Numerical study of bioconvection saturated with nanofluid containing gyrotactic microorganisms confined within Hele-Shaw cell

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ABSTRACT. *The onset of bioconvection contains both nanoparticles and gyrotactic microorganisms confined within a Hele-Shaw cell is investigated by incorporating the effects of Brownian diffusion and thermophoresis by using the zero flux nanoparticle boundary conditions. The linear analysis is based on the normal mode technique and the resulting equations are solved numerically by the higher order Galerkin weighted Residual method. The critical Hele-Shaw Rayleigh number is presented in terms of bioconvection parameters, nanofluid parameters, and Hele-Shaw parameters. It is found that the highly promoted disturbance in the presence of gyrotactic microorganisms enhances heat transfer in nanofluids. Gyrotactic microorganisms enhance the bioconvection and this effect is larger if both the concentration and average speed of microorganisms have larger values. Wavenumber is the function of Hele-Shaw Rayleigh number, bioconvection Péclet number and Gyrotaxis number. A comparison is also made between the different bioconvection Péclet number and bioconvection Hele-Shaw number. The present study may found applications in bio-convection nanotechnological devices.*

RÉSUMÉ. *Le début de la bioconvection contenant à la fois des nanoparticules et des micro-organismes gyrotactiques confinés dans une cellule de Hele-Shaw est étudié en incorporant les effets de la diffusion brownienne et de la thermophorèse en utilisant les conditions limites de flux nul nanoparticule. L'analyse linéaire est basée sur la technique du mode normal et les équations résultantes sont résolues numériquement par la méthode résiduelle pondérée de Galerkin d'ordre supérieur. Le nombre critique de Hele-Shaw Rayleigh est présenté en termes de paramètres de bioconvection, de paramètres de nanofluide et de paramètres de Hele-Shaw. Il est constaté que la perturbation hautement valorisée en présence de micro-organismes gyrotactiques améliore le transfert thermique dans les nanofluides. Les micro-organismes gyrotactiques améliorent la bioconvection et cet effet est plus important si la concentration et la vitesse moyenne des micro-organismes ont des valeurs plus grandes. Le numéro d'onde est la fonction du numéro de Hele-Shaw Rayleigh, du numéro de Péclet de bioconvection et du numéro de Gyrotaxis. Une comparaison est également faite entre les différents nombres de*

Péclet de bioconvection et nombres de Hele-Shaw de bioconvection. La présente étude pourrait être appliquée dans les appareils nanotechnologiques à bioconvection.

KEYWORDS: *nanofluid, Hele-Shaw cell, thermophoresis, Brownian motion, bioconvection, gyrotactic microorganism.*

MOTS-CLÉS: *nanofluid, cellule de Hele-Shaw, thermophorèse, mouvement Brownien, bioconvection, Micro-organisme gyrotactique.*

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1. Introduction

Bioconvection is a phenomenon that occurs when convection instability is induced by self-propelled up swimming microorganisms that are denser than cell fluid. Due to up-swimming, the microorganisms involved; such as gyrotactic microorganisms, like algae, tend to focus on the upper portion of the fluid layer thus causing a top-heavy density stratification that becomes unstable. Platt (1961) introduced the term bioconvection and studied the moving polygonal patterns in dense cultures of Tetrahymena. Childress, Levadowsky, and Spiegel, (1975) studied the linear stability of the stratified layer when the virtual thickness is greater compared with the depth of the subsurface layer. Pedley, Hill, and Kessler (1988) introduced the theoretical bio-convective model for the gyrotactic microorganism. Later, Hill, Pedley, and Kessler (1989) discussed the overstability and oscillatory modes of a suspension of microorganisms. Bioconvection modes for different types of micro-organisms have been reported in a large number of articles Hillesden and Pedley (1996), Hill and Häder (1996), Bees and Hill (1997), Ghorai and Hill (1999). Kuznetsov and Avramenko (2002) gave the numerical results which show macroscopic fluid circulation is induced due to large permeability. Later, Kuznetsov and Avramenko (2003) investigated the behavior of gyrotaxis and found that the eccentricity of the cells depends on Darcy number, while rates of cell deposition and declogging do not depend on Darcy number. The correlation between the temperature gradient in the fluid layer and the density difference induced by the up swimming microorganisms was studied by Kuznetsov (2006).

Choi (1995) defined a new class of fluid which consists of nano-sized particles and the base fluid, known as a nanofluid. Das *et al.* (2003) considered 1–4% alumina nanoparticles of less than 10 nm in size and achieved a 10-30% increment in thermal conductivity. Incorporating the effect of thermophoresis and Brownian motion, Buongiorno (2006) developed a mathematical model for nanofluid. Using Buongiorno model, Tzou (2008) and Nield and Kuznetsov (2009, 2010) investigated the thermal Rayleigh instability of nanofluid. This model has also been used by Nield and Kuznetsov (2010) who studied the effect of double diffusion; Yadav *et al.* (2015) who studied the effect of viscosity variation and thermal conductivity; Saini and Sharma (2017) who studied the thermal instability in Rivlin-Erickson Elastico-Viscous nanofluid with the effect of throughflow. These referred papers provide the numerical range for parameters and present mathematical formulation of the conservation equations. Baehr and Stephan (2006) were perhaps the first who gave the concept of physically realistic boundary conditions (zero nanoparticle flux on the boundaries). After the work of Baehr and Stephan (2006), Nield and Kuznetsov (2014a, 2014b) revised their work by using more realistic boundary conditions. Due to the vast range of applications, nanofluids have attracted the attention of many researchers in the recent past. They are widely used in engineering (such as cooling, microheat pipes, microchannel heat sinks, microreactors), biomedical (such as cancer therapy, sterilization of medical suspensions), process industries, polymer coatings, aerospace tribology, microfluid delivery devices etc (Ebrahimi, Sabbaghzadeh, Lajevardi, and Hadi, 2010; Fan, Chen, Ding, Plucinski, and Lapkin, 2010).

Kuznetsov (2010) developed a new theory which incorporates the Brownian motion and thermophoresis in nanofluid bioconvection. The top-heavy distribution of nanoparticles lowers the Rayleigh number, while bottom heavy distribution increases the Rayleigh number. Kuznetsov (2011) observed that adding the microorganisms to a nanofluid increase the stability of a suspension. On the other hand, (Tham, Nazar, and Pop, 2013) considered the mixed convection flow over a solid field and found that gyrotactic has a strong influence of the velocity of microorganisms transport rate. Nanofluid with bioconvection may find useful applications in different bio-microsystems, such as; enzyme biosensors, chip-size micro-devices for evaluating nanoparticle toxicity, evaluating toxic and inflammatory responses of the lung to silica nanoparticles, mass transport enhancement and mixing etc. (Huh *et al.* 2010; Sokolov, Goldstein, Feldchtein, and Aranson 2009).

The first detailed observations of Hele-Shaw convection were carried out in Hele-Shaw (1898). Wooding (1960) investigated the instability in vertical Hele-Shaw cell of viscous liquid. Hartline and Lister (1977) discussed thermal convection in Hele-Shaw cell and found that the system of equations is identical to the equations of a fluid flow through porous media. (Kvernvold, 1979; Aniss, Souhar, and Brancher, 1995; Yadav and Kim, 2015) studied the problem related to mechanics approximated by Hele-Shaw cell. The present study focuses on analytical and numerical investigations of the effect of gyrotactic microorganisms confined in a Hele-Shaw cell with realistic boundary conditions. The present study found applications bio-convection nanotechnological devices.

2. Problem formulation

We consider a horizontal nanofluid layer with gyrotactic microorganism confined between in the Y-direction by vertical impermeable boundaries at $Y^*=0$ and $Y^*=b<<d$ (see Fig.1). We take temperatures T^*_0 and T^*_d ($T^*_0>T^*_d$) at the lower and upper boundary respectively. The base fluid is water so that microorganisms can stay alive in it. Nanoparticles have no effect on the velocity and direction of gyrotactic microorganisms. Suspension of nanoparticles is assumed to be dilute, stable and do not to agglomerate. We use the Brinkman model and the Oberbeck–Boussinesq approximations are used.

The dimensionless governing equations for a water-based nanofluid containing nanoparticles and gyrotactic microorganisms within Hele-Shaw cell approximation are written below (Pedley *et al.*, 1988; Nield and Kuznetsov, 2009; Yadav and Lee, 2016).

$$\nabla.\mathbf{V}=0 \tag{1}$$

$$\frac{H_s}{\Pr}\left(\frac{\partial}{\partial t}+\mathbf{V}.\nabla\right)\mathbf{V}=-\nabla p+H_s\nabla^2\mathbf{V}-\mathbf{V}-R_m\hat{\mathbf{k}}+R_H T\hat{\mathbf{k}}-R_n\phi\hat{\mathbf{k}}-\frac{R_b}{L_b v}n\hat{\mathbf{k}} \tag{2}$$

$$\left(\frac{\partial}{\partial t}+\mathbf{V}.\nabla\right)T=\nabla^2 T+\frac{N_B}{L_e}\nabla\phi.\nabla T+\frac{N_A N_B}{L_e}\nabla T.\nabla T \tag{3}$$

$$\left(\frac{\partial}{\partial t}+\mathbf{V}.\nabla\right)\phi=\frac{1}{L_e}\nabla^2\phi+\frac{N_A}{L_e}\nabla^2 T \tag{4}$$

$$\frac{\partial n}{\partial t}=-\nabla.\left(n\mathbf{V}+n\frac{Q_b}{L_b}\hat{\mathbf{p}}-\frac{1}{L_b}\nabla n\right) \tag{5}$$

where v is the dimensionless velocity, $\hat{p}$ is the unit vector indicating swimming direction of microorganism, $\hat{k}$ is the vertically upward unit vector. On the boundaries, the temperature is assumed to be constant and the nanoparticles flux is assumed to be zero. The dimensionless boundary conditions are:

$$w=\frac{dw}{dz}=0, T=1,\ \frac{\partial\phi}{\partial z}+N_A\frac{\partial T}{\partial z}=0,\ \ Q_b n=\frac{dn}{dz} \qquad \text{at } z=0 \tag{6a}$$

$$w=\frac{dw}{dz}=0,\ T=0,\ \frac{\partial\phi}{\partial z}+N_A\frac{\partial T}{\partial z}=0,\ Q_b n=\frac{dn}{dz} \qquad \text{at } z=1 \tag{6b}$$

Dimensionless variables in the equations are as follows:

$$t=\frac{t^*\alpha_m}{d^2}, p=\frac{p^*K}{\mu\alpha_m},(x,y,z)=\frac{(x^*,y^*,z^*)}{d},\mathbf{V}(u,v,w)=\frac{\mathbf{V}^*(u^*,v^*,w^*)d}{\alpha_m},$$
$$T=\frac{T^*-T_d^*}{T_0^*-T_d^*},\phi=\frac{\phi^*-\phi_0^*}{\phi_0^*},n=n^*\theta \text{ where } \alpha_m=\frac{k_m}{(\rho c)_f} \tag{7}$$

Here t^* is the time, α_m is the thermal diffusivity of nanofluid, p^* is the pressure, K=b^2/12 is the permeability of the Hele-Shaw cell, μ is the viscosity, V^* is the velocity, T^* is the temperature of nanofluid, ϕ^* is the nanoparticles volume fraction, n^* is the microorganism concentration, θ is the average volume of microorganism, $(\rho c)f$ is the volumetric heat capacity for the nanofluid, k_m is the thermal conductivity of nanofluid.

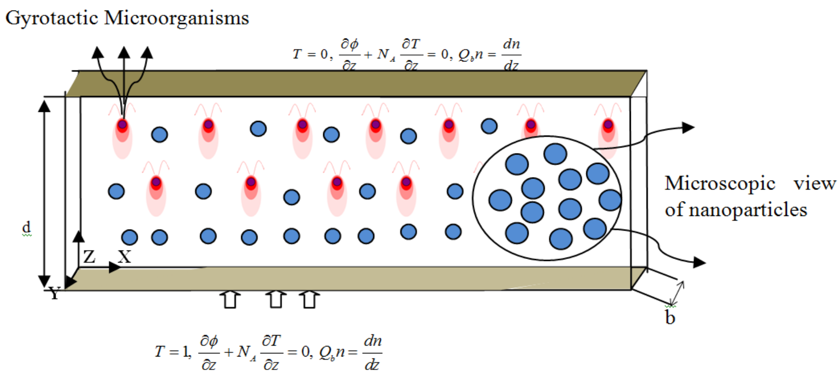

Figure 1. Schematic diagram of the problem

The dimensionless parameters in Eqs. (1)-(6) namely, the Prandtl number P_r, Hele-Shaw number H_S, Hele-Shaw Rayleigh number R_H, basic density Rayleigh number R_m, bioconvection Rayleigh number R_b, nanoparticle Rayleigh number R_n, Lewis number L_e, bioconvection Lewis number L_b, bioconvection Péclet number Q_b and modified diffusivity Ratio N_B, are defined as

$$\Pr = \frac{\mu}{\rho_f \alpha_m}, R_H = \frac{\rho g \beta_T K d (T_0^* - T_d^*)}{\mu \alpha_m}, R_m = \frac{[\rho_P \phi_0^* + \rho_{f_0}(1-\phi_0^*)] g d K}{\mu \alpha_m},$$
$$H_s = \frac{K}{d^2}, R_b = \frac{\Delta \rho g v K d}{\mu D_m}, R_n = \frac{\left\{(\rho_P - \rho)\phi_d^*\right\} g K d}{\mu \alpha_m}, L_e = \frac{\alpha_m}{D_B}, L_b = \frac{\alpha_m}{D_m},$$
$$Q_b = \frac{W_c \hat{\mathbf{p}} d}{D_m}, N_A = \frac{D_T (T_0^* - T_d^*)}{D_B T_d^* \phi_0^*}, N_B = \frac{(\rho c)_p \phi_0^*}{(\rho c)_f} \tag{8}$$

where g is the gravity vector, β_T is the volumetric thermal expansion coefficient, ρ_p is density of nanoparticles, ρ_{f_0} is the density of the nanofluid, $(\rho c)p$ is the volumetric heat capacity for the nanoparticles, $(\rho c)f$ is the volumetric heat capacity for the nanofluid, $\Delta\rho=\rho_{cell}-\rho_f$ is the difference between cell density and a fluid density, D_m is the diffusivity of microorganism, D_B is the Brownian diffusion coefficient, $W_c\hat{p}$ is the vector of microorganism's average swimming velocity relative to the nanofluid (W_c is assumed to be constant), D_T is the thermophoresis diffusion coefficient.

The basic state of nanofluid is assumed to be time-independent and is described by

$$\mathbf{V} = \mathbf{V_b} = (0,0,0), p = p_b(z), T = T_b(z), \phi = \phi_b(z), n = n_b(z) \tag{9}$$

The Eqs. (1)-(5) are simplified as

$$\frac{d^2T_b}{dz^2}+\frac{N_B}{L_e}\frac{d\phi_b}{dz}\frac{dT_b}{dz}+\frac{N_AN_B}{L_e}\left(\frac{dT_b}{dz}\right)^2=0 \tag{10}$$

$$\frac{d^2\phi_b}{dz^2}+N_A\frac{d^2T_b}{dz^2}=0 \tag{11}$$

$$nQ_b-\frac{dn}{dz}=0 \tag{12}$$

Eq. (12) is integrated, then the solution of n_b is obtained

$$n_b(z)=v\ \exp(Q_bz) \tag{13}$$

Here v is the integration constant is given by

$$v=\frac{\bar{n}Q_b}{\exp(Q_b)-1} \tag{14}$$

Where $\bar{n}=\int_0^1 n_b(z)dz$ is the average dimensionless concentration of microorganisms in the nanofluid layer. On solving the Eqs. (10)-(11), the solutions are

$$T_b=1-z,\ \phi_b=\phi_0+N_Az\ . \tag{15}$$

Considering, in the usual manner, Perturbations are superimposed on the basic solution, as follows:

$$\mathbf{V}=\mathbf{V}',T=T_b+T',\phi=\phi_b+\phi',n=n_b+n,'\ p=p_b+p' \tag{16}$$

Substituting Eq. (16) in Eqs. (10)-(12) and utilizing Eq. (15) and neglecting the product of prime quantities

$$\nabla.\mathbf{V}'=0 \tag{17}$$

$$\frac{H_s}{\mathrm{Pr}}\frac{\partial\mathbf{V}'}{\partial t}=-\nabla p'+H_s\nabla^2\mathbf{V}'-\mathrm{V}'+R_HT'\hat{\mathbf{k}}-R_n\phi'\hat{k}-\frac{R_b}{L_bv}n'\hat{\mathbf{k}} \tag{18}$$

$$\frac{\partial T'}{\partial t}-w'=\nabla^2T'+\frac{N_B}{L_e}\left(N_A\frac{\partial T'}{\partial z}-\frac{\partial\phi'}{\partial z}\right)-\frac{2N_AN_B}{Le}\frac{\partial T'}{\partial z} \tag{19}$$

$$\frac{\partial \phi'}{\partial t} + N_A w' = \frac{1}{L_e}\nabla^2\phi' + \frac{N_A}{L_e}\nabla^2 T' \tag{20}$$

$$\frac{\partial n'}{\partial t} = -\nabla.\left(n_b\left(\mathbf{V}' + \frac{Q_b}{Lb}\hat{\mathbf{p}}\right) + n'\frac{Q_b}{Lb}\hat{\mathbf{k}} - \frac{1}{L_b}\nabla n' \right) \tag{21}$$

Applying the procedure outlined in Pedley, Hill, and Kessler (1988) for average swimming direction vector, the perturbation of the swimming direction of a gyrotactic microorganism can be expressed as

$$\frac{\partial n'}{\partial t} = -w'\frac{\partial n_b}{\partial z} - \frac{Q_b}{L_b}\frac{\partial n'}{\partial z} + GQ_b n_b\left((1-\alpha_0)\left(\frac{\partial^2 w'}{\partial x^2} + \frac{\partial^2 w'}{\partial y^2}\right) + (1+\alpha_0)\frac{\partial^2 w'}{\partial z^2}\right) + \frac{1}{L_b}\nabla^2 n' \tag{22}$$

In Eq. (22), α_0 is the measure of cell eccentricity, $G=BD/H^2$ is the Gyrotaxis number.

Eliminating the horizontal components of velocity and pressure Eqs. (17)-(18) are reduced to the following equation.

$$\frac{H_s}{\Pr}\frac{\partial}{\partial t}\nabla^2 w' - H_s\nabla^4 w' + \nabla^2 w' = R_H\nabla_H^2 T' - R_n\nabla_H^2\varphi' - \frac{R_b}{L_b v}\nabla_H^2 n' \tag{23}$$

In Eq. (23), ∇_H^2 is the 2D Laplacian operator in the horizontal plane.

The boundary conditions are

$$w' = Dw' = 0, T' = 0,\ \frac{\partial \phi'}{\partial z} + N_A\frac{\partial T'}{\partial z} = 0, \quad n'Q_b = \frac{dn'}{dz} \quad at\ z = 0, z = 1 \tag{24}$$

Due to the absence of two opposite agencies who affect instability oscillatory convection cannot occur, therefore in absence of oscillatory convection analyzing the disturbances into the normal modes in the following form

$$[w', T', \phi', n']y = [W(z), \Theta(z), \Phi(z), N(z)]\exp(ik_x x + ik_y y) \tag{25}$$

For spherical microorganisms, α_0 is set to be zero (The value of α_0 lie between 0 to 0.5 Pedley and Kessler, 1987). Substituting Eq. (25) in Eqs. (19), (20), (22) and (23), the equations for the amplitudes W, Θ,Φ, N are obtained

$$W+\left(D^2-\frac{N_A N_B}{L_e}D-a^2\right)\Theta-\frac{N_B}{L_e}D\Phi=0 \tag{26}$$

$$-N_A W+\frac{N_A}{L_e}(D^2-a^2)\Theta+\frac{1}{L_e}(D^2-a^2)\Phi=0 \tag{27}$$

$$\frac{1}{L_b}a^2N+\frac{Q_b}{L_b}DN-\frac{1}{L_b}D^2N+e^{Q_b z}Qv[W+Ga^2W-GD^2W]=0 \tag{28}$$

$$\left[H_s(D^2-a^2)^2-(D^2-a^2)\right]W-R_H a^2\Theta+R_n a^2\Phi-\frac{R_b}{L_b v}a^2N=0 \tag{29}$$

where $D=d/dz$ and, $a=\sqrt{k_x^2+k_y^2}$ isdimensionless horizontal wave number.

3. Solution of stability problem

3.1. Single-term galerkin method

The system of equations (26)-(29) along with thermal, nanoparticle volume fraction, microorganism concentration conditions given by Eq. (24) are solved by using the single-term Galerkin-type weighted residuals method. Accordingly Θ,Φ,W and N are taken as:

$$\Theta=\sum_1^n P_s\Theta_s\,,\Phi=\sum_1^n Q_s\Phi_s\,,W=\sum_1^n R_sW_s\,,N=\sum_1^n S_sN_s$$

Where P_s, Q_s, R_s, and S_s, are constants and we choose base functions as

$$\Theta_s=z^s(1-z),\Phi_s=-N_A z^s(1-z),W_s=z^s(1-z)^2,$$
$$N_s=(s-Q_b)z^{s-1}+(Q_b-s+1)z^s \tag{30}$$

For a First approximation (N=1), the Eigenvalue equation is

$$R_H=\frac{\begin{array}{c}(10Q_b^4+a^2(120-10Q_b^2+Q_b^4))(28(10+a^2)(H_s(504+24a^2+a^4)+12+a^2)\\-27a^2N_A(1+L_e)R_{n,H})-1260(3Q_b^2-28)(10+a^2)A_1R_{b,H}\end{array}}{27a^2(10Q_b^4+a^2(120-10Q_b^2+Q_b^4))} \tag{31}$$

where,

$$A_1 = \frac{4\begin{pmatrix} -132\left(-1+e^{Q_b}\right)\left(1+a^2G\right)+66\left(1+e^{Q_b}\right)Q_b+\left(-1+e^{Q_b}\right) \\ \left(-13+48G\right)Q_b^2-\left(1+e^{Q_b}\right)\left(-1+24G\right)Q_b^3+4\left(-1+e^{Q_b}\right)GQ_b^4 \end{pmatrix}}{Q_b^4}$$

The corresponding critical wave number is obtained as

$$a_c = \left[\frac{\gamma^{2/3}+1782\beta\lambda^{1/3}G+3175524\beta^2G^2-119Q_b^4\left(\gamma^{1/3}+3564\beta G-119Q_b^4\right)}{\gamma^{1/3}Q_b^4}\right]^{1/2} \tag{32}$$

where,

$$\beta = e^{Q_b}-1,$$

$$\alpha = 882\sqrt{30}\sqrt{Q_b^{12}\begin{pmatrix} 5658783768\beta^3G^3-1133662068\beta^2G^2Q_b^4 \\ +75704706\beta GQ_b^8+4149271Q_b^{12} \end{pmatrix}},$$

$$\gamma = \alpha+5658783768\beta^3G^3-1133662068B^2G^2Q_b^4+75704706\beta GQ_b^8+9983701Q_b^{12}$$

Eq. (32) shows that critical wave number depend on bioconvection Rayleigh number, bioconvection Péclet number, Gyrotaxis number, and Hele-Shaw number.

For the case when (R_n=0, R_H=0 and $Q_b \to 0$) then $Q_bR_b \to 720$ which is the same as the value given in Sparrow *et al.* (1960). For the case of regular fluid (R_n=0, R_b=0) and $D_a=\infty$, Rayleigh number attains its minimum value of 1751.851 at a = 3.12. This critical value of Hele-Shaw Rayleigh number is 2.5% greater than the value obtained by the Rayleigh-Benard Problem Chandrasekhar (1961). In a regular fluid for the cases, D_a=0.1, and D_a=1 Rayleigh number attains its minimum values of 220.672, 1795.67 at a = 3.145, a = 3.112. These values of critical Hele-Shaw Rayleigh number are 2.61%, and 2.48% greater than the value obtained by Guo and Kaloni (1995).

To study the effect of various parameters, we examine the behavior of $\frac{\partial R_a}{\partial H_s}, \frac{\partial R_a}{\partial L_e}, \frac{\partial R_a}{\partial N_A}, \frac{\partial R_a}{\partial R_n}, \frac{\partial R_a}{\partial Q_b}$, and $\frac{\partial R_a}{\partial R_b}$, analytically as

$$\frac{\partial R_H}{\partial H_s} = \frac{28(10+a^2)(504+24a^2+a^4)}{27a^2} \tag{33}$$

$$\frac{\partial R_H}{\partial R_{n,H}} = -N_A(1+L_e), \quad \frac{\partial R_H}{\partial L_e} = -N_AR_{n,H}, \quad \frac{\partial R_H}{\partial N_A} = -(1+L_e)R_{n,H} \tag{34}$$

$$\frac{\partial R_H}{\partial R_{b,H}} = -\frac{560(10+a^2)A_1}{3a^2} \tag{35}$$

$$\frac{\partial R_H}{\partial Q_b} = -\frac{560(10+a^2)R_{b,H}A_2}{3a^2} \tag{36}$$

Where

$$A_2 = \frac{\begin{array}{l} 4(-4\tilde{\beta}\tilde{\gamma}\tilde{\eta} + Q_b(-66\tilde{\beta}\tilde{\gamma}\left(-1+e^{Q_b}\left(1+2a^2G\right)\right) + Q_b(20a^2\tilde{\gamma}\tilde{\eta} \\ +\tilde{\beta}\left(6\tilde{\eta}+\tilde{\gamma}\left(26-96G+8e^{Q_b}\left(5+12G\right)\right)\right) \\ +\tilde{\gamma}Q_b\left(\begin{array}{l}\tilde{\beta}\left(3-72G-2e^{Q_b}\left(5+12G\right)\right) \\ +Q_b\left(-4\left(\left(10+a^2\right)\tilde{\eta}+4\tilde{\beta}G\right)+\tilde{\beta}e^{Q_b}\left(1-8G+4GQ_b\right)\right)\end{array}\right)\Bigg)\Bigg)\Bigg) \end{array}}{\tilde{\beta}^2 Q_b^5}$$

$$\tilde{\beta} = \left(10Q_b^4 + a^2\left(120-10Q_b^2+Q_b^4\right)\right),\ \tilde{\gamma} = \left(-28+3Q_b^2\right),$$
$$\tilde{\eta} = -132\left(-1+e^{Q_b}\right)\left(1+a^2G\right) + 66\left(1+e^{Q_b}\right)Q_b$$
$$+\left(-1+e^{Q_b}\right)\left(-13+48G\right)Q_b^2 - \left(1+e^{Q_b}\right)\left(-1+24G\right)Q_b^3 + 4\left(-1+e^{Q_b}\right)GQ_b^4$$

It is clear from the Eqs. (32)-(33) that the Hele-Shaw number has a stabilizing effect and nanofluid parameters (L_e, N_A, R_n) have a destabilizing effect on the bioconvection. The behavior of bioconvection Péclet number and bioconvection Hele-Shaw Rayleigh number cannot be studied directly. To simplify the expression, assuming that the minimum of Hele-Shaw Rayleigh number occurs approximately at 3.12 and the value of a Gyrotactic number is taken as G = 0.03. Under these values, Eqs. (34)-(35) simplify as follows

$$\frac{\partial R_H}{\partial R_{b,H}} = \ -0.22(\mathrm{Q}_b = 0.1),\ -2.55(\mathrm{Q}_b = 1),\ -1.04\times10^4(\mathrm{Q}_b = 10) \tag{37}$$

It shows that bioconvection Péclet number has a destabilizing effect corresponding to all swimmers and this effect is more predominant for fast swimmers $(Q_b = 10)$.

$$\frac{\partial R_H}{\partial R_{b,H}} = \ -0.22(\mathrm{Q}_b = 0.1),\ -2.55(\mathrm{Q}_b = 1),\ -1.04\times10^4(\mathrm{Q}_b = 10) \tag{38}$$

It is observed that for all three values of bioconvection Péclet number, the bioconvection Rayleigh number has a destabilizing effect.

3.2. Revisited non-oscillatory instability: Six-term Galerkin weighted method

Using the single-term Galerkin weighted method it is not possible to obtain exact analytical solutions, therefore we using Six-term Galerkin weighted method. On using six-term Galerkin approximation, we have obtained a system of twenty-four linear algebraic equations in the twenty-four unknowns. This system of homogeneous algebraic equations can have a nontrivial solution if and only if

$$\det\begin{bmatrix} A & B & C & 0 \\ D & E & F & 0 \\ G & H & I & J \\ 0 & 0 & K & L \end{bmatrix} = 0 \tag{39}$$

Where,

$$A = \begin{bmatrix} < D\Theta_1 D\Theta_1 - \frac{N_A N_B}{L_e}\Theta_1\Theta_1 - a^2\Theta_1\Theta_1 > & \cdots & < D\Theta_1 D\Theta_6 - \frac{N_A N_B}{L_e}\Theta_1\Theta_6 - a^2\Theta_1\Theta_6 > \\ \vdots & \cdots & \vdots \\ < D\Theta_6 D\Theta_1 - \frac{N_A N_B}{L_e}\Theta_6\Theta_1 - a^2\Theta_6\Theta_1 > & \cdots & < D\Theta_6 D\Theta_1 - \frac{N_A N_B}{L_e}\Theta_6\Theta_1 - a^2\Theta_6\Theta_1 > \end{bmatrix},$$

$$B = \begin{bmatrix} < -\frac{N_B}{L_e}\Theta_1 D\Phi_1 > & \cdots & < -\frac{N_B}{L_e}\Theta_1 D\Phi_6 > \\ \vdots & \ddots & \vdots \\ < -\frac{N_B}{L_e}\Theta_6 D\Phi_1 > & \cdots & < -\frac{N_B}{L_e}\Theta_6 D\Phi_6 > \end{bmatrix},$$

$$C = \begin{bmatrix} < \Theta_1 W_1 > & \cdots & < \Theta_1 W_6 > \\ \vdots & \ddots & \vdots \\ < \Theta_6 W_1 > & \cdots & < \Theta_6 W_6 > \end{bmatrix},$$

$$D = \begin{bmatrix} < \frac{N_A}{L_e}\left(D\Phi_1 D\Theta_1 - a^2\Phi_1\Theta_1\right) > & \cdots & < \frac{N_A}{L_e}\left(D\Phi_1 D\Theta_6 - a^2\Phi_1\Theta_6\right) > \\ \vdots & \ddots & \vdots \\ < \frac{N_A}{L_e}\left(D\Phi_6 D\Theta_1 - a^2\Phi_6\Theta_1\right) > & \cdots & < \frac{N_A}{L_e}\left(D\Phi_6 D\Theta_6 - a^2\Phi_6\Theta_6\right) > \end{bmatrix},$$

$$E = \begin{bmatrix} \frac{1}{L_e}\left(D\Phi_1 D\Phi_1 - a^2\Phi_1\Phi_1\right) & \cdots & \frac{1}{L_e}\left(D\Phi_1 D\Phi_6 - a^2\Phi_1\Phi_6\right) \\ \vdots & \ddots & \vdots \\ \frac{1}{L_e}\left(D\Phi_6 D\Phi_1 - a^2\Phi_6\Phi_1\right) & \cdots & \frac{1}{L_e}\left(D\Phi_6 D\Phi_6 - a^2\Phi_6\Phi_6\right) \end{bmatrix},$$

$$F = \begin{bmatrix} < -N_A\Phi_1 W_1 > & \cdots & < -N_A\Phi_1 W_6 > \\ \vdots & \ddots & \vdots \\ < -N_A\Phi_6 W_1 > & \cdots & < -N_A\Phi_6 W_6 > \end{bmatrix},$$

$$G=\begin{bmatrix} <-R_H a^2 W_1\Theta_1> & \cdots & <-R_H a^2 W_1\Theta_6> \\ \vdots & \ddots & \vdots \\ <-R_H a^2 W_6\Theta_1> & \cdots & <-R_H a^2 W_6\Theta_6> \end{bmatrix},$$

$$H=\begin{bmatrix} <-R_{nH} a^2 W_1\Phi_1> & \cdots & <-R_{nH} a^2 W_1\Phi_6> \\ \vdots & \ddots & \vdots \\ <-R_{nH} a^2 W_6\Phi_1> & \cdots & <-R_{nH} a^2 W_6\Phi_6> \end{bmatrix},$$

$$I=\begin{bmatrix} <H_s(D^2W_1D^2W_1-2a^2DW_1DW_1+a^4W_1W_1)-(DW_1DW_1-a^2W_1W_1)> & \cdots & <H_s(D^2W_1D^2W_6-2a^2DW_1DW_6+a^4W_1W_6)-(DW_1DW_6-a^2W_1W_6)> \\ \vdots & \ddots & \vdots \\ <H_s(D^2W_6D^2W_1-2a^2DW_6DW_1+a^4W_6W_1)-(DW_6DW_6-a^2W_6W_6)> & \cdots & <H_s(D^2W_6D^2W_6-2a^2DW_6DW_6+a^4W_6W_6)-(DW_6DW_6-a^2W_6W_6)> \end{bmatrix}$$

$$J=\begin{bmatrix} <-\frac{R_{bH}}{L_b\nu}a^2W_1N_1> & \cdots & <-\frac{R_{bH}}{L_b\nu}a^2W_1N_6> \\ \vdots & \ddots & \vdots \\ <-\frac{R_{bH}}{L_b\nu}a^2W_6N_1> & \cdots & <-\frac{R_{bH}}{L_b\nu}a^2W_6N_6> \end{bmatrix},$$

$$K=\begin{bmatrix} <e^{Qz}Qv((1+Ga^2)N_1W_1-GN_1D^2W_1)> & \cdots & <e^{Qz}Qv((1+Ga^2)N_1W_6-GN_1D^2W_6)> \\ \vdots & \ddots & \vdots \\ <e^{Qz}Qv((1+Ga^2)N_6W_1-GN_6D^2W_1)> & \cdots & <e^{Qz}Qv((1+Ga^2)N_6W_6-GN_6D^2W_6)> \end{bmatrix}$$

$$L=\begin{bmatrix} <-\frac{1}{L_b}(DN_1DN_1-QN_1N_1-a^2N_1N_1)> & \cdots & <-\frac{1}{L_b}(DN_1DN_6-QN_1N_6-a^2N_1N_6)> \\ \vdots & \ddots & \vdots \\ <-\frac{1}{L_b}(DN_6DN_1-QN_6N_1-a^2N_6N_1)> & \cdots & <-\frac{1}{L_b}(DN_6DN_6-QN_6N_6-a^2N_6N_6)> \end{bmatrix}$$

Using the six-term Galerkin weighted residual method for the case of regular fluid (R_n=0, R_b=0) and D_a=∞, Rayleigh number attains its minimum value of 1707.76 at a=3.116. This critical value of Rayleigh number is same as obtained by Rayleigh-Bernard Problem (Chandrasekhar, 1961). In a regular fluid for the cases, D_a=0.1, and D_a=1 Rayleigh number attains its minimum values of 215.67, 1751.87 at a=3.15, and a=3.12. This critical values of Rayleigh number same as the value obtained by Guo and Kaloni (1995). Thus the use of a six-term Galerkin weighted residual method reduces the error by more than a factor of two.

To see the effect of swimming speed of gyrotactic microorganisms on the onset of bioconvection, the values of $R_{H,c}$ for different values of bioconvection Rayleigh number and Hele-Shaw number are compared in Table 1. It is observed that faster swimming bacterial species have the more destabilizing effect than the slowly swimming bacterial species.

4. Results and discussion

Based on the data presented in Pedley, Hill, and Kessler (1988), Guo and Kaloni (1995), Buongiorno (2006), the following values of the parameter for the alumina-water nanofluid are utilized:

$$\phi_0 = 0.01,\ \phi_1 - \phi_0 = 0.001, T_d = 300K, T_0 - T_c = 1\text{K},\ \rho_{f0} = 10^3\text{Kg/m}^3,\ \rho_p = 4\times10^3\text{Kg/m}^3,$$
$$(\rho c)_p = 3.1\times10^6\text{J/m}^3, (\rho c)_f = 4\times10^6\text{J/m}^3, \alpha_f = 2\times10^{-7}\text{m}^2/\text{s},\ \text{D}_B = 4\times10^{-11}\text{m}^2/\text{s},$$
$$\text{D}_T = 6\times10^{-11}\text{m}^2/\text{s}, D_m = 5\times10^{-8}\text{m}^2/\text{s}, \mu = 10^{-3}\text{pas, and } \beta = 3.4\times10^{-3}\text{1/K}$$

We have fixed the parameters as: $L_e = 500, L_b = 4, Q_b = 3.0, R_n = 1, R_b = 3.0, N_A = 2, N_B = 0.01, H_s = 0.9$. The values of Q_b are in the range between 0.1 to 10. In Fig.2, the effect of bioconvection Rayleigh number on Hele-Shaw Rayleigh number is analyzed graphically with respect to (a) $Q_b = 0.1$ (b) $Q_b = 1$ (c)$Q_b = 10$. Hele-Shaw Rayleigh number attains its minimum value at $a = 3.12$ in all cases. The critical Hele-Shaw Rayleigh number decreases with increase in the bioconvection Rayleigh number, hence its effect is to accelerate the onset of bioconvection. This result is expected from the physical point of view also because an increase in bioconvection Rayleigh number enhances the concentration of gyrotactic microorganisms at the top and develops top-heavy density stratification. From Figs. (2a)-(2c) it is also found that a bioconvection Péclet number accelerates the onset of bioconvection. Fast swimmers produce stronger disturbance, it thus facilitates the development of bioconvection resulting in a lower Rayleigh number at a larger value of bioconvection Péclet number.

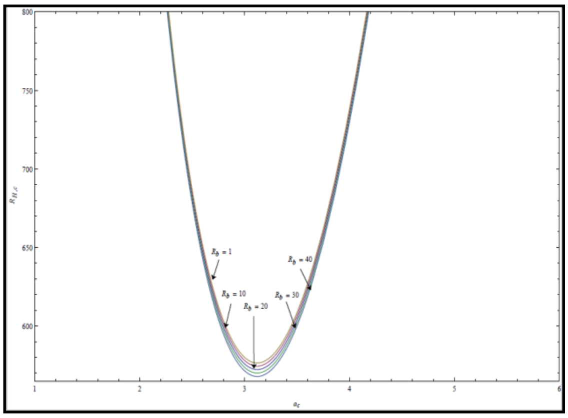

Figure 2(a). Variation of $\boldsymbol{R_{H,c}}$ with $\boldsymbol{a_c}$ for slow swimmers

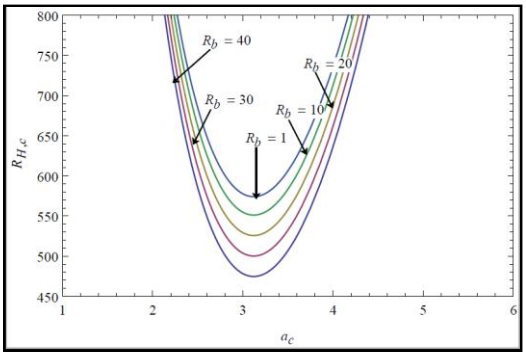

Figure 2(b). Variation of $\boldsymbol{R_{H,c}}$ *with* $\boldsymbol{a_c}$ *for intermediate swimmers*

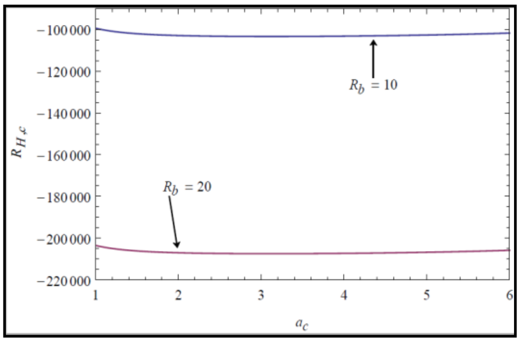

Figure 2(c). Variation of $\boldsymbol{R_{H,c}}$ *with* $\boldsymbol{a_c}$ *for fast swimmers*

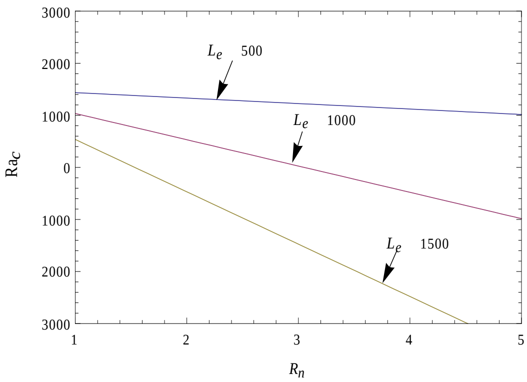

Figure 3. Variation of critical Hele-Shaw Rayleigh number with nanoparticle Rayleigh number for different values of Lewis number

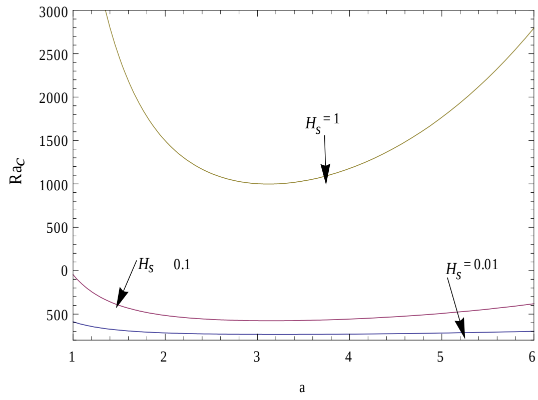

Figure 4. Variation of critical Hele-Shaw Rayleigh number with wave number for different values of Hele-Shaw number

Fig. 3 shows the variations of R_H when $L_e = 500,1000,1500$. It is observed that critical Hele-Shaw Rayleigh number decreases with increasing value of Lewis number. Lewis number reduces the mass diffusivity of the nanofluid which increases the nanoparticle volume fraction and subsequently increases the amount of heat transfer. Thus, Lewis number accelerates the onset of bioconvection. Nanoparticle Rayleigh number accelerate the onset of bioconvection, this result is expected from a physical point of view also, because an increase in a volumetric fraction increases the Brownian motion of nanoparticles which produce a destabilizing effect, as shown in Fig. 3.

Fig. 4 shows the variations of R_H when $H_S = 0.01,0.1,1$. Hele-Shaw Rayleigh number increases with increasing values of Hele-Shaw number. This result is expected physically because an increase in permeability of the Hele-Shaw cell, increase the width of the Hele-Shaw cell, which slows down forming of bioconvection pattern. Therefore, Hele-Shaw number hinders the development of bioconvection.

To see the effect of swimming speed of gyrotactic microorganisms on the onset of bioconvection, the values of $R_{H,c}$ for different values of bioconvection Rayleigh number and Hele-Shaw number are compared in Table 1. It is observed that fast swimming bacterial species have a more destabilizing effect than the slow-swimming bacterial species.

Table 1. Comparative results of the RH,c for Rb and Hs with G=0.03, Le=500, NA=2, and Rn=1 for the (i) Q=0.1, (ii) Q=1, and (iii) Q=10 using one-term and six-term weighted residual method

H_s	R_b	$R_{H,c}$		
		Slowly Swimmers (Q=0.1)	Intermediate Swimmers (Q=1)	Faster Swimmers (Q=10)

		One-term	Six-term	One-term	Six-Term	One-term	Six-term
0.1	10	-783.32	-785.18	-808.83	-809.78	-107781	-107764
	30	- 787.41	-789.81	-857.82	-860.63	-321781	-321724
	50	-791.33	-793.98	-908.80	-911.20	-535781	-535684
1	10	791.57	750.99	768.17	727.17	-106227	-106227
	30	787.88	746.67	717.81	676.84	-320206	-320187
	50	783.93	742.87	666.43	625.07	-534206	-534147
∞	10	745.52	706.46	722.48	682.96	-106252	-106292
	30	741.98	702.64	671.41	631.46	-320252	-320292
	50	737.37	698.48	620.54	580.96	-534252	-534292

Table 2 shows that critical wave number increases with increasing values of bioconvection Péclet number, bioconvection Rayleigh number and Gyrotaxis number, thus higher concentration and swimming speed of microorganisms reduce the size of cells.

Table 2. Numerical values of α_c for different values of R_b, G, Q_b using six-term Galerkin weighted residual method

G	Q_b	R_b	α_c
0.01			3.116
0.02			3.116
0.03			3.117
	0.1		3.113
	3		3.116
	10		3.125
		1	3.118
		20	3.124
		30	3.127

5. Conclusions

In this study, we investigate the convection of nanofluid containing gyrotactic microorganism, occupying a vertically oriented Hele-Shaw cell. The main conclusions of the present study are as follows:

1. The Lewis number, bioconvection Rayleigh number accelerates the onset of bioconvection.

2. The Hele-Shaw number hinders the development of bioconvection and stabilizes the onset of bioconvection.

3. The nanoparticle Rayleigh number destabilizes the onset of convection. It may be quality to the fact that increasing the volume fraction of nanoparticles, increase the irregular random motion of the particle and subsequently increase the amount of heat transfer.

4. The speed of microorganisms accelerates the onset of bioconvection.

5. The critical wave number is insensitive to the variation of nanoparticle Hale-Shaw Rayleigh number, Lewis number, modified particle density increment, and modified diffusivity ratio. It shows that Brownian motion and thermophoresis of nanoparticles do not change the cell size.

6. The critical wave number is strongly dependent on Hele-Shaw Rayleigh number, Gyrotaxis number, and Bioconvection Péclet number.

7. The gyrotactic microorganisms increase the critical wave number. Therefore in the presence of microorganisms the size of convection cells becomes smaller.

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