediction, and ocean dynamics [9]. This method is built on approximating the series solutions for differential equations in terms of classical orthogonal polynomials (Legendre, Chebyshev, Hermit, Jacobi, ...), say  $\sum a_k \phi_k$ . There are three well-known versions used as popular techniques to determine the expansion coefficients, namely collocation, tau, and Galerkin methods. Classical orthogonal polynomials are used successfully and extensively for the numerical solution of differential equations in spectral methods (see [6-18]).

In this paper, we present a new numerical method for solving Black–Scholes equation based on backward-difference method and Galerkin spectral method. The main aim of this method is to reduce the solution of this problem to the solution of a system of algebraic equations. The advantage of using spectral Galerkin method lies in the fact that spectral accuracy this method over other methods.

The layout of this paper is as follows: Section 2, presents some preliminaries. In Section 3, we introduce a new method based on the semi-discretization backward-difference method and Jacobi-Galerkin method. Some theoretical results, show the convergence and stability of the proposed method. The convergence analysis will be given in Section 4. The accuracy of the proposed method is shown by considering numerical example Section 5. Finally, a conclusion is drawn in Section 6.

## 2. PRELIMINARIES

In this section, we review the basic properties of the Jacobi polynomials and generalized Jacobi polynomials (GJPs) that will use in this paper.

## 2.1 The Jacobi polynomials

The Jacobi polynomials,  $J_n^{(a,b)}(x)$ , are eigenfunctions of the singular Sturm–Liouville problem.

$$\frac{d}{dx} \left( (1-x)^{a+1} (1+x)^{b+1} \frac{d}{dx} y(x) \right) + \gamma (1-x)^a (1+x)^b y(x) = 0, \quad x \in [-1,1], \quad (2)$$

where, corresponding eigenvalues are  $\gamma_n^{(a,b)} = n(n+a+b+1)$ , and satisfy the following orthogonality in  $L_{\omega^{(a,b)}}^2[-1,1]$ :

$$\int_{-1}^1 \omega^{(a,b)}(x) J_i^{(a,b)}(x) J_j^{(a,b)}(x) dx = \lambda_j^{(a,b)} \delta_{i,j}, \quad (3)$$

where,  $\omega^{(a,b)}(x) = (1-x)^a (1+x)^b$  is the Jacobi weight function and,

$$\lambda_i^{(a,b)} = \frac{2^{a+b+1} \Gamma(i+a+1) \Gamma(i+b+1)}{(2i+a+b+1)! \Gamma(i+a+b+1)},$$

which is the normalization factor, and  $\delta_{i,j}$  is the kronecker delta function. These polynomials can be computed by the following three-terms recursion relation:

$$J_0^{(a,b)}(x) = 1, \quad J_1^{(a,b)}(x) = \frac{a+b+2}{2}x + \frac{a-b}{2}, \quad (4)$$

$$J_{n+1}^{(a,b)}(x) = (a_n^{(a,b)}x - b_n^{(a,b)})J_n^{(a,b)}(x) - c_n^{(a,b)}J_{n-1}^{(a,b)}(x), \quad n = 1, 2, \dots, \quad (5)$$

where,

$$a_n^{(a,b)} = \frac{(2n+a+b+1)(2n+a+b+2)}{2(n+1)(n+a+b+1)},$$

$$b_n^{(a,b)} = \frac{(b^2 - a^2)(2n+a+b+1)}{2(n+1)(n+a+b+1)(2n+a+b)},$$

$$c_n^{(a,b)} = \frac{(n+a)(n+b)(2n+a+b+2)}{(n+1)(n+a+b+1)(2n+a+b)}.$$

## 2.2 Generalized Jacobi polynomials

For real numbers  $\alpha, \beta$  and  $r \in \mathbf{Z}$ , we define the space:  $W_{\omega^{\alpha,\beta}} = \{u | u \text{ is measurable on } I \text{ and } \|u\|_{\omega^{\alpha,\beta},r} < \infty\}$ , equipped with the norm and semi-norm.

$$\|u\|_{\omega^{\alpha,\beta},r} = \left( \sum_{k=0}^r \|\partial_x^k u\|_{\omega^{\alpha+k,\beta+k}}^2 \right)^{1/2},$$

$$|u|_{\omega^{\alpha,\beta},r} = \|\partial_x^r u\|_{\omega^{\alpha+r,\beta+r}},$$

where,  $\|u\|_{\omega}^2 = \int v^2 \omega dx$ . We denote the Generalized Jacobi polynomials (GJPs) on  $[-1,1]$  by  $G_n^{(a,b)}(x)$  and define it as [19]:

$$G_n^{(a,b)}(x) = \begin{cases} (1-x)^{-a} (1+x)^{-b} J_{n-n_0}^{(-a,-b)}(x), & n_0 = -(a+b), \quad a, b \leq -1, \\ (1-x)^{-a} J_{n-n_0}^{(a,b)}(x), & n_0 = -a, \quad a \leq -1, \quad b > -1, \\ (1+x)^{-b} J_{n-n_0}^{(a,b)}(x), & n_0 = -b, \quad a > -1, \quad b \leq -1, \\ J_{n-n_0}^{(a,b)}(x), & n_0 = 0, \quad a, b > -1, \end{cases} \quad (6)$$

For all  $n \geq n_0$ . An important property of the GJPs is that for  $a, b \in \mathbf{Z}^+$ , we have,

$$D^i G_n^{(-a,-b)}(-1) = 0, \quad i = 0, \dots, a-1,$$

$$D^j G_n^{(-a,-b)}(1) = 0, \quad j = 0, \dots, b-1.$$

In this paper, we set  $a = b = -1$ , and the shifted GJPs  $\hat{G}_n^{(-1,-1)}(x)$  on arbitrary interval  $[l_m, l_M]$  as:

$$\hat{G}_n^{(-1,-1)}(x) := G_n^{(-1,-1)}\left(\frac{2x - l_m - l_M}{l_M - l_m}\right), \quad (7)$$

with the homogenous boundary conditions:

$$\hat{G}_n^{(-1,-1)}(l_m) = \hat{G}_n^{(-1,-1)}(l_M) = 0, \quad n \geq 2. \quad (8)$$

It can also be easily show that  $\{\hat{G}_n^{(-1,-1)}(x); n \geq 2\}$  construct a complete orthogonal system in  $L_{\omega^{-1,-1}}^2$  (see Ref. [19]). Define,

$$B := \text{span}\{\hat{G}_2^{(-1,-1)}(x), \hat{G}_3^{(-1,-1)}(x), \dots, \hat{G}_N^{(-1,-1)}(x)\},$$

And consider the orthogonal projection  $\pi_N^{-1,-1}: L_{\omega^{-1,-1}}^2 \rightarrow B$  defined by:

$$\langle u - \pi_N^{-1,-1}u, v \rangle_{\omega^{-1,-1}} = 0, \quad \forall v \in B.$$

In the following theorem, we estimate the projection errors which are useful in the error analysis of spectral-Galerkin methods.

**Theorem 1:** [19] Assume that  $u \in W_{\omega^{-1,-1},r}$ ,  $0 \leq \mu \leq r$  and  $C$  a generic positive constant independent of any function and  $N$ ,

$$\|u - \pi_N^{-1,-1}u\|_{\omega^{-1,-1},\mu} \leq CN^{\mu-r} |u|_{\omega^{-1,-1},r} \quad (9)$$

## 3. DESCRIPTION OF THE METHOD

At the first, we convert the problem (1) from backward to forward problem (a problem with initial conditions). To do this, change of variable  $\tau = T - t$  in Eq. (1) yields:

$$\hat{P}_\tau(S, \tau) = \frac{1}{2} \sigma^2 S^2 \hat{P}_{SS}(S, \tau) + rS \hat{P}_S(S, \tau) - r \hat{P}(S, \tau), \quad (10)$$

with the initial condition:

$$\hat{P}(S, 0) = \max(E - S, 0),$$

And the nonhomogeneous boundary conditions,

$$\hat{P}(0, \tau) = E e^{-r\tau}, \quad \hat{P}(S, \tau) = 0, \quad \text{as } S \rightarrow \infty,$$

where,  $P(S, t) = P(S, T - \tau) = \hat{P}(S, \tau)$ . Assume  $S_M := S_\infty < \infty$ , where is the suitably chosen positive number. Now, due to the application of homogeneous boundary conditions (8) of the GJPs, we must transform Eq. (10) into a homogeneous boundary condition problem. Assume that:

$$\hat{P}(S, \tau) = W(S, \tau) + Ee^{-r\tau}\left(1 - \frac{S}{S_M}\right), \quad (11)$$

where,  $W(S, \tau)$  is a new unknown function, then we get:

$$\hat{P}_\tau(S, \tau) = W_\tau(S, \tau) - rEe^{-r\tau}\left(1 - \frac{S}{S_M}\right), \quad (12)$$

$$\hat{P}_\tau(S, \tau) = W_\tau(S, \tau) - rEe^{-r\tau}\left(1 - \frac{S}{S_M}\right), \quad (13)$$

$$\hat{P}_{SS}(S, \tau) = W_{SS}(S, \tau). \quad (14)$$

Substituting (11) and (12) into (10) yields:

$$W_\tau(S, \tau) = \frac{1}{2}\sigma^2 S^2 W_{SS}(S, \tau) + rSW_S(S, \tau) - rW(S, \tau) - \frac{rEe^{-r\tau}S}{S_M}, \quad (15)$$

with the initial condition,

$$W(S, 0) = \max(E - S, 0) - E\left(1 - \frac{S}{S_M}\right), \quad (16)$$

and the homogeneous boundary conditions.

$$W(0, \tau) = W(S_M, \tau) = 0. \quad (17)$$

### 3.1 A semi-discretization in time for Eq. (15)

Given the discretization of the time interval  $[0, T]$  with step size  $h = T/M$ ,

$$0 = \tau_0 < \tau_1 < \dots < \tau_M = T.$$

By applying the backward-difference method for the left side of Eq. (15) we obtain:

$$\begin{aligned} & \frac{W(S, \tau_{i+1}) - W(S, \tau_i)}{h} + O(h) \\ &= \frac{1}{2}\sigma^2 S^2 W_{SS}(S, \tau_{i+1}) + rSW_S(S, \tau_{i+1}) \\ & - rW(S, \tau_{i+1}) - \frac{rEe^{-r\tau_{i+1}}S}{S_M}, \end{aligned}$$

for  $i = 0, 1, \dots, M - 1$ . Taking the  $F_i(S)$ , approximation of the exact solution  $W(S, \tau_i)$  for  $i = 0, 1, \dots, M$ , we obtain an ordinary differential equation:

$$DF_{i+1}(S) = R_i(S), \quad (18)$$

with homogeneous boundary condition as follows:

$$F_{i+1}(0) = F_{i+1}(S_M) = 0, \quad i = 0, 1, \dots, M,$$

where,

$$\begin{aligned} DF_{i+1}(S) &:= h \frac{1}{2}\sigma^2 S^2 F_{i+1}''(S) + rhSF_{i+1}'(S) - (1 \\ & + rh)F_{i+1}(S), \\ R_i(S) &:= h \frac{rEe^{-r\tau_{i+1}}S}{S_M} - F_i(S). \end{aligned}$$

For each time step  $i$  with homogeneous boundary conditions. It is clear that for the first step,  $i = 0$ , from initial condition of (15), we have:

$$F_0(S) = \max(E - S, 0) - E\left(1 - \frac{S}{S_M}\right).$$

The unknown functions  $F_i(S)$ ,  $i = 1, \dots, M$  are approximated in the finite dimensional space,  $B$ , as follows:

$$F_{i,N}(S) \simeq \sum_{j=2}^N c_{i,j} \hat{G}_j^{(-1,-1)}(S). \quad (19)$$

Substituting (19) into (18) yields the Galerkin residual functions (see [3]),  $Res_i(S)$ , as follows :

$$Res_i(S) = DF_{i+1,N}(S) - R_i(S), \quad i = 0 \dots M - 1. \quad (20)$$

The application of the Galerkin method for each time step  $i = 0, 1, \dots, M - 1$ , yields the following  $(N - 2)$  set of linear algebraic equations in the unknown expansion coefficients,  $c_{i,j}$ , namely,

$$\int_0^{S_M} Res_i(S) \hat{G}_j^{(-1,-1)}(S) \omega^{-1,-1}(S) dS = 0, \quad (21)$$

$$i = 0, 1, \dots, M - 1, \quad j = 2, \dots, M,$$

or equivalently,

$$\begin{aligned} & \langle DF_{i+1,N}(S), \hat{G}_j^{(-1,-1)}(S) \rangle_{\omega^{-1,-1}} \\ &= \langle R_i(S), \hat{G}_j^{(-1,-1)}(S) \rangle_{\omega^{-1,-1}}, \end{aligned} \quad (22)$$

with weight function  $\omega^{-1,-1} = (S_M - S)^{-1}S^{-1}$ . This system of algebraic equation can be solved for the unknown coefficients  $c_{i,j}$  and  $F_{i,N}(S)$  calculated for  $i = 1, \dots, M$ .

## 4. CONVERGENCE ANALYSIS

At the first, in order to obtain the convergence of the solution  $F_i(S)$  to  $W(S, \tau_i)$  we begin by studying the stability and consistency of above time semi-discrete method. For consistent numerical approximations, stability and convergence are equivalent, therefore, we obtain the convergence of time semi-discrete method. (For more details, see the papers [19-21]).

**Theorem 2:** [21] Assume  $K_{i+1}(S) := \frac{rEe^{-r\tau_{i+1}}S}{S_M}$  and  $L(S) := \max(E - S, 0) - E\left(1 - \frac{S}{S_M}\right)$ . Then, the time semi-discrete problem (18) is unconditionally stable, i.e., if,

$$\hat{L}(S) = L(S) + \hat{\delta}(S), \quad \hat{K}_{i+1}(S) = K_{i+1}(S) + \hat{\delta}_{i+1}(S)$$

This provides a perturbed sequence  $\hat{F}_i(S)$ , whose distance from the original sequence  $F_i(S)$ , is uniformly bounded by the maximum size of the perturbation, i.e.,

$$\|F_i(S) - \hat{F}_i(S)\|_\infty \leq C(\hat{\delta}(S) + \max_{j=0, \dots, i-1} \|\hat{\delta}_j(S)\|_\infty).$$

**Theorem 3:** [13] If,

$$\left| \frac{\partial^{i+j} W(S, \tau)}{\partial^i S \partial^j \tau} \right| \leq C, \quad 0 \leq j \leq 3, \quad 0 \leq i \leq 4,$$

Then, the local truncation error  $e_n$ , and global truncation error  $E_n$  associated to the semi-discrete method (18) satisfies:

$$\| e_n \|_{\infty} \leq Ch^2, \quad \| E_n \|_{\infty} \leq Ch.$$

**Theorem 4:** [13] The time semi-discrete method (18) is first order convergent, i.e.,

$$\| W(S, \tau_i) - F_i(S) \|_{\infty} \leq Ch.$$

The following theorem shows the convergence of the full discretization method which is obtained by:

$$\begin{aligned} \| W(S, \tau_i) - F_{i,N}(S) \|_{\omega^{-1,-1,2}} &\leq \\ \| W(S, \tau_i) - F_i(S) \|_{\omega^{-1,-1,2}} &+ \\ \| F_i(S) - F_{i,N}(S) \|_{\omega^{-1,-1,2}} & \end{aligned}$$

and Theorems 1 and 4, respectively. The idea of the proof comes from Refs. [19-21].

**Theorem 5:** Let  $W(S, \tau)$  be the solution of the initial boundary value problem (15)-(17) and  $F_{i,N}(S)$  be the Galerkin approximation to the solution  $F_i(S)$  in each time steps  $i = 0, \dots, M$ . Then,

$$\| W(S, \tau_i) - F_{i,N}(S) \|_{\infty} \leq C(h + N^{2-r}).$$

It is seen that the temporal and spatial rate of convergence are  $O(h)$  and  $O(N^{2-r})$ , respectively, where  $r$  is an index of regularity of the underlying function.

## 5. NUMERICAL RESULTS

Consider Eq. (1) with parameters  $E = 10$ ,  $T = 0.25$ ,  $r = 0.1$ ,  $\sigma = 0.4$  and  $0 \leq S \leq 20$ . The exact solution is,

$$P(S, \tau) = E e^{-r(T-\tau)} N(-d_2) - SN(-d_1),$$

**Table 1.** The absolute errors (Abs.err) and values of  $S$  for  $N = 25$  at time  $t = T$

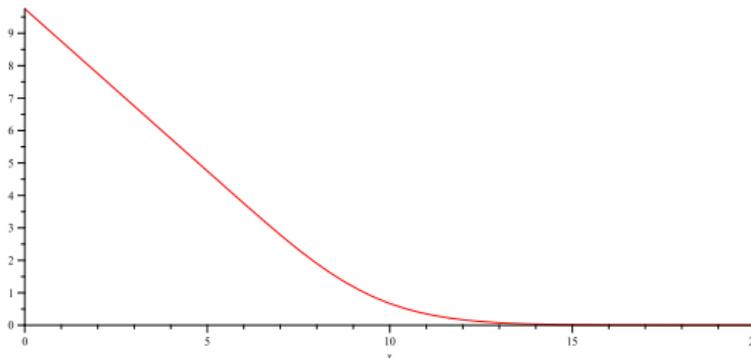
S	Abs.err	S	Abs.err
1	5.126000 e-05	11	7.229574 e-04
2	2.758800 e-04	12	2.024524 e-04
3	2.660140 e-04	13	1.626119 e-04
4	2.244770 e-04	14	3.512227 e-04
5	5.347300 e-05	15	3.037570 e-04
6	2.884300 e-04	16	2.424921 e-04
7	6.534150 e-04	17	1.357483 e-04
8	2.637970 e-04	18	7.445729 e-05
9	4.073430 e-04	19	4.032019 e-06
10	9.394778 e-04	20	1.129299 e-04

where,

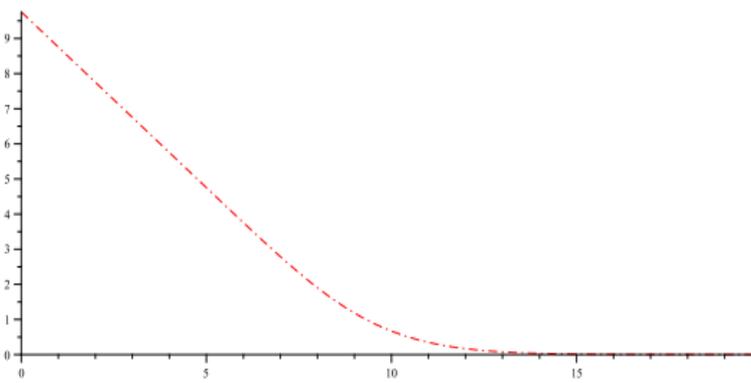
$$N(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^y e^{-\frac{u^2}{2}} du,$$

$$d_1 = \frac{\ln\left(\frac{S}{E}\right) + \left(r + \frac{\sigma^2}{2}\right)(T - \tau)}{\sigma\sqrt{T - \tau}},$$

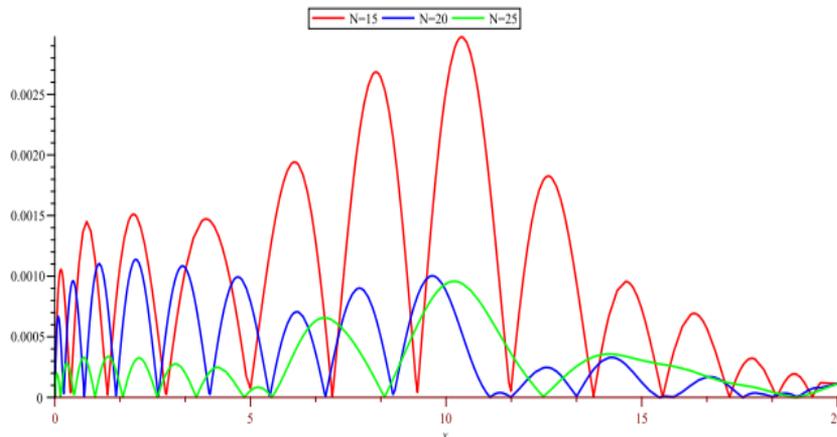
$$d_2 = \frac{\ln\left(\frac{S}{E}\right) + \left(r - \frac{\sigma^2}{2}\right)(T - \tau)}{\sigma\sqrt{T - \tau}} = d_1 - \sigma\sqrt{T - \tau}.$$



**Figure 1.** Exact solution at time  $t=T$



**Figure 2.** Approximate solution at time  $t=T$  and  $N=25$



**Figure 3.** Absolute error for various  $N=15,20,25$

The function  $N(y)$  is the cumulative probability distribution function for a standardized normal distribution. Figures 1 and 2 show the exact solution and numerical solution in  $t = T$  for  $N = 25$ . The absolute error is shown for various  $N = 15,20,25$  in Figure 3. In Table 1, the results for  $N = 25$  for various values of  $S$  are presented in Table 1. By these results, we observe increase the accuracy of our proposed method by growing as the number of basis functions  $N$  in approximate solutions (19).

## 6. CONCLUSION

In the paper, by utilizing the backward-difference and shifted Jacobi-Galerkin method we reduce the problem to the set of system of linear algebraic equations. We also obtain the error estimation for the method. The solution obtained using the proposed technique shows that this method can solve the Black-Scholes equation for put options effectively.

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