"SMALLER SPECIMENS OF THE SAME PROPERTIES ARE LESS VULNERABLE THAN LARGER ONES": A GENERAL RULE THAT EMERGES FROM "TAIL STATISTICS"

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ABSTRACT

In reality, values of a specimen (its strength, etc.) have certain probability distributions. Deterministic values in reality are also uncertain and have error margins. Thus, averages and standard deviations are used for estimates in engineering and nature, and major factors in estimations are based on the nominal value and the upper and lower allowed (survival, design, standards) limits. Observations beyond those limits indicate mostly a failure. However, probability distributions in nature and engineering show a relatively long "tail" beyond the limits of several standard deviations. The values of deviations from the mean (or target) can be very large, which can endanger the whole specimen. The phenomena at the "tail" domain (and not in the domain up to the limits) are the main reason for the fact that larger specimens are generally more vulnerable than smaller ones, having the same mean and probability distribution. The present paper extends this concept to relevant phenomena in nature and in technology.

Keywords: distribution tails, probability distributions, size effect, statistics, strength of specimen, upper standard limit.

1 INTRODUCTION

Stephen Jay Gould [1] has used a-symmetrical probability distributions to show that most of the durable living creatures in the cosmos are relatively simple and robust microorganisms. Evolution of those small entities leads into complex forms (such as the appearance of *Homo sapiens*) as a result of accidental (random) conditions that do not allow the almost everlasting durability that characterizes germs. One reason, as explained in Rosenhouse [2], is that larger bodies include statistically larger faults than smaller ones, no matter what kind of distribution is involved in each case, but, provided the distribution has a "tail" (or "tails") behind the mean which is (are) not truncated.

This principle is in accordance with the observation of Leonardo da Vinci (1452 – 1519) [3] who described in his notebook (in the Codex Atlanticus) measurements of the strength of wires of various lengths. He found that the shorter the wire, the stronger it will be, since wires are one-dimensional (and since wires of his time were of low quality and due to larger flaws in the longer wire). This result has led to the weakest link concept, where a wire in a discrete presentation by N chain elements has a smallest strength distribution like that of a sample of size N (see the note at the end of the paper). When the weakest link of the chain is overloaded by the ultimate stress σ_{cr}, and breaks, the whole chain fails.

If $p_1(\sigma_{cr})$ is the probability distribution of the ultimate strength of a unit length wire (up to $\sigma_{cr,max}$) and $p(D)$ is probability of distribution of the diameter D (up to D_{max}), then the probability that the unit length wire would resist a certain axial force F_0, is:

$$\hat{p}_1\left(F_{cr} = \frac{\pi\sigma_{cr}D^2}{4} > F_0\right) = \int_0^{\sigma_{cr,max}} p(\sigma_{cr}) \int_{D_0}^{D_{max}} p(D)\,d(D)\,d\sigma_{cr}; \quad D_0 = \sqrt{\frac{4F_{cr}}{\pi\sigma_{cr}}} \quad . \tag{1}$$

The relation of the result of eqn. (1) with that of a wire of 10 units of length is:

$$\hat{p}_k = \hat{p}_1^k.$$

(2)

If k = 10, then for $\hat{p}_1 = 0.9$, $\hat{p}_{10} = 0.9^{10} = 0.35$. It means that $\hat{p}_1 = 0.9$ is about 2.5 times stronger than $\hat{p}_{10} = 0.35$ [4]. The Leonardo size effect was validated by other researchers, such as Lloyd (1830 – see [5, 6]), Le Blank (1839) and Parsons [7].

The statistical theory of extreme value helps in formulating the problem for a given failure probability density distribution function $p(\sigma)$ for each element, with a cumulative distribution $P(\sigma)$ – basic distribution. Then, respectively, the distribution and the cumulative distribution of failure probability of the Leonardo chain (wire) are:

$$p_N(\sigma) = np(\sigma)\left\{1 - P(\sigma)\right\}^{N-1};$$

$$P_N(\sigma) = \int_{-\infty}^{\sigma} p(\sigma)dp = 1 - \left\{1 - P(\sigma)\right\}^{N}.$$

(3)

Now, in three dimensions if, for example, a stone is hit during a comminution process, it will be cut at its weakest surfaces. As an illustration, we consider the case that two parts are obtained out of the whole stone then each should be stronger than the whole stone (because the weakest weakness surface disappeared). If we go on and break each of the remaining two stones, each into two parts, following the same reasoning, the resulting four smaller stones will be stronger than the original two stones of the former stage and so on (see Fig. 1). Each stage of comminution reduces surfaces of weakest weaknesses, and thus leads to new smaller parts that are growing in strength at each stage. This process can go down even to the atomic scale. Research results have led to many specific theoretical and experimental results about comminution, some of which are detailed by Beke [8]. This mechanism is due to statistical reasons, assuming that specimens have lower minimal local strength with increase of volume. In this context, Fisher and Tippett [9] developed a mathematical distribution, which followed research by Tippett about the length effect on the strength of long fibers. It is known as Weibull distribution after Weibull [10] who developed independently an extreme value distribution of the same shape to describe the size effect on fatigue fracture of metals. Since then and up to now, many modifications appeared in the literature. In fact, the theory of size effect in quasi-brittle materials has led to only one possible distribution. Frécht [11] presented the stability postulate of extreme value statistics to verify that large size asymptotic behavior conforms to strength distribution of quasi-brittle materials, which was solved mathematically.

Frécht has shown that the survival probability $f_N(\sigma)$ for a structure of a very large N must satisfy asymptotically the functional equation, which depends on the strength σ:

$$\left\{f(\sigma)\right\}^{N} = f(a_N\sigma + b_N), \ a_N, b_N \ - \ \text{functions of the size N.}$$

(4)

Fisher and Tippett [9] in their extreme value statistics paper about textile fibers have proven that only three distributions can satisfy the recursive functional relation for $f(\sigma)$: Frécht, Gumbel and Weibull distributions. Since Frécht and Gumbel distributions allow for negative values of the argument and have no threshold, which prevents their use for strength of materials – the only remaining one is that of Weibull.

This last argument is important, for example, in numerical analysis by the stochastic finite element method [12]. So, for this specific problem, only one type of distribution could be used. For other cases, other distributions are suitable. Yet, for the argument of the present paper, which presents a general approach and suits many kinds of phenomena, the choice of distributions is practically unlimited.
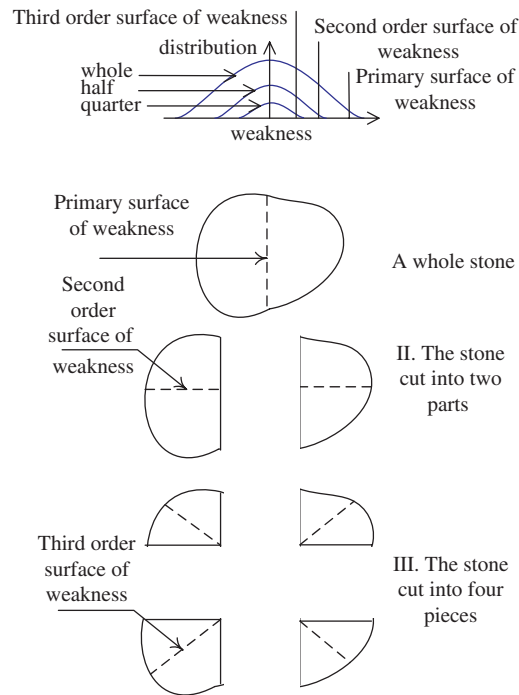
Figure 1: Illustration of a gradual comminution of a stone at its surfaces of weakness.

In this context, one should not miss the achievement of Galileo Galilei (1564–1642) who introduced to the size effect the scaling approach [13]. It followed his observation that larger boats are more vulnerable than the small ones. "The reason why they employed stocks, scaffolding and bracing of larger dimensions for launching a big vessel than they do for a small one; and he answered that they did this in order to avoid the danger of the ship parting under its own heavy weight [*vasta mole*], a danger to which small boats are not subject."

Galileo Galilei assumed a similar size effect on strength to structures and living organisms – the larger is weaker if the other properties of the bodies, but size are kept, since the smaller are more robust. They are stronger in proportion to the size (see the "first day" dialogue in the book).

Another point that should be noticed in scaling is the strength of structures built of thin elements. In this case, the stress is proportional also to its two dimensional cross sections. Now, if we increase each dimension of a body by factor ten, then its volume will increase by factor 1000 and so its weight, since it is three dimensional, but its cross sections by 100. Hence, the weight/stress factor will increase by 10, which makes the larger structure more vulnerable.

One conclusion up to now is that it is wrong to use small specimen strength results for larger specimens [14].

2 THE THEORY FOR NARTURAL POLYNOMIAL DISTRIBUTIONS

Ollendorff *et al*. [15] have used for wake-up analysis a distribution of sensitivity (wake up) data to events x (noise levels) of different strengths that exist naturally in common population of sample

size N (total number of noise events (see the note at the end of the paper). Such data appear currently in many publications [16]. The differential change of a distribution of a discrete form is:

$$Np(x) = \frac{\Delta N}{\Delta x}, \tag{5a}$$

and for a continuous distribution,

$$Np(x) = \frac{dN}{dx}, \tag{5b}$$

with p(x) as the probability density and N as the sample size. The probability density can be defined generally in mathematical terms of the series:

$$p(x) = -\sum_{j=0}^{n}\left[A_{j+1}\left(x - B_{j+1}\right)^{j}\right]. \tag{6}$$

A and B are constants to be solved by curve fitting based on measured data.

Yet, this natural pattern can be biased to some extent by external effects, and truncated by upper and lower limits. To prove the argument of the paper, the following Gaussian distribution is applied, but other distributions such as exponential, lognormal, Poisson, Rayleigh as well as extreme value theory distributions that deal with very low probability of size and frequency (Frécht, Gombel and Weibull for heavy, medium and short tails, respectively) can be used to show the argument of the paper, which is general in its consequences.

The discrete Gaussian distribution is:

$$p(x_i) = \frac{1}{\sigma\sqrt{2\pi}}\exp\left[\frac{-\left(x_i - m\right)^2}{2\sigma^2}\right], \quad m = \frac{\sum_{i=1}^{N}x_i}{N}, \quad \sigma = \sqrt{\frac{\sum_{i=1}^{N}\left(x_i - m\right)^2}{N}}, \tag{7}$$

x_i is a certain chosen value i of x, m is the mean or nominal value and σ is the standard deviation. It means dependence on two parameters, m and σ only and being symmetrical. This kind of curve usually leads to a better curve fitting at the average, and stronger deviation from measured values at the "tails" of the curve (see Fig. 2). However, since we deal with a comparison between a large and a small specimen of a similar distribution, this fact does not affect our proof, which suits any "tail."

The correlation between the series form (eqn. (6)) and the Gaussian distribution (eqn. (7)) is obtained by the following expansion:

$$e^{-\beta} = 1 - \beta + \frac{\beta^2}{2!} - \frac{\beta^3}{3!} + \frac{\beta^4}{4!} - \cdots; \quad \beta = \frac{\left(x - m\right)^2}{2\sigma^2}. \tag{8}$$

An important point is that the averages of the size of the faults in the small and the large speci-mens should be the same as well as the shape of the distribution, to avoid different conditions that may cause a shift and distortion of the whole distribution of the specimen and prevent an objective test.

The percentage of a population that falls within a given standard deviation can be calculated. Representative results are given in the second row in the following Table 1.
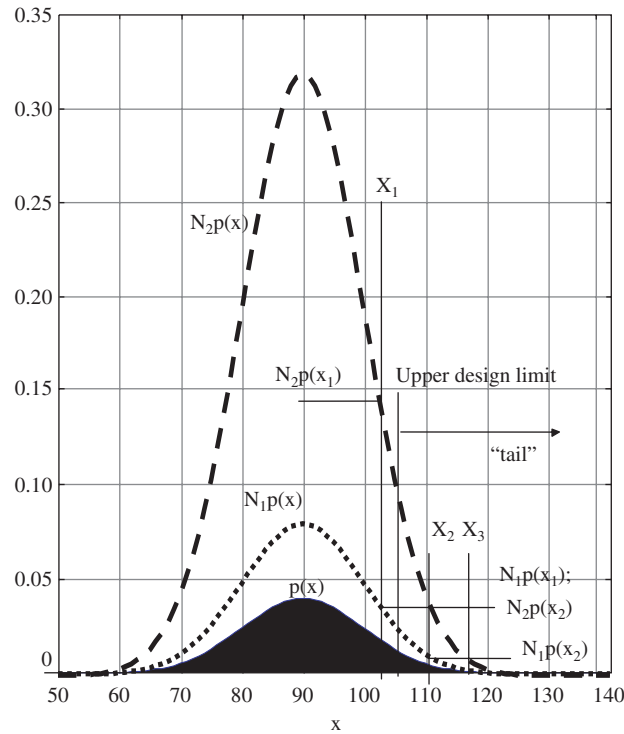
Figure 2: A continuous Gaussian probability density distribution, p(x), and distributions in the small (1) and the large specimens (2).

Table 1: The relations between standard deviation, population percentage and defective parts per million in a sample.

Standard deviation	±0.01σ	±1σ	±2σ	±3σ	±4σ	±4.5σ (long term)	±6σ (short term)
Population percentage	0.8	68.27	95.45	99.73	99.9937	99.99966	99.9999998
DPMO		317300	45500	2700	63	3.4	0.002

DPMO – defects per million opportunities.

3 A COMPARISON BETWEEN THE STRENGTH OF SMALL AND LARGE SPECIMENS OF THE SAME PROPERTIES

If we move from the probability distribution curve to the distribution of the faults in the specimen, the probability density of the ith specimen is $p_i(x)$, with $i = 1$ for the smaller specimen and $i = 2$ for the larger specimen. The number of events in each specimen is N_i, $i = 1, 2$ for the smaller and larger specimens, respectively, and the distributions in the specimens become:

$$y_i = N_i p_i(x)dx, \quad i = 1, 2, \ N_1 \ll N_2. \tag{9}$$

For the two specimens: $m_1 = m_2 = m$ and $p_1(x) = p_2(x)$.

Let x_1 be the strength of events of the smaller specimen, and x_2 be the strength of events of the larger specimen with the same number of events in the plot of y_i. Assuming a Gaussian distribution of y_i yields the expression that links between x_1 and x_2. The condition for that is:

$$N_2 p(x_2)dx = N_1 p(x_1)dx. \tag{10}$$

Hence,

$$\frac{N_2}{\sigma\sqrt{2\pi}}\exp\left[\frac{-(x_2-m)^2}{2\sigma^2}\right] = \frac{N_1}{\sigma\sqrt{2\pi}}\exp\left[\frac{-(x_1-m)^2}{2\sigma^2}\right],$$

$$m = \int_{-\infty}^{\infty} xp(x)dx; \quad \sigma^2 = E\left(x^2\right) - m^2; \quad E\left[x^2\right] = \int_{-\infty}^{\infty} x^2 p(x)dx.$$

Some more algebra leads to the solution:

$$x_2 - m = \sqrt{2\sigma^2 \ln\left(\frac{N_2}{N_1}\right) + (x_1-m)^2}. \tag{11}$$

It is deduced from eqn. (11) that $x_2 \gg x_1$ for $N_2 \gg N_1$ for the same number of events on the distribution curves of y_i. At the far end of the distribution "tail" (Fig. 2), where the distribution of the small specimen practically vanishes, events in the larger specimen still exist. Those additional faults are extremely defective and dangerous. They are the reason for the larger specimen being weaker than the smaller one.

The "size effect" has been connected to strength of materials theory. Marriotte (1620–1684) [17] investigated pressurized vessels to find the critical strain (stress), in parallel to Hooke [18]. What is more relevant here is the Griffith [19] fracture theory about the critical stress required for crack propagation in a brittle material. Kim *et al*. [14] have shown that larger specimens involve on the extreme, statistically larger faults in the right tail (where the left tail practically did not exist). On the other hand, Griffith has shown that for an elliptical crack of half length (radius), a, and specific energy, γ, the critical stress becomes:

$$\sigma_{cr} = \sqrt{\left(\frac{2\gamma E^*}{\pi a}\right)}; \quad \begin{array}{l} E = \text{Young's modulus for plane stress.} \\ E^* = E/\left(1-\nu^2\right) \text{ for plane stress; } \nu - \text{Poisson's ratio.} \end{array} \tag{12}$$

Since the critical stress is inversely proportional to "a," the larger specimen is weaker.

This rule can carry a more abstract and general form. Since our statement is useful for a variety of distribution "tails", the distribution in Fig. 2 has both left and right tails and usually one is the "good guy" (the "positive tail") and the second is the "bad guy" (the "negative tail"). By investigating how many medals each participating country is expected to win in the Olympic Summer Games, Beranrd and Busse [20] developed after other researchers such as Ball [21] and Johnson and Ali [22] a prediction theory. They determined two parameters that influence mostly the results, namely population size, which gives a better pool, and gross national product per capita (GNP) or gross domestic product (GDP). It should be noted that beyond a certain optimal value, an increase in GDP yields a negative influence [23]. Organizational ability is another parameter and so on. However, most important is the fact that larger populations have chances to win under the same conditions more medals than smaller populations. On the other hand, following Tolstoy, Turchin's [24] controversial

concept claims that imperial nations are doomed to collapse because as size increases, they lose their united social corporation. He borrows from the 14th century Ibn Khaldun the concept of "asabiya" (group feeling) that weakens as the empire grows. Turchin adds to his model a statistical theory that leads to life cycles of imperial nations. Hence, these last illustrations show "size effects" at the opposite distribution "tails" of populations.

4 A THEORY FOR IMPROVED PRODUCTS

The quality of items produced in industry can be changed by what we call total quality control, as Feigenbaum [25] defined: "Total quality control is an effective system for integrating the quality development, quality maintenance and quality improvement for integrating quality development, quality maintenance, and quality improvement efforts of the various groups in an organization so as to enable production and service at most economical levels which allow full customer satisfaction." This statement leads directly to the approach of "design for six sigma," including its many versions [26]. Originally, the method was formulated by Bill Smith from Motorola in the 1980s, following the 1920s methodologies (e.g., Shewhart, Deming, Juran) Generally, design for six sigma means a short process that leads to six standard deviations between the mean (target) and the nearest specification limit (USL – upper specification limit, and LSL – lower specification limit). For long processes, it is taken as 4.5σ + an expected 1.5σ drift (see Table 1). It means 0.002 defects per million (practically no failure) in a short process and 3.4 defects per million in the long process.

The following abstract example shows how the effect of increase of specimen size can be used to find strength of materials (by acoustic emission), agricultural products testing, comminution of stones, environmental state evaluation (as by increasing the size of an airport, size of nations, and so on).

Given the following data: $m = 100$ units, $\sigma = 20$ units, $x_1 = 115$ units, $N_2/N_1 = 4$, with the question: What is the value of x_2 for the same number of events per dx or Δx?

Using eqn. (11) for $x_1 - m = 15$ units results in:

$$x_2 - m = \sqrt{2 \times 20^2 \ln(4) + 225},$$
$$x_2 = 100 + 36.52 = 136.52 \text{ units.}$$

This is a remarkable increase in the severity of events or defects as compared with x_1. Now, if x_2 has to satisfy the six-sigma criterion, σ should be restricted to the maximum:

$$\hat{\sigma} = \frac{36.52}{6} = 6.08 \text{ units.}$$

This can be achieved in industry by improving production procedures. In nature, evolution does it at a relatively low rate.

Let us return now to the sample size effect (see the note at the end of the paper). As the sample size increases, the standard deviation error estimate has much slower convergence than that of the mean. It means that defects rate estimation is strongly influenced by the uncertainty in standard deviation, which means that the analysis of the number of defects per million opportunities leads to an additional error.

It is possible to include in our argument another category of distributions, which involves "long tails." It means observations of high amplitudes followed by a long tail of observations of low amplitude. Those include the power law, Zipf law, Pareto and general Levy distributions, etc. [27]. A great part of the total number of events is included in the long tail distributions. In such distributions,

"the winner takes all" and the 80/20 rule (20% of the population holds 80% of the total wealth) govern. This means that more diversity and freedom in choice make inequality more extreme. This is not the whole story, since, the internet and the on line marketing deal with the importance of the tail of the distribution, which can constitute in such cases a major share of the given volume of objects (see Fig. 3). Consequently, the term "long tail" refers mostly to distributions that have thicker tails compared with that of the exponential distribution.

This modern view about statistics can be useful commercially, as, for example, online bookstores, know very well that the "long tail" of less popular or rare books is quantitatively not less important than the small number of best sellers [28], which makes the strategies of on line sales different from those of the classical shops.

Some people claim the long tail philosophy emerged at the late 1700s. Dave Free [29] introduced Eli Whitney as the original entrepreneur of the "long tail" due to the development of interchangeable parts invention. This invention had an important influence on the American production and marketing. At that time, also Honoré Blanc (1778) and others promoted the use interchangeable parts. The long tail approach (in fact, Zipf's law [30] with indication of the base and the rest of the tail as separate entities) was introduced by Anderson [31, 32].

Kilkki [33] formulated a model for such long tail, with parameters that can be obtained by curve fitting of real data. The variables and parameters of this specific mathematical model are the rank – x (the location of the item in the best sellers list, which includes a certain amount of copies; the best seller is ranked as x = 1); the number of objects that cover half of the whole volume – N_{50}; the factor that defines the form of the function – α (has to be less than 1); the total volume – β. Finally, the mathematical form of the model is:

$$F(x) = \frac{\beta}{\left(\dfrac{N_{50}}{x}\right)^{\alpha} + 1}. \tag{13}$$
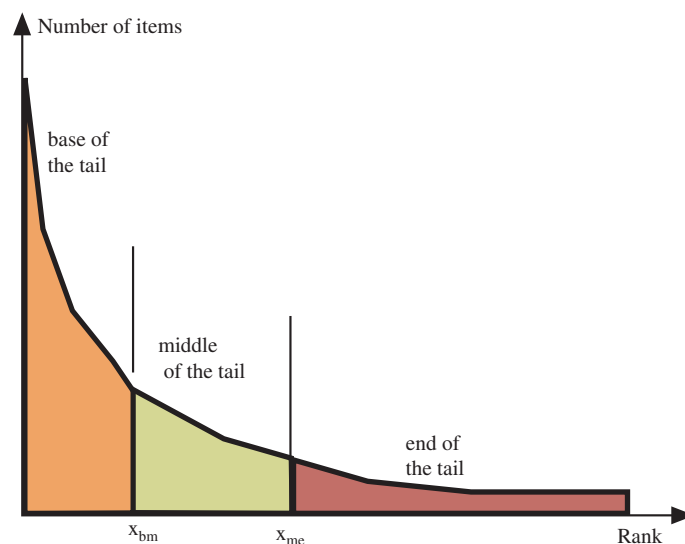

Figure 3: The "long tail" split into three parts.

As an example to the Amazon sales Kilkki assumes: $N_{50} = 400{,}000$, $\alpha = 0.48$, $\beta = 1.2$. The form of the real tail can be determined by the difference between two consecutive values of the cumulative distribution, which is: $F(x) - F(x - 1)$.

The "tail statistics" that is presented here differs from the popular "long tail statistics" since in the long tail statistics is interested in the volume of the tail or parts of it, while the tail statistics, as it appears here is interested in the severity of events at the distant parts of the "negative tail," since the horizontal scale in the tail statistics measures the severity of single events (and in some cases even one event can do the difference). However, in both kinds of statistics, a careful study of the distribution tail statistics becomes a major issue.

As in the "long tail statistics," also the regular tail shown in Fig. 2, for example, can be split into several parts. See Fig. 3 for the long tail statistics, where Kilkki recommends splitting the tail domains for the "long tail statistics" at:

$$x_{bm} = N_{50}^{2/3}; \quad x_{me} = x_{bm}^{2} = N_{50}^{4/3}. \tag{14}$$

The "tail statistics" can be split in a different way. Then, after division, events at the end of the tail can be considered most dangerous, while those at the base of the "negative tail" can be considered not too risky (depends on the case) and between these extremes exist the cases of a certain risk, but not the maximum. This argument is also true for the "positive tail," where we find the "gold medal winners."

For certain reasons, the tail can drop at relatively high rank numbers – a situation that does not appear theoretically. However, in the long tail practice, there is no sales potential beyond a certain high level rank. Nielsen [34] has shown this effect by using a log–log scale.

In the classical tail statistics, this situation occurs at the upper limit of the effect of the individual event.

5 CONCLUSION

In spite of the fact that the average and the distribution of properties can be the same for large and small specimens of the same material and conditions, it is shown in this paper that specimens of larger size include more defects that are more dangerous than those in smaller specimens. It means that the larger specimen will be at higher risk. This is the main reason why larger specimens are more vulnerable, even if we do not add to it the scaling effect.

The explanation provided here can be used in a large variety of statistical problems, where the distributions include a mean band that includes most of the observations and decaying distribution tails.

It has to be noted here that in contrast to the "long tail statistics," the "tail statistics" involves usually a distribution of two tails (see Figs. 2 and 3) and the effect of the lower tail (at low x numbers), which is opposite to the effect the upper tail (compared to the average) depends on the kind of problem.

In contrast to defects, if we try to select, for example, medal winners for two populations under the same conditions, it is most probable that larger population will provide them, with some statistical exceptions that depend on other factors.

NOTE

Sample size is the number of observations in a sample ([35], p. 16). If we test, for example, a metal specimen by acoustic emission the sample size of a larger specimen will be larger than the sample size of a smaller one under the same conditions. The same will happen if we sample populations

using a sample size proportional to their magnitude. One possibility is to consider the whole population in each case. Otherwise, the sampling should obey relevant statistical rules. As an example, an incidental simple sampling of a certain population is defined as a collection of observations which is independent of that population.

REFERENCES

[1] Gould, S.J., *Full House*, Crown Publishers: New York, 1996.

[2] Rosenhouse, G., A universal law that explains why larger samples are more vulnerable than smaller ones. *5th Int. Conf. on Quality, Reliability, and Maintenance – QRM*, ed. G.J. McNulty, Oxford: UK, pp. 315–318, 2004.

[3] da Vinci, Leonardo, References: McCurdy, E., *The Notebooks of Leonardo da Vinci*, 1500s, p. 546, 1945; or Ravaisson-Mollien, C., *Les Manuscrits de Leonard de Vinci*, translated to French by Institut de France (1881–1891), Vol. 3.

[4] Lund, J.R. & Byrne, J.P., Leonardo da Vinci's tensile strength tests: implications for the discovery of engineering mechanics. *Civil Eng. and Env. Syst.*, **18(3)**, pp. 243–250, 2001. (Permission of Gordon and Breach Sci. Publishers.)

[5] Rossmanith, H.P., The importance of engineering fracture mechanics in structural integrity: a short history of fracture mechanics. *Technology, Law and Insurance,* **2(4)**, pp. 195–229, 1997.

[6] Todhunter, I. & Pearson, K., *A History of the Theory of Elasticity and the Strength of Materials from Galilei to the Present Time*, Cambridge, 1886.

[7] Parsons, W.B., *Engineers and Engineering in the Renaissance*, MIT Press: Cambridge, MA, 1939.

[8] Beke, B.B., *Principles of Comminution*, Akademiai Kiado: Budapest, 1964.

[9] Fisher, R.A. & Tippett, L.H.C., Limiting forms of the frequency distribution of the largest and smallest member of a sample. *Proc. Cambridge Philosophical Society*, **24**, pp. 180–190, 1928.

[10] Weibull, W., The phenomenon of rupture in solids. *Proc. Roy. Swedish Inst. of Engng Research*, **153**, pp. 1–55, 1939.

[11] Frécht, M., Sur la loi de probabilité de l'écart maximum. *Ann. Soc. Plon. Math.*, **6**, p. 93, 1927.

[12] Novák, D, Vorechovský, M., Pukl, R. & Červenka, V., Statistical nonlinear fracture analysis: size effect of concrete beams. *Int. Conference on Fracture Mechanics of Concrete Structures FraMCos 4*, eds. De Borst *et al*. Balkema: Cachan, France, and Rotterdam, 2001.

[13] Galilei, Galileo, Limceo. *Diuscorsi e Dimonstrrazioni Matematiche*, intorno á due Nuove Scienze, Elsevirii: Leiden, 1638; or Galilei, Galileo, *Dialogues Concerning Two New Sciences*, translated by Henry Crew & Alfonso de Slavio, 1954,

[14] Kim, H.C., Elvin, D.E.C. & Rosenhouse, G., The source-strength specimen size effects on acoustic emission energy release during work-hardening of NaCl single crystals. *Phil. Mag. A*, **40(1)**, pp. 49–64, 1979.

[15] Ollendorff, F., Schaudinischky, H. & Rosenhouse, G., Noise level threshold for wakening and wake-up dependence on noise level. *Acustica*, **32(2)**, pp. 100– 03, 1975.

[16] Ficon, *Federal Agency Review Selected Airport Noise Analysis Issues*, Federal Interagency Committee on Noise: Washington, DC, 1992.

[17] Mariotte, E., *Traité du Mouvement des Eaux*, posthumously edited by M. de la Hire, English trans. by J.T. Desvaguliers, p. 249, 1718; or *Maroitte Collected Works*, 2nd edn., p. 1686, 1740.

[18] Hooke, R., *De Potentia Restitutiva*; or *Of Spring Explaining the Power of Springing Bodies*, p. 23, 1678 (1686).

[19] Griffith, A.A., The phenomena of rupture and flow of solids. *Philosophical Trans. Roy. Soc. of London, Series A*, **221**, pp. 163–198, 1921.

[20] Bernard, A.B. & Busse, M.R., Who wins the Olympic Games: economic resources and medal totals. *Review of Economics and Statistics*, **86(1)**, pp. 413–417, 2004.

[21] Ball, D.W., Olympic Games competition: structural correlates of national success. *Journal of Comparative Sociology*, **15**, pp. 186–200, 1972.

[22] Johnson, D.K.N. & Ali, A., Coming to play or coming to win: participating and success at the Olympic Games, Wellesley College Working paper 2000-10, 2000.

[23] Hoffmann, R., Ging, L.C. & Ramasamy, B., The socio-economic determinants of international soccer performance. *J. Appl. Economics*, **V(2)**, pp. 253–272, 2002.

[24] Turchin, P., *War and Peace and War: The life Cycle of Imperial Nations*, Pi Press: New York, 2006.

[25] Feigenbaum, A.V., *Quality Control: Principles, Practice, and Administration*, McGraw-Hill, New York, 1951; re-issue: *Total Quality Control*, McGraw-Hill Professional, 2004.

[26] Mikel, H. & Schroeder, R., *Six Sigma*, Random House: New York, 2002.

[27] Shirky, C., Power laws, weblogs and inequality (Chapter 3). *Extreme Democracy*, eds. J. Lebkowsky & M. Radcliffe, Lulu Marketplace: US, 2005.

Brynjolfsson, E., Hu, J.Y. & Smith, M.D., Consumer surplus in the digital economy: estimating the value of increased product variety at online book sellers. *Management Science*, **49(11)**, pp. 1580–1596, 2003.

[28] Free, D., The original "long tail" entrepreneur. http://www.seedsofgrowth.com/the-original-long-tail-entrepreneur, September 21, 2006, accessed January 2008. Also: http://en.wikipedia.org/wiki/Interchangeable_parts, accessed January 2008.

[29] Li, W., Zipf's law. http://www.nslij-genetics.org/wli/zipf, accessed January 2008. See also: http://planemath.org/encyclopedia/Zipfs Law.html, accessed January 2008.

[30] Anderson, C., The long tail. *Wired*, Issue 12.10, October 2004. http://www.wired.com/wired/archive/12.10/tail.html, accessed January 2008.

[31] Anderson, C. *The Long Tail*, Random House: London, 2006.

[32] Kilkki, K., A practical model for analyzing long tails. *First Monday*, **12(5)**, 2007. http://www.firstmonday.org/issues/issue12_54/kilkki, accessed January 2008.

[33] Nielsen, J., Data visualization of web states: logarithmic charts and the dropping tail. http://www.useit.com/alertbox/visualizing-traffic-analysis.html, August 14, 2006, accessed January 2008.

[34] Evans, M., Hastings, N. & Peacock, B., *Statistical Distributions*, 3rd edn., Wiley: New York, 2000.