Geometries, invariances, and SNR interpretations of matched and adaptive subspace detectors[1]

Géométries, invariances et interprétations par le rapport signal à bruit de détecteurs en sous-espaces adaptés ou adaptatifs

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abstract

Matched subspace detectors generalize the matched filter by accommodating signals that are only constrained to lie in a multidimensional subspace. There are four of these detectors, depending upon knowledge of signal phase and noise power. The *adaptive subspace detectors* generalize the matched subspace detectors by accommodating problems where the noise covariance matrix is unknow, and must be estimated from training data. In this paper we review the geometries and invariances of the matched and adaptive subspace detectors. We also establish that every version of a matched or adaptative subspace detectors can be interpreted as an estimator of output signal-to-noise ratio (SNR), in disquise.

1. Introduction

The matched filter tests for the presence of a signal in a noisy measurement by resolving the projection of a measurement onto the signal template, in a coordinate system whitened by the noise covariance matrix. This statistic can be generalized to cases where the signal phase is unknown, and to cases where the noise scaling is unknown [1, 2, 3, 4]. In the latter case, the *coherence* or *direction cosine* between the measurement and signal is computed. When the signal is known only to lie in a multidimensional linear subspace, then no such template may be used. The theory of "matched subspace detectors" (MSDs) shows that the resolution onto a template is replaced by a projection onto the signal subspace [5, 6]. The MSDs are not *ad hoc* variations of the matched filter; rather they are Uniformly Most Powerful (UMP) Invariant Neyman-Pearson tests, and Generalized Likelihood Ratio Tests (GLRT) [5, 6, 7]. When the noise covariance matrix is unknown, then it must be estimated. This produces variations on the MSDs, which are called "adaptive subspace detectors" (DSDs), and which employ a sample covariance matrix estimated from training data. In this paper we review the geometries, invariances, and optimalities of the MSDs and ASDs. Their distributions are treated in [5]. Then we reinterpret the MSDs and ASDs as estimators of output sigal-to-noise ratio (SNR). The key to this interpretation is a coordinate rotation to align the measurement with respect to an orthogonal basis defined by the signal and the measurement.

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2. Matched subspace detectors: geometries and invariances

The matched subspace detectors have clearly stated optimalities and invariances and have evocative geometrical interpretations [5, 6, 7]. They are uniformly most powerful (UMP) for detecting a subspace signal in Gaussian noise among all detectors required to be invariant to relevant transformations of the measurement [5]. They are also all Generalized Likelihood Ratio Tests (GLRT) [6]. In the paragraphs to follow we review their geometries and invariances.

2.1. Matched filter

The simplest of the four MSDs is the *matched filter*, proportional to a weighted inner product between a signal template $\underline{\psi}$ and the measurement vector $\underline{y}$:

$$n = \frac{\underline{\psi}^* \mathbf{R}^{-1} \underline{y}}{\sqrt{\underline{\psi}^* \mathbf{R}^{-1} \underline{\psi} \sigma}} \gtrless \eta \tag{1}$$

The vectors $\underline{\psi}$ and $\underline{y}$ are N-dimensional and the matrix $\mathbf{R}$ is $N \times N$. The threshold η is chosen so that the probability of falsely choosing H_1 (signal present) in a test of H_0 (signal absent) : $\underline{y} : CN[\underline{0}, \sigma^2 \mathbf{R}]$ versus H_1 (signal present) : $\underline{y} : CN[\mu \underline{\psi}, \sigma^2 \mathbf{R}]$ is α. In this problem, the signal $\underline{\psi}$ and its phase, and the noise scaling and covariance $\sigma^2 \mathbf{R}$ are known. Only the siganl gain μ is unknown; the detectors is UMP over all values of μ.

The whitened version of this problem is to test YH_0 : $\underline{z}$: $CN[\underline{0}, \sigma^2 \mathbf{I}]$ versus H_1 : $\underline{z}; CN[\mu \phi, \sigma^2 \mathbf{I}]$ where $\underline{z} = \mathbf{R}^{-1/2} \underline{y}$ is the whitened measurement and $\phi = \mathbf{R}^{-1/2} \underline{\psi}$ is the whitened signal. In this coordinate system, the matched filter is

$$n = \frac{\underline{\phi}^* \underline{z}}{\sqrt{\underline{\phi}^* \phi \sigma}} \gtrless \eta, \tag{2}$$

The statistic n measures the *resolution* of the measurement $\underline{y}$ onto the subspace $\langle \underline{\psi} \rangle$, in a coordinate system whitened by $\mathbf{R}^{-1/2}$. Alternatively, it is the resolution of the whitened measurement $\underline{z}$ onto the whitened signal $\underline{\phi}$. The detector is invariant to translations of the measurement $\underline{z}$ in the orthogonal subspace $\langle \underline{\phi} \rangle^{\perp}$. Thus, as illustrated in Figure 1, the invariance set of transformed measurements is a plan, where the projection of the measurement $\underline{z}$ onto the subspace $\langle \underline{\phi} \rangle$ is constant.

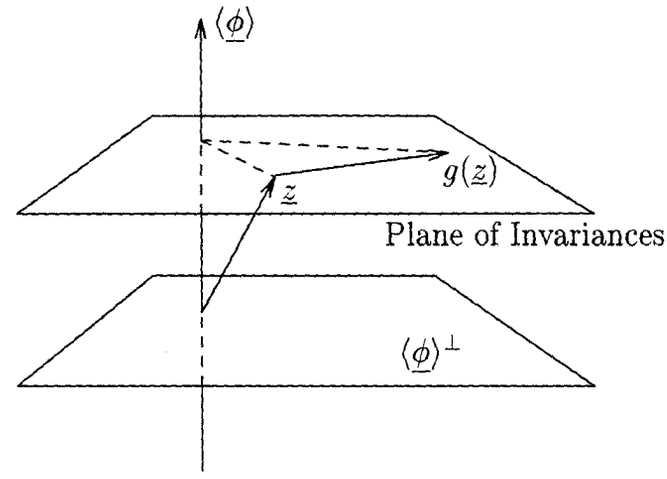

Figure 1. – Invariances of Matched Filter. The invariance set is the surface of the plane, where the projection of $\underline{z}$ onto the whitened signal subspace is constant.

2.2. Matched subspace detector

When the phase of the signal template $\underline{\psi}$ is unknown, then the UMP detector for detecting a subspace signal in Gaussian noise, amoung all detectors required to be invariant to rotations in the whitened signal subspace and translations in the orthogonal subspace, is given by magnitude squaring the matched filter [5] :

$$\chi^2 = \frac{|\underline{\psi}^* \mathbf{R}^{-1} \underline{y}|^2}{(\underline{\psi}^* \mathbf{R}^{-1} \underline{\psi}) \sigma^2} \gtrless \eta. \tag{3}$$

In this problem, the signal subspace $\langle \psi \rangle$ and the noise scaling and covariance $\sigma^2 \mathbf{R}$ are known. The phase of the sigal and the signal gain μ are unknown.

In whitened coordinates, the matched subspace detector is [5, 6]

$$\chi^2 \frac{|\underline{\phi}^* \underline{z}|^2}{(\phi^* \phi) \sigma^2} = \frac{\underline{z}^* \mathbf{P}_{\phi} \underline{z}}{\sigma^2} \gtrless \eta, \tag{4}$$

where $\mathbf{P}_{\phi}$ is the projection onto the subspace $\langle \underline{\phi} \rangle$:

$$\mathbf{P}_{\phi} = \underline{\phi} (\underline{\phi}^* \underline{\phi})^{-1} \underline{\phi}^*. \tag{5}$$

The statistic χ^2 computes the *energy* of the measurement $\underline{y}$ in the subspace $\langle \underline{\psi} \rangle$, in a coordinate system whitened by $\mathbf{R}^{-1/2}$. The detector is invariant to rotations of the $\underline{z}$ in the subspace $\langle \underline{\phi} \rangle$, and to translations in the orthogonal subspace $\langle \underline{\phi} \rangle^{\perp}$. Thus, as illustrated in figure 2, the invariance set of transformed measurements is the surface of two planes, where the energy of the measurement $\underline{z}$ in the subspace $\langle \underline{\phi} \rangle$ is constant.

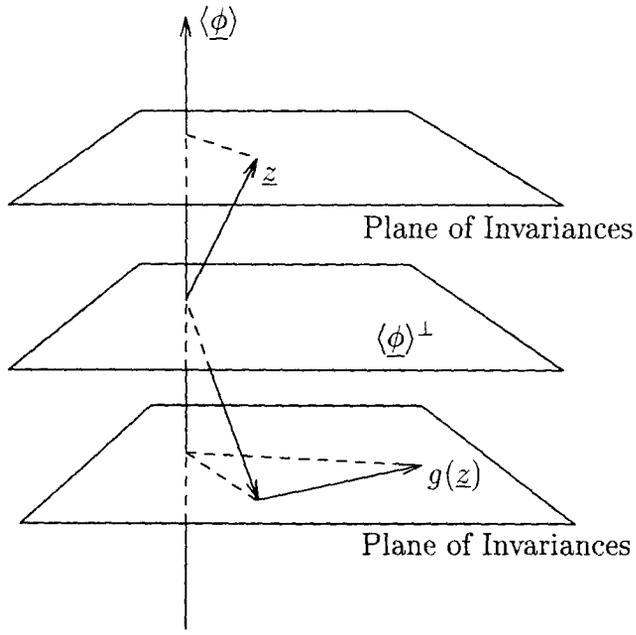

Figure 2. – Invariances of Matched Subspace Detector. The invariance set is the surface of the two planes, where the energy of $\underline{z}$ in the whitened signal subspace is constant.

2.3. CFAR matched filter

When the phase of the signal vector $\underline{\psi}$ is known, but the noise scaling is not, then the UMP detector among all detectors required to be invariant to measurement scaling and rotations in the orthogonal subspace $\langle\underline{\phi}\rangle^{\perp}$, is given by normalizing the matched filter by the magnitude of the measurement vector in whitened coordinates [5] :

$$\cos = \frac{\underline{\psi}^{*}\mathbf{R}^{-1}\underline{y}}{\sqrt{\underline{\psi}^{*}\mathbf{R}^{-1}\underline{\psi}}\sqrt{\underline{y}^{*}\mathbf{R}^{-1}\underline{y}}} = \frac{\underline{\phi}^{*}\underline{z}}{\sqrt{\underline{\phi}^{*}\underline{\phi}}\sqrt{\underline{z}^{*}\underline{z}}} \gtrless \eta \qquad (6)$$

In this problem, the signal $\underline{\psi}$ and phase and the noise covariance $\mathbf{R}$ are known, but the complex signal gain μ and the noise scaling σ^2 are unknown. The statistic *cos* measures the *coherence*, or *cosine*, between the measurement $\underline{y}$ and the subspace $\langle\underline{\psi}\rangle$ in a coordinate system whitened by $\mathbf{R}^{-1/2}$. It is also the cosine between the whitened measurement $\underline{z}$ and the whitened signal $\underline{\phi}$. The detector is invariant to rotations of the measurement $\underline{z}$ in the orthogonal subspace $\langle\underline{\phi}\rangle^{\perp}$, and to scaling of the measurement. Thus, as illustrated in Figure 3, the invariance set of transformed measurements is the surface of a cone, where the angle that the measurement $\underline{z}$ makes with the signal $\underline{\phi}$ is constant.

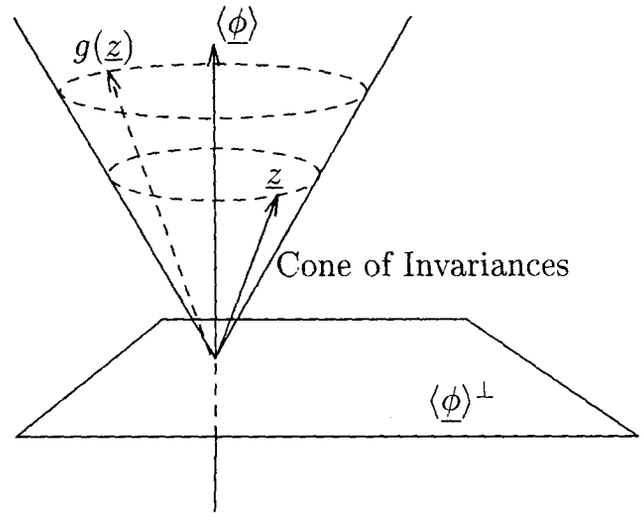

Figure 3. – Invariances ofn CFAR Matched Filter. The variance set is the surface of the cone, where the angle $\underline{z}$ makeswith the whitened signal is constant.

2.4. CFAR matched subspace detector

When neither the phase of the signal vector $\underline{\psi}$ nor the noise scaling is known, The UMP detector among all detectors required to be invariant to measurement scalings and subspace rotations in $\langle\underline{\phi}\rangle$ and $\langle\underline{\phi}\rangle^{\perp}$, is given by normalizing the matched filter by the magnitude-squared of the measurement vector in whitened coordinates [5] :

$$\beta = \frac{|\underline{\psi}^{*}\mathbf{R}^{-1}\underline{y}|^2}{(\underline{\psi}^{*}\mathbf{R}^{-}1\underline{\psi})(\underline{y}^{*}\mathbf{R}^{-}1\underline{y})} = \frac{|\underline{\phi}^{*}\underline{z}|^2}{(\underline{\phi}^{*}\underline{\phi})(\underline{z}^{*}\underline{z})} = \frac{\underline{z}^{*}\mathbf{P}_{\phi}\underline{z}}{\underline{z}^{*}\underline{z}} \gtrless \eta \qquad (7)$$

In this problem, the signal subspace $\langle\underline{\psi}\rangle$ and the noise covariance $\mathbf{R}$ are known. The phase of the signal, signal gain μ, and the noise scaling σ^2 are unknown. This detector was first advocated in a coherent "*t*" form in the 1970's [1, 2, 3, 4]. It has also been suggested by Conte, Lops, and Ricci, who derived it as a limiting-case GLRT for detecting signals in compound-Gaussain noise of Known covariance structure [8, 9]. the statistic β measures the *squared coherence*, or *cosine-squared* between the measurement $\underline{y}$ an the subspace $\langle\underline{\psi}\rangle$ in a coordinate system whitened by $\mathbf{R}^{-1/2}$. It is also the fraction of the whitened measurement energy that lies in the signal subspace $\langle\underline{\phi}\rangle$. The detector is invariant to rotations of the measurement $\underline{z}$ in the subspace $\langle\underline{\phi}\rangle$, to rotations in the orthogonal subspace $\langle\underline{\phi}\rangle^{\perp}$, and to scaling of the measurement. Thus, as illustrated in Figure 4, the invariance set of transformed measurements is the surface of a double cone, where fractional energy of $\underline{z}$ in the subspace $\langle\underline{\phi}\rangle$ is constant.

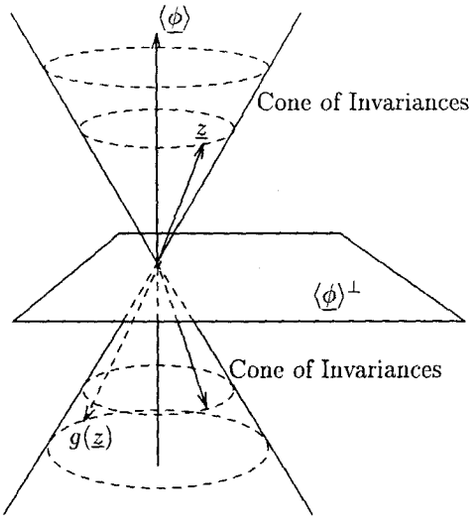

Figure 4. – Invariances ofn CFAR MatchedSubspace Detector. The invariance set is the surface of the double cone, where the fractional energy of $\underline{z}$ in the whitened,signal suibspace is constant.

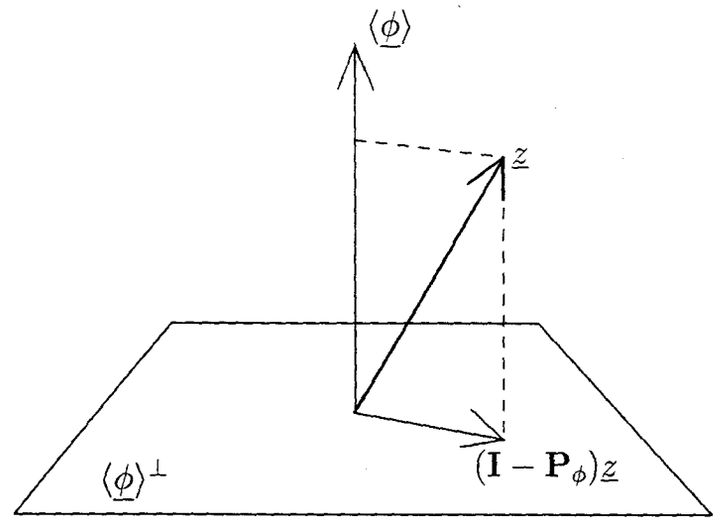

Figure 5. – Decomposition of $\mathbb{N}^n$ into signal subspace $\langle\underline{\phi}\rangle$ and subcomponents of the orthogonal subspace $\langle\underline{\phi}\rangle^{\perp}$.

3. Matched subspace detectors: SNR interpretations

In order to interpret the MSDs in terms of output SNRs we will need to resolve the measurement energy $\underline{z}^*\underline{z}$ into its constituent parts. To this end, we will construct an orthogonal span for the plane defined by the signal $\underline{\phi}$ and the measurement $\underline{z}$ and complete N-dimensional space by constructing a unitary subspace perpendicular to his plane. Thus we define the rotation matrix

$$\mathbf{U} = [\underline{u}_{\phi}, \underline{u}_z, \mathbf{V}], \tag{8}$$

where $\underline{u}_{\phi}$ is a unit vector in the direction of $\underline{\phi}$ and $\underline{u}_z$, is a unit vector in the direction of $(\mathbf{I}-\mathbf{P}_{\phi})\underline{z}$:

$$\underline{u}_{\phi} = \underline{\phi}\frac{1}{\sqrt{\underline{\phi}^*\underline{\phi}}}; \tag{9}$$

$$\underline{u}_z = (\mathbf{I}-\mathbf{P}_{\phi})\underline{z}\frac{1}{\sqrt{\underline{z}^*(\mathbf{I}-\mathbf{P}_{\phi})\underline{z}}}.$$

The matrix $\mathbf{V}$ is a rank $(N-2)$ unitary matrix with the property $\mathbf{V}^*\mathbf{U} = [\underline{0}, \underline{0}, \mathbf{I}]$. This decomposition of complex Euclidean space is illustrated in Figure 5. The idea is to insert : $\mathbf{U}\mathbf{U}^* = \mathbf{I}$ into $\underline{z}^*\underline{z}$ to reveal the energy components in the signal subspace and the orthogonal subspace.

3.1. Matched susbspace detectors

The matched filter of Equation 1 may be written as,

$$n = \frac{\underline{\phi}^*\underline{z}}{\underline{\phi}^*\underline{\phi}\sigma}\sqrt{\underline{\phi}^*\underline{\phi}} = \frac{\overline{\mu}}{\sigma}\sqrt{\underline{\phi}^*\underline{\phi}}, \tag{11}$$

where $\overline{\mu}$ is a *sample estimate* of the signal gain :

$$\overline{\mu} = \frac{\underline{\phi}^*\underline{z}}{\underline{\phi}^*\underline{\phi}}. \tag{12}$$

Note $\overline{\mu} : CN[\mu, \sigma^2/\underline{\phi}^*\underline{\phi}]$ and $n : CN[\frac{\mu}{\sigma}\sqrt{\underline{\phi}^*\underline{\phi}}, 1]$. In this form the matched filter is seen to be an estimator of input (voltage) SNR, namely $\overline{\mu}/\sigma$, times voltage gain $\sqrt{\underline{\phi}^*\underline{\phi}}$, meaning it is an estimate of the output (voltage) SNR :

$$n = \overline{\text{VSNR}} = \frac{\overline{\mu}}{\sigma}\sqrt{\underline{\phi}^*\underline{\phi}}. \tag{13}$$

In a similar manner, the matched subspace detector of Equation 3 can be written as an estimate of the output SNR,

$$\chi^2 = \frac{|\underline{\phi}^*\underline{z}|^2}{(\underline{\phi}^*\underline{\phi})\sigma^2} = \frac{|\overline{\mu}|^2}{\sigma^2}\underline{\phi}^*\underline{\phi} = \overline{\text{SNR}}, \tag{14}$$

where $|\overline{\mu}|^2/\sigma^2$ is a sample estimated of the input SNR and $\underline{\phi}^*\underline{\phi}$ is the matched filter gain. Note that the estimate of input SNR may be written as

$$\frac{|\overline{\mu}|^2}{\sigma^2} = \frac{|\underline{\phi}^*\underline{z}|^2}{(\underline{\phi}^*\underline{\phi})^2\sigma^2} = \frac{\underline{z}^*\mathbf{P}_{\phi}\underline{z}}{\underline{\phi}^*\underline{\phi}\sigma^2}, \tag{15}$$

which shows it to be the signal energy per sample, or signal power $\underline{z}^*\mathbf{P}_{\phi}\underline{z}/\underline{\phi}^*\underline{\phi}$, divided by the per sample noise power σ^2.

3.2. CFAR matched susbpace detectors

The CFAR matched filter of Equation 6 may be written as

$$\cos = \frac{\underline{\phi}^* \underline{z}}{\underline{\phi}^* \underline{\phi} \sqrt{\underline{z}^* \underline{z}}} \sqrt{\underline{\phi}^* \underline{\phi}} = \frac{\overline{\mu}}{\sqrt{\underline{z}^* \underline{z}}} \sqrt{\underline{\phi}^* \underline{\phi}}, \tag{16}$$

Now the basis $\langle \mathbf{U} \rangle$ of Equation 8 may be used to resolve the energy term $\underline{z}^* \underline{z}$ as

$$\begin{aligned} \underline{z}^* \underline{z} &= \frac{|\underline{\phi}^* \underline{z}|^2}{(\underline{\phi}^* \underline{\phi})^2} \underline{\phi}^* \underline{\phi} + \underline{z}^* (\mathbf{I} - \mathbf{P}_\phi) \underline{z} \\ &= |\overline{\mu}|^2 \underline{\phi}^* \underline{\phi} + \frac{\underline{z}^* (\mathbf{I} - \mathbf{P}_\phi) \underline{z}}{N-1} (N-1). \end{aligned} \tag{17}$$

We go one step further in our decomposition of $\underline{z}^* \underline{z}$ by calling the second term the sample estimate of the noise variance, $\overline{\sigma}^2$:

$$\underline{z}^* \underline{z} = |\overline{\mu}|^2 \underline{\phi}^* \underline{\phi} + \underline{\sigma}^2 (N-1) \tag{18}$$

In fact, this sample estimate $\overline{\sigma}^2 \underline{z}^* (\mathbf{I} - \mathbf{P}_\phi) \underline{z} / _{N-1}$ is χ^2- distributed with mean value σ^2. With this decomposition of $\underline{z}^* \underline{z}$ we may write the CFAR matched subspace detector as

$$\cos = \frac{\overline{\overline{\mathrm{VSNR}}}}{\sqrt{|\overline{\overline{\mathrm{VSNR}}}|^2 + N - 1}} \tag{19}$$

where the sample estimate of output SNR is

$$\overline{\overline{VSNR}} = \frac{\underline{\phi}^* \underline{z}}{\sqrt{\underline{\phi}^* \underline{\phi}}} \sqrt{\frac{1}{\underline{z}^* (\mathbf{I} - \mathbf{P}_\phi) \underline{z} / _{N-1}}} = \frac{\overline{\mu}}{\overline{\sigma}} \sqrt{\underline{\phi}^* \underline{\phi}} \tag{20}$$

The double over-bar indicates that we are estimating output SNR by two sample estimates : one for $\overline{\mu}$ and one for $\overline{\sigma}$.

To complete the interpretation of coherence, we write it as

$$\cos = \frac{t}{|t|^2 + 1}, \tag{21}$$

where t, within a scale constant, is the classical t-statistic

$$t = \frac{\cos}{1 - |\cos|^2} = \frac{\overline{\overline{\mathrm{VSNR}}}}{\sqrt{N-1}} \tag{22}$$

So, the CFAR matched filter is a monotonic function of the statistician's classical t test; it is simply the estimated output SNR, divided by $\sqrt{N-1}$.

Finally, the CFAR matched subspace detector of Equation 7 takes the canonical form

$$\beta = \frac{|\overline{\mu}|^2 \underline{\phi}^* \underline{\phi}}{|\mu|^2 \underline{\phi}^* \underline{\phi} + \overline{\sigma}^2 (N-1)}, \tag{23}$$

where $\overline{\sigma}^2$ is the *sample estimate* of the noise variance scaling, σ^2 :

$$\overline{\sigma}^2 = \frac{\underline{z}^* \mathbf{I} - (\mathbf{P}_\phi) \underline{z}}{N-1}. \tag{24}$$

This formula for squared coherence may also be written in terms of the *sample estimate* of the output SNR :

$$\beta = \frac{\overline{\overline{\mathrm{SNR}}}}{\overline{\overline{\mathrm{SNR}}} + N - 1};$$

$$\overline{\overline{\mathrm{SNR}}} = (N-1) \frac{\underline{z}^* \mathbf{P}_\phi \underline{z}}{\underline{z}^* (\mathbf{I} - \mathbf{P}_\phi) \underline{z}} = \frac{|\overline{\mu}|^2}{\overline{\sigma}^2} \underline{\phi}^* \underline{\phi} \tag{25}$$

To complete the interpretation of squared coherence, we write it as

$$\beta = \frac{F}{F+1}, \tag{26}$$

where F, within a scale constant, is the classical F-statistic

$$F = \frac{\beta}{1-\beta} = \frac{\overline{\overline{\mathrm{SNR}}}}{N-1}. \tag{27}$$

So, the coherence detector is a beta version of the statistician's classical F test; in its F form it is simply the estimated output SNR, divided by $N-1$.

Let' summarize : The four matched subspace detectors described here are the uniformly-most-powerful (UMP) invariant detectors for a subspace signal in Gaussian noise. They have the appropriate invariances and evocative geometries. They all have simple interpretations in terms of estimated output SNR. These geometries and SNR interpretations are different than those published by Picinbono [10], but there is an idea in common, namely a decomposition, of the measurement space into the direct sum of a signal subspace and an orthogonal subspace.

4. Adaptive subspace detectors: geometries and invariances

The matched subspace detectors require prior knowledge of the noise covariance matrix $\mathbf{R}$. This information is usually not known, meaning that it must be estimated and used correctly in an adaptive detector. A seemingly *ad hoc* approach to this problem is to simply replace the noise covariance by a *sample* covariance matrix, constructed from a sequence of M i.i.d. $CN[\underline{0}, \mathbf{R}]$ training vectors $\underline{y}_1, \underline{y}_2, \ldots, \underline{y}_M$:

$$\mathbf{S} = \frac{1}{M} \sum_{i=1}^{M} \underline{y}_i \underline{y}_i^*. \tag{28}$$

These training vectors are independent of the test vector $\underline{y}$, the measurement to be tested for H_0 versus H_1. They share the same noise structure $\mathbf{R}$ as $\underline{y}$, but not necessarily the same noise scaling (in the adaptive context, σ is a relative scaling of the test vector relative to the training vectors) [11].

4.1. Coherence adaptive matched filter

Using the Sample covariance, an adaptative matched filter (AMF) is given by :

$$\widehat{n} = \frac{\underline{\psi}^* \mathbf{S}^{-1} \underline{y}}{\sqrt{\underline{\psi}^* \mathbf{S}^{-1} \underline{\psi} \sigma}} \gtrless \eta. \tag{29}$$

This statistic measures the resolution of the measurement $\underline{y}$ onto the signal $\underline{\psi}$ in an *adaptively* witened coordinate system. For purposes of interpretation, it is useful to write the coherent AMF as

$$\widehat{n} = \frac{\underline{\phi}^* \widehat{\mathbf{I}}^{-1} \underline{z}}{\sqrt{\underline{\phi}^* \widehat{\mathbf{I}}^{-1} \underline{\phi} \sigma}} = \frac{\underline{\widehat{\phi}}^* \underline{\widehat{z}}}{\sqrt{\underline{\widehat{\phi}}^* \underline{\widehat{\phi}} \sigma}}, \tag{30}$$

where the matrix $\widehat{\mathbf{I}}$ is the *sample estimate* of identity and where $\underline{\widehat{\phi}}$ and $\underline{\widehat{z}}$ are resolutions of $\underline{\phi}$ and $\underline{z}$ in the coordinate system $\widehat{\mathbf{I}}^{-1/2}$:

$$\widehat{\mathbf{I}} = \mathbf{R}^{-1/2} \mathbf{S} \mathbf{R}^{-*/2}; \quad \underline{\widehat{\phi}} = \widehat{\mathbf{I}}^{-1/2} \underline{\phi}; \quad \underline{\widehat{z}} = \widehat{\mathbf{I}}^{-1/2} \underline{z}. \tag{31}$$

The AMF is invariant to translations of the measurement $\underline{\widehat{z}}$ in the orthogonal subspace $\langle \underline{\widehat{\phi}} \rangle^{\perp}$. Thus, the invariance set is the surface of a plane, where the projection of the measurement $\underline{\widehat{z}}$ onto the signal $\langle \underline{\widehat{\phi}} \rangle$ is constant. So Figure 1 still applies.

4.2. Adaptive subspace detector

The adaptive subspace detector (ASD) is given by the magnitude-squared of the coherence AMF :

$$\widehat{\chi^2} = \frac{|\underline{\psi}^* \mathbf{S}^{-1} \underline{y}|^2}{(\underline{\psi}^* \mathbf{S}^{-1} \psi y) \sigma^2} \gtrless \eta. \tag{32}$$

This detector was first proposed by Robey, et al. [12] and by Chen and Reed [13]. It is a simplification of the Kelly detector [14], wich is the actual GLRT corresponding to χ^2 for the adaptive case (for more explanation, see [7]. The ASD measures the energy of the measurement $\underline{y}$ contained in the subspace $\langle \underline{\psi} \rangle$, in the adaptively whitened coordinate system. the ASD may be rewritten as

$$\widehat{\chi^2} = \frac{|\underline{\phi}^* \widehat{\mathbf{I}}^{-1} \underline{z}|^2}{\underline{\phi}^* \widehat{\mathbf{I}}^{-1} \underline{\phi} \sigma^2} = \frac{|\underline{\widehat{\phi}}^* \underline{\widehat{z}}|^2}{\underline{\widehat{\phi}}^* \underline{\widehat{\phi}} \sigma^2} \tag{33}$$

where $\underline{\widehat{\phi}}$ and $\underline{\widehat{z}}$ are adaptively whitened versions of ψ and y. this statistic is invariant to rotations of the measurement $\underline{\widehat{z}}$ in the subspace $\langle \underline{\widehat{\phi}} \rangle$, and to translations in the perpendicular subspace $\langle \underline{\widehat{\phi}} \rangle^{\perp}$. Thus, the invariance set is the surface of two planes, where the energy of the measurement $\underline{\widehat{z}}$ in the subspace $\langle \underline{\widehat{\phi}} \rangle$ is constant, consistent with F igure 2.

4.3. CFAR adaptive matched filter

The CFAR adaptive matched filter is

$$\widehat{\cos} = \frac{\underline{\psi}^* \mathbf{S}^{-1} \underline{y}}{\sqrt{\underline{\psi}^* \mathbf{S}^{-1} \underline{\psi}} \sqrt{\underline{y}^* \mathbf{S}^{-1} \underline{y}}} \tag{34}$$

It measures the coherence, or direction cosine, of the measurement $\underline{y}$ with the signal $\underline{\psi}$ in the adaptive whitened coordinate system. This detector may be rewritten in adaptively whitened coordinates as

$$\widehat{\cos} = \frac{\underline{\phi}^* \widehat{\mathbf{I}}^{-1} \underline{z}}{\sqrt{\underline{\phi}^* \widehat{\mathbf{I}}^{-1} \underline{\phi}} \sqrt{\underline{y}^* \widehat{\mathbf{I}}^{-1} \underline{y}}} = \frac{\underline{\widehat{\phi}}^* \underline{\widehat{z}}}{\sqrt{\underline{\widehat{\phi}}^* \underline{\widehat{\phi}}} \sqrt{\underline{\widehat{z}}^* \underline{\widehat{z}}}} \tag{35}$$

It is invariant to rotations of the measurement $\underline{\widehat{z}}$ in the perpendicular subspace $\langle \underline{\widehat{\phi}}^{\perp} \rangle$, to scaling of the sample covariance $\mathbf{S}$, and to a different scaling of the measurement [15]. That is,

$$\widehat{\cos}(c_2 \underline{y}, c_1^2 \mathbf{S}) = \widehat{\cos}(\underline{y}, \mathbf{S}). \tag{36}$$

Thus, the invariance set is the surface of a cone, where the angle that the measurement $\underline{\widehat{z}}$ makes with the signal $\underline{\widehat{\phi}}$ is constant, consistent with figure 3.

4.4. CFAR adaptive subspace detector

The CFAR ASD is given by

$$\widehat{\beta} = \frac{|\underline{\psi}^* \mathbf{S}^{-1} \underline{y}|^2}{(\underline{\psi}^* \mathbf{S}^{-1} \underline{\psi})(\underline{y}^* \mathbf{S}^{-1} \underline{y})} \gtrless \widehat{\eta}. \tag{37}$$

This statistic measures the squared coherence, or cosine-squared, between the measurement $\underline{y}$ and the subspace $\langle \underline{\psi} \rangle$ in an adaptively whitened coordinate system. In the adaptively whitened coordinates, the statistic may be rewritten as

$$\widehat{\beta} = \frac{|\underline{\phi}^* \widehat{\mathbf{I}}^{-1} \underline{z}|^2}{(\underline{\phi}^* \mathbf{I}^{-1} \underline{\phi})(\underline{z}^* \widehat{\mathbf{I}}^{-1} \underline{z})} = \frac{|\underline{\widehat{\phi}}^* \underline{\widehat{z}}|^2}{(\underline{\widehat{\phi}}^* \underline{\widehat{\phi}})(\underline{\widehat{z}}^*) \underline{\widehat{z}}} \tag{38}$$

It is invariant to rotations of the measurement $\underline{\widehat{z}}$ in the subspace $\langle \underline{\widehat{\phi}} \rangle$, to rotations in the perpendicular subspace $\langle \underline{\widehat{\phi}} \rangle^{\perp}$, to scaling of the sample covariance $\mathbf{S}$, and to a different scaling of the

measurement $\underline{y}$. Thus, the invariance set is the surface of a double cone, where the fractional energy of the measurement $\widehat{\underline{z}}$ in the subspace $\langle\widehat{\underline{\phi}}\rangle$ is constant, consistent with Figure 4. These are fundamental invariances for incoherent signal detection when signal phase and channel attenuation are unknown.

5. Adaptive subspace detectors: SNR interpretations

As with the matched subspace detectors, the adaptive subspace detectors can be rewritten in terms of maximum likehood estimates of the signal gain μ and noise scaling σ, yielding interpretations of the statistics in terms of estimated output SNR. These maximum likelihood estimates are obtained in the adaptative scenario by considering the *joint* density function of both the test and training data vectors:

$$f_1(\underline{y},\underline{y}_1,\ldots,\underline{y}_M) = \frac{1}{\pi^N \det(\sigma^2\mathbf{R})}\exp\{-\frac{1}{\sigma^2}(\underline{y}-\mu\underline{\psi})^\dagger \mathbf{R}^{-1}(\underline{y}-\mu\underline{\psi})\}\prod_{m=1}^{M}\frac{1}{\pi^N\det(\mathbf{R})}\exp\{-\underline{y}_m^\dagger\mathbf{R}^{-1}\underline{y}_m\} \quad (39)$$

Maximizing this likelihood function over μ and σ yields the following maximum likelihood estimates for the signal gain and noise scaling [11] :

$$\widehat{\mu} = \frac{\widehat{\underline{\phi}}^*\widehat{\underline{z}}}{\widehat{\underline{\phi}}^*\widehat{\underline{\phi}}}; \qquad |\widehat{\mu}|^2 = \frac{\widehat{\underline{z}}^*\mathbf{P}_{\widehat{\phi}}\widehat{\underline{z}}}{\widehat{\underline{\phi}}^*\widehat{\underline{\phi}}} \quad (40)$$

$$\widehat{\sigma} = \sqrt{\widehat{\underline{z}}^*(\mathbf{I}-\mathbf{P}_{\widehat{\phi}})\widehat{\underline{z}}\cdot\frac{L}{MN}} \quad (41)$$

Here L is an over-training parameter, $L = M - N + 1$.

These estimates only depend on two vectors, $\widehat{\underline{z}}$ and $\widehat{\underline{\phi}}$. Moreover, the angle between $\widehat{\underline{z}}$ and $\widehat{\underline{\phi}}$ is invariant to rotation of the coordinate system in which it is measured. Therefore, we may construct an orthogonal span for the two-dimensional subspace defined by the signal $\widehat{\underline{\phi}}$ and the measurement $\widehat{\underline{z}}$, and complete N-dimensional space by constructing the perpendicular subspace. To this end, we follow the procedure surrounding Equation 8 to construct $\widehat{\mathbf{U}}$ and apply it to $\widehat{\underline{z}}$ and $\widehat{\underline{\phi}}$. This allows us to produce the following SNR interpretations.

5.1. Adaptive subspace detectors

The adaptive matched filter may be written as

$$\widehat{n} = \frac{\widehat{\mu}}{\sigma}\sqrt{\widehat{\underline{\phi}}^*\widehat{\underline{\phi}}} = \widehat{\mathrm{VSNR}}, \quad (42)$$

which is an adaptive sample estimate of the output voltage signal-to-noise. Similarly, the noncoherent adaptive subspace detector may be written as

$$\widehat{\chi^2} = \frac{|\widehat{\mu}|^2}{\sigma^2}\widehat{\underline{\phi}}^*\widehat{\underline{\phi}} = \widehat{\mathrm{SNR}}, \quad (43)$$

which is an adaptive sample estimate of the output signal-to-noise ratio.

5.2. CFAR adaptive subspace detectors

The CFAR adaptive matched filter may be written as

$$\widehat{\cos} = \frac{\widehat{\mu}\sqrt{\widehat{\underline{\phi}}^*\widehat{\underline{\phi}}}}{\sqrt{|\widehat{\mu}|^2\widehat{\underline{\phi}}^*\widehat{\underline{\phi}}+\widehat{\sigma}^2\frac{MN}{L}}}. \quad (44)$$

As in Section 3.2, $\widehat{\cos}$ may be rewritten in terms of signal gain noise scaling estimates as

$$\widehat{\cos} = \frac{\widehat{t}}{|\widehat{t}|^2+1} = \frac{\widehat{\mathrm{VSNR}}}{\sqrt{|\widehat{\mathrm{VSNR}}|^2+\frac{MN}{L}}};$$

$$\widehat{\mathrm{VSNR}} = \frac{\widehat{\underline{\phi}}^*\widehat{\underline{z}}}{\sqrt{\widehat{\underline{\phi}}^*\widehat{\underline{\phi}}}}\sqrt{\frac{1}{\widehat{\underline{z}}^*(\mathbf{I}-\mathbf{P}_{\widehat{\phi}})\widehat{\underline{z}}\frac{MN}{L}}} = \frac{\widehat{\mu}}{\widehat{\sigma}}\sqrt{\widehat{\underline{\phi}}^*\widehat{\underline{\phi}}}, \quad (45)$$

where $\widehat{t}$ is the *adaptive* t-statistic

$$\widehat{t} = \frac{\widehat{\cos}}{1-|\widehat{\cos}|^2} = \sqrt{\frac{L}{MN}}\widehat{\mathrm{VSNR}} \quad (46)$$

So the adaptive $\widehat{t}$ is just a scaled version of the adaptively estimated output voltage SNR, and vice-versa. The double hat indicates that the estimate of VSNR uses estimates of both μ and σ.

Finally, the CFAR ASD may be written as

$$\widehat{\beta} = \frac{|\widehat{\mu}|^2\widehat{\underline{\phi}}^*\widehat{\underline{\phi}}}{|\widehat{\mu}|^2\widehat{\underline{\phi}}^*\widehat{\underline{\phi}}+\widehat{\sigma^2}\frac{MN}{L}} \quad (47)$$

Again as in Section 3.2, $\widehat{\beta}$ may be written as

$$\widehat{\beta} = \frac{\widehat{F}}{\widehat{F}+1} = \frac{\widehat{\mathrm{SNR}}}{\widehat{\mathrm{SNR}}+\frac{MN}{L}};$$

$$\widehat{\mathrm{SNR}} = \frac{MN}{L}\frac{\widehat{\underline{z}}^*\mathbf{P}_{\widehat{\phi}}\widehat{\underline{z}}}{\widehat{\underline{z}}^*(\mathbf{I}-\mathbf{P}_{\widehat{\phi}})\widehat{\underline{z}}} = \frac{|\widehat{\mu}|^2}{\widehat{\sigma}^2}\widehat{\underline{\phi}}^*\widehat{\underline{\phi}}, \quad (48)$$

where $\widehat{F}$ is the *adaptive* F-statistic

$$\widehat{F} = \frac{\widehat{\beta}}{1-\widehat{\beta}} = \frac{L}{MN}\widehat{\widehat{\mathrm{SNR}}}. \qquad (49)$$

So, the CFAR ASD is a beta version of an *adaptive* F test, which is simply the adaptively estimated output SNR, divided by MN/L. The approach of substituting a sample covariance for the known covariance was independently suggested in [8, 9, 15]. The adaptive CFAR statistic of Equation 48 has since been shown to be the Generalized Likelihood Ratio Test over the joint likelihood of the test and training data [11].

6. conclusions

In this paper we have reviewed the theory of matched and adaptive subspace detectors [7] for the case where the signal to be detected lies in the one-dimensional complex subspace $\langle\underline{\psi}\rangle$. The story generalizes completely to the case where the subspace $\langle\Psi\rangle$ is multidimensional, thus making it applicable to matched field processing, detection in multi-path, etc.

The most succinct summary statement for the matched subspace detectors is that they are uniformly-most-powerful-invariant and generalized likelihood ratio tests. Beyond this, their geometries and invariances are evocative, and they may all be interpreted in terms of estimated output SNRs. For a quite different perspective, see [10], which is one of the first detection papers to develop insights from signal and orthogonal subspace decompositions.

Of the adaptive subspace detectors discussed in this paper, only the two CFAR tests are truly generalized likelihood ratio tests. But all four have evocative geometries and invariances, and all four may be interpreted in terms of estimated output SNRs.

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