Advances in Modelling and Analysis A
Vol. 55, No. 1, March, 2018, pp. 20-34
Journal homepage: http://iieta.org/Journals/AMA/AMA_A

Analytical solution to the mathematical models of HIV/AIDS with control in a heterogeneous population using Homotopy Perturbation Method (HPM)

David Omale[1*], Saikat Gochhait[2]

[1] Mathematical Sciences Department, Kogi State University, Anyigba, India
[2] Symbiosis Institute of Telecom Management, Symbiosis International: Deemed University, Pune, India.

Corresponding Author Email: davidubahi@gmail.com

https://doi.org/10.18280/ama_a.550103

Received: 20 February 2018
Accepted: 26 March 2018

Keywords:
homotopy, perturbation, HIV/AIDS control, condom, therapeutic dose model

ABSTRACT

A mathematical model for the control for HIV/AIDS was formulated, using vaccine, condom, therapeutic dose and public health campaign'; The models are compartmental in nature and non-linear. We use Homotopy perturbation method to solve the model. It was discovered that, any non-linear equation to be easily solved using this method, it is explicit and converge easily.

1. INTRODUCTION

Non linear phenomena are fundamentally importance in different fields of science and engineering. Most nonlinear models related to real life problems are too difficult to solve either analytically or numerically, therefore the present study is an attempt in developing methods to obtain analytical solution for the set of problems. The Homotopy perturbation method is one of such method, it is straight forward and it is applicable to non-linear and linear problems. It is also applicable to both partial differential equations and ordinary differential equations. The Homotopy perturbation method (HPM) was proposed by He in 1998 and was developed and improved upon by him, Homotopy perturbation method is the combination of the traditional perturbation method and homotopy method. The perturbation method is based on the existence of small parameter. Human immunodeficiency virus infection/acquired immunodeficiency syndrome (HIV/AIDS) is a disease of the human immune system caused by infection with human immunodeficiency virus (HIV) Sepkowitz [4].

1.1 Signs and symptoms of HIV/AIDS

There are three main stages of HIV infection: acute infection, clinical latency and AIDS. H.H.S.(2010).
The initial period following the contraction of HIV is called acute HIV, primary HIV or acute retroviral syndrome. Many individuals develop an influenza-like illness or a mono nucleosis like illness 2-4 week post exposure while others have no significant symptoms, WHO [5]. Symptoms occurs in 40-90% of cases and the sign most commonly include at this stage fever, large tender lymph nodes, throat inflammation, a rash, headache and/or sores of the mouth and genitals, some people also may develop opportunistic infection at this stage. WHO [5]. Gastrointestinal symptoms such as nausea vomiting or diarrhea may occur as a neurological symptoms of peripheral neuropathy Guillain Barre Syndrome. The duration of the symptoms varies, but is usually one to two weeks. Vogel .et al [6].

The initial symptoms are followed by a stage called clinical latency, a symptomatic HIV or chronic HIV without treatment. This second stage of the natural history of HIV infection can last from about three to over 20 years, Evian [7] and on the average about 18years [8]. While typically there are few or no symptoms at first, near the end of this stage many people experience: fever, weight loss, gastro intestinal problems and muscle pains. Between 50-70% of the people also develop persistent generalized lymphadenopathy; characterized by unexplained non-painful enlargement of more than one group of lymph nodes for over three or six months.

Although most HIV -1 infected individuals have a detectable viral load and in the absence of treatment will eventually progress to AIDS, a small proportion about 5% retain high level of CD4-T cell (T-helper cells) without antiretroviral therapy for more than 5 years [9]. These individuals are classified as HIV controller or long-term non progressors (LTNP). [9]. Another group is those who also maintain a low or undetectable viral load without anti-retroviral treatment. These group are known as elite controllers or elite suppressors. They represent approximately 1 in 300 infected persons [10].

The last stage of HIV is the Acquired Immunodeficiency Syndrome (AIDS). It is defined in terms of either a $CD4^+$ T cell count below 200 cells per ml or the occurrence of specific disease in association with an HIV infection (Holmes et al 2003). Without treatment about half of people infected with HIV develop AIDS within ten years [11]. The most common initial conditions that alert one of the presence of AIDS are pneumocystis pneumonia (40%), cachexia in form of HIV wasting syndrome (20%) and esophageal candidacies [11]. Other common signs include recurring respiratory tract infection.

People with AIDS have an increased risk of developing various viral induced cancers including Kaposi Sarcoma, Burkett's lymphoma, primary central nervous system lymphoma and cervical cancer [6]. These set of people often have systematic symptoms such as prolonged fevers, sweats (particularly at night) swollen lymph nodes, chills weakness and weight loss. Diarrhoea is another common symptom present in about 90% of people with AIDS [12]. They can also be affected by diverse psychiatric and neurological symptoms independent of opportunistic infection, cancer and finally death [13].

1.2 Mathematical model formulation

The development of our model is based on the following assumptions that

1. The diseases HIV/AIDS is killing continuously
2. Individual who contact this disease will definitely die of the disease if untreated or on control drug.
3. There is no medicine right now for total cure of this particular disease, therefore infected individual will live with the disease in his/her life time. Individual on HIV drug will remain on the drug forever.
4. Individual who is faithful to the drug will not die of HIV/AIDS
5. There is no vaccine with 100% efficacy to prevent HIV/AIDS
6. The available vaccines are imperfect; and so the vaccine will wane with time.
7. That not all the people within the sexually active population are willing to use condom whenever they have sex.
8. There are no vertical transmissions of the diseases.
9. That campaign reduces the rate of transmission; because those who are properly informed will reduce their exposure to infection whenever they meet any infectious opportunity.

We develop and analyze a mathematical model for HIV/AIDS transmission dynamics and control improving on the existing models as discoursed in our literature review. This is done by incorporating vaccination coverage, condom usage, campaign and therapeutic doses. The model is defined as a set of ordinary differential equations based on our assumptions about the dynamics of HIV/AIDS, and some biological interventions.

The interaction between the classes is describe as follows: The susceptible is divided into three groups: (S) represent the number of individuals not yet infected with the virus (HIV/AIDS) virus but are susceptible to the disease and its recruitment is not vaccinated, denoted by π, the other susceptible group is the vaccinated susceptible population denoted by (V), when the susceptible population, as a result of public enlightment campaign get vaccinated at the rate δ_1 and its recruitment is vaccinated at a proportion P, the vaccine has the ability to reduce the infection rate by a factor $(1-\theta_1)k$ where θ_1 is the vaccine efficacy. When the efficacy is low, the infection may occur at the rate $(1-\theta_1)k$, θ_1, measure the efficacy of the vaccine such that $0 \le \theta_1 \le 1$, If $\theta_1 = 1$,vaccine is completely effective in preventing the population from infections, if the θ_1 is equal to 0 the vaccine is useless, as the whole population will be infected if they interact with infected population. The third susceptible class are those who use condom at the δ_2 and its recruitment is denoted by ω, the failure rate in protecting an individual is denoted by ε, in that case the condom users will be susceptible again. The effectiveness of the condom is denoted by φ, such that $0 \le \varphi \le 1$, If $\varphi = 1$, the condom is very effective and it can prevent the population from the infections, but if the condom efficacy is equal to zero (0), the condom is useless. The waning rate of the vaccine is denoted by θ and the individual become susceptible again. Exposed class (E) is made of individuals who have contracted the infection at the early stage, but are not capable of infecting others in the population yet, the exposed individual will become infectious at the rate ϕ the public health campaign is denoted by C, the rate at which the infectious individual through effective public health campaign go for treatment is τ, the non effectiveness of therapy is denoted by σ_1, σ_2 such that $1 \le \sigma_1, \sigma_2 \le 1$, the infectious individual progress to full blown AIDS at the rate η, the delay rate in developing symptom is e, (A) is the population of individual with clinical AIDs, it is a function of (I), (I_2) and (A_2) developing disease symptoms. The susceptible may become infectious at the rate of infection k, the force of infection is given by

$$k = \frac{n_1\beta_1 I_2 + n_2\beta_2 A_2 + n_3\beta_3 A_T}{N}$$

where
n =number of sex partners
β_1 =transmission rate from infectious individual not receiving treatment
β_2 =transmission rate from infectious individual receiving treatment
β_3 transmission rate of AIDS individual who is undergoing therapy, (HAARTS)

Table 1. State variable of the HIV/AIDS with control strategies

***S*(*t*)**	Number of susceptible at time *t*
***V*(*t*)**	Number of preventive vaccinated individual at time *t*
***H*(*t*)**	Number of susceptible that are condom users at time *t*.
***E*(*t*)**	Latent/exposed individuals at time *t*
***I*(*t*)**	Infectious individuals at time *t* not receiving any treatment
***I*2(*t*)**	Number of infectious individuals who are undergoing treatment
***A*(*t*)**	Number of individuals with full blown AIDS.
***AT*(*t*)**	Number or proportion of full blown AIDS who are undergoing therapy.
***A*2(*t*)**	Proportion of full blown AIDS who are not receiving the therapy.

In the force of infection $\beta_1 > \beta_2 > \beta_3$. This show that β_1 contribute much on the transmission of the infection due to the fact that they are not receiving treatment, so they are not protected, β_2 contribute much less on the transmission of the infection due to their HIV status, they have acquired HIV/AIDS but receiving treatment so their viral load will be significantly reduced, unless if they desist from taking their daily pills. β_3 Is expected to contribute least to the infection, since they just acquired the full virus and are aware of the AIDS status and they are receiving the daily therapy. There is natural death rate (μ) in the whole compartments, but there is an HIV/AIDS induced death rate in the (A) and (A_2) classes. (A) And A_2 are the same if proportion of (A) class stop receiving treatment.

The total population at any time t is given by

$$N(t)=S(t)+V(t)+H(t)+E(t)+I(t)+I_2(t)+A_2(t)+A(t)+A_T(t)$$

The population are homogeneously mixed and each susceptible individual has equal chances to acquire HIV infection when individual come in contact with an infectious individuals.

The full description of the variables and parameters to be used in the model are as follows in table 1 and table 2.

Table 2. Parameter descriptions

π	Population recruited into the susceptible class.
P	Proportion of susceptible recruited individual with lost preventive vaccination
ω	Proportion of susceptible recruited individual that uses condom
μ	Per capita death rate (Nature death)
α_i	Disease induced death rate
δ_1	Preventive Vaccination rate in the population
δ_2	Rate of condom usage in the population
θ	Waning rate of the vaccine
ε	Improper condom usage
φ	Condom efficacy or effectiveness
θ_1	Vaccination efficacy rate
ϕ	Progression rate of latent individual to infectious class.
c	Public health campaign rate
σ_1, τ_c	Rate of non effectiveness of the drug.
τ	Treatment rate of infectious individual
η	Rate of progression to full blown AIDS
e	Reduction in developing symptom.
r_1	Rate at which those in the AID class receive treatment due to effectiveness of public health campaign
k	Effective contact rate of the susceptible with the infectious classes and called force of infection.
δ_2	The rate at which the susceptible individual uses condom effectively
r	Rate at which unvaccinated and those who voluntarily refused to use condom become exposed to the infections.
r_2	Rate at which proportion of those in *A* class refused to receive the therapy and remain with AIDS.
α_2	Disease induced death rate of those who refused therapy as AIDS individuals.

1.3 Flow diagram illustrating the interactions of the different compartments

$$\frac{dS}{dt}=\pi-\delta_1 cS-rkS-\delta_2 cS+\varepsilon H-\mu S+pV$$

$$\frac{dV}{dt}=\delta_1 cS-(1-\theta_1)kV-pV-\mu V$$

$$\frac{dH}{dt}=-(1-\varphi)kH-\varepsilon H+\delta_2 cS-\mu H$$

$$\frac{dE}{dt}=(1-\varphi)kH+rkS+(1-\theta_1)kV-\phi E-\mu E$$

$$\frac{dI}{dt}=\phi E+(1-\sigma_1)I_2-\tau cI-\eta cI-\mu I$$

$$\frac{dI_2}{dt}=\tau cI-(1-\sigma_1)I_2-\mu I_2$$

$$\frac{dA_T}{dt}=r_1 cA-(1-\sigma_2)A_T-\mu A_T$$

$$\frac{dA}{dt}=\eta cI+(1-\sigma_2)A_T-r_1 cA-\alpha A-\mu A-r_2 A$$

$$\frac{dA_2}{dt}=r_2 A-(\alpha_2+\mu)A_2$$

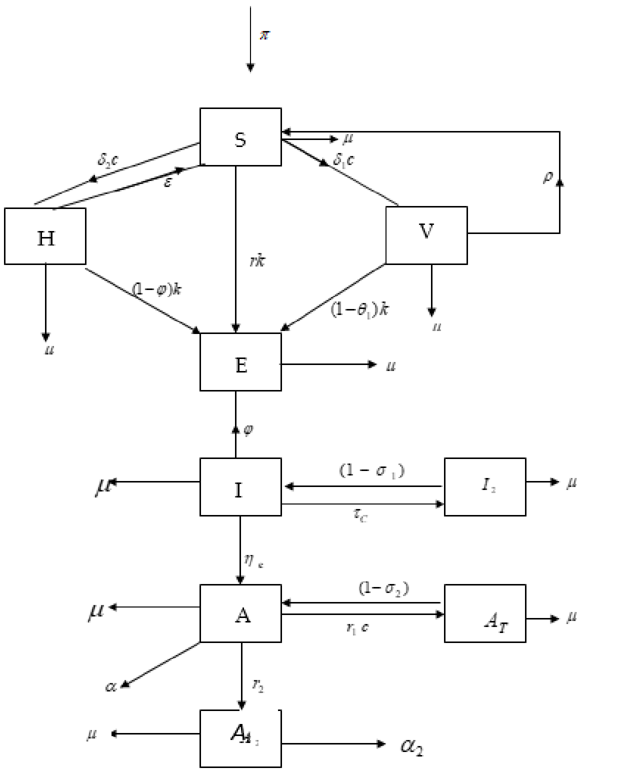

Figure 1. Model Flow Diagram for the main model
From our assumptions and the flow chart we obtain the system of ordinary differential equations

where k is the effective contact rate given as

$$k = \frac{n_1\beta_1 I_2 + n_2\beta_2 A_2 + n_3\beta_3 A_T}{N}$$

With the following initial conditions

$$S(0)>0, V(0)>0, H(0)>0, E(0)>0, I(0)>0, I_2(0)>0, A_2(0)>0, A_T(0)>0, A(0)>0$$

With the effective contact rate

$$\beta_1 > \beta_2 > \beta_3$$

But $N = S + V + H + E + I + I_2 + A + A_T + A_2$

2. BASIC IDEA OF HE'S HOMOTOPY PERTURBATION METHOD

To demonstrate the basic ideas of He's homotopy perturbation method we consider the non linear differential equation.

$$A(u) - f(r) = 0 \qquad r \in \Omega \tag{1}$$

With the boundary condition

$$B\left(u, \frac{\partial u}{\partial n}\right) = 0 \qquad r \in \Gamma \tag{2}$$

where
A is a general differential operator,
B a boundary operator
$f(r)$ a known analytical function and
Γ is the boundary of the domain
Ω respectively, Taghipour (2011)
The general operator A can be divided into two parts L and N where L is the linear part and N is the non linear part respectively. Equation (1) can therefore be written as;

$$L(u) + N(u) - f(r) = 0 \tag{3}$$

We now construct a homotopy $V(r,p)$ such that
$V(r,p): \Omega X[0,1] \to R$ which satisfies

$$H(r,p) = (1-p)[L(v) - L(u_0)] + p[A(v) - f(r) = 0]$$
$$P \in [0,1], \qquad r \in \Omega \tag{4}$$

OR

$$H(r,p) = L(v) - L(u_0) + pL(u_0) + [N(v) - f(r)] = 0 \tag{5}$$

where $P \in [0,1]$ is an embedding parameter, while u_0 is an initial approximation of equation (1) which satisfies the boundary conditions.

Obviously from equation (4) and (5) we have

$$H(u,0) = L(v) - L(u_0) = 0 \tag{6}$$

$$H(u,1) = A(v) - f(r) = 0 \tag{7}$$

The changing process of p from zero to unity is just that of $V(r,p)$ from $u_0(r)$ to $u(r)$.

In topology this is called deformation while $L(v) - L(u_0)$, $A(v) - f(r)$ are called Homotopy.

According to Homotopy perturbation method (HPM) we can first use the embedding parameter (p) as a small parameter and assume solution for equation (4) and (5) which can be expressed as;

$$V = v_0 + pv_1 + p^2 v_2 + \ldots \tag{8}$$

Setting $p = 1$ we will obtain an approximate solution of equation (8) as

$$U = \lim_{p \to 1} v = v_0 + v_1 + v_2 + \ldots \tag{9}$$

Equation (9) is the analytical solution of (1) by Homotopy perturbation method.

He (2003), (2006) makes the following suggestion for convergence of (9) (1). the second derivation of $N(v)$ with respect to V must be small because the parameter p must be relatively large i.e. $P \to 1$

(2). the norm of $L^{-1}\dfrac{\partial N}{\partial V}$ must be smaller than one so that the series converge.

3. SOLUTION OF THE MODELS USING HPM

Consider the systems of non linear ordinary differential equations given as;

$$\frac{dS}{dt} = \pi + \varepsilon H - \delta_1 cS - \delta_2 cS + PV - \mu S - rkS \quad (10)$$

$$\frac{dV}{dt} = \delta_1 cS - (1-\theta_1)kV - PV - \mu V \quad (11)$$

$$\frac{dH}{dt} = \delta_2 cS - \varepsilon H - (1-\varphi)kH - \mu H \quad (12)$$

$$\frac{dE}{dt} = rkH + (1-\varphi)kH + (1-\theta_1)kV - \phi \mathrm{E} - \mu \mathrm{E} \quad (13)$$

$$\frac{dI}{dt} = \phi \mathrm{E} + (1-\sigma_1)I_2 - \tau cI - \eta c_3 I - \mu I \quad (14)$$

$$\frac{dA}{dt} = \eta c_3 I + (1-\sigma_1)A_T - r_1 cA - r_2 A - \alpha A - \mu A \quad (15)$$

$$\frac{dI_2}{dt} = \tau cI - (1-\sigma_1)I_2 - \mu I_2 \quad (16)$$

$$\frac{dA_T}{dt} = r_1 cA - (1-\sigma_2)A_T - \mu A_T \quad (17)$$

$$\frac{dA_2}{dt} = r_1 A - \alpha_2 A_2 - \mu A_2 \quad (18)$$

$$S(0)=0, V(0)=0, H(0)=0, E(0)=0, I(0)=0,$$
$$A(0)=0, I_2(0)=0, A_T(0)=0, and\ A_2(0)=0$$

The parameters and the variables in the models are as defines in our discussion. We therefore simplify these equations further;

$$\frac{ds}{dt} = \pi + \varepsilon H - (\delta_1 c - \delta_2 c - \mu - rk)S + PV \quad (19)$$

$$\frac{dv}{dt} = \delta_1 cS - ((1-\theta_1)k - P - \mu)V \quad (20)$$

$$\frac{dH}{dt} = (\delta_2 c)S - (\varepsilon - (1-\varphi)k - \mu)H \quad (21)$$

$$\frac{dE}{dt} = (r + (1-\varphi))kH + (1-\theta_1)Vk - (\phi + \mu)\mathrm{E} \quad (22)$$

$$\frac{dI}{dt} = \phi \mathrm{E} + (1-\sigma_1)I_2 - (\tau c + \eta c_3 + \mu)I \quad (23)$$

$$\frac{dA}{dt} = \eta c_3 I + (1-\sigma_1)A_T - (r_1 c + r_2 + \alpha + \mu)A \quad (24)$$

$$\frac{dI_2}{dt} = \tau cI - ((1-\sigma_1) + \mu)I_2 \quad (25)$$

$$\frac{dA_T}{dt} = r_1 cA - ((1-\sigma_2) + \mu)A_T \quad (26)$$

$$\frac{dA_2}{dt} = r_1 A - (\alpha_2 + \mu)A_2 \quad (27)$$

Let

$$q_1 = \delta_1 c + \delta_2 c + rk + \mu$$
$$q_2 = (1-\theta_1)k + \rho + \mu$$
$$q_3 = \delta_2 c + \omega$$
$$q_4 = \varepsilon + (1-\varphi)k + \mu$$
$$q_5 = (r + (1-\varphi))k + (1-\varphi)V$$
$$q_6 = \phi + \mu$$
$$q_7 = \tau c + \eta c_3 + \mu$$
$$q_8 = r_1 c + r_2 + \alpha + \mu$$
$$q_9 = (1-\sigma_1) + \mu$$
$$q_{10} = (1-\sigma_2) + \mu$$
$$q_{11} = \alpha + \mu$$
$$q_{12} = \theta + \rho$$
$$q_{13} = (1-\theta_1)$$

Therefore, our models will now become

$$\frac{ds}{dt} = \pi + \varepsilon H + \rho V - q_1 S \quad (28a)$$
$$= \pi + \varepsilon H + (\rho)V - q_1 S \equiv \pi + \varepsilon H + \rho V - q_1 S$$

$$\frac{dV}{dt} = \delta_1 cS - q_2 V \quad (28b)$$

$$\frac{dH}{dt} = q_3 S - q_2 H \quad (28c)$$

$$\frac{dE}{dt} = q_5 H + q_{13} E - q_6 \quad (28d)$$

$$\frac{dI}{dt} = \phi E + (1-\sigma_1)I_2 - q_7 I \quad (28e)$$

$$\frac{dA}{dt} = \eta c_3 I + (1-\sigma_2)A_T - q_8 A \quad (28f)$$

$$\frac{dI_2}{dt} = \tau c I - q_9 I_2 \quad (28g)$$

$$\frac{dA_T}{dt} = r_1 c A + q_{10} A_T \quad (28h)$$

$$\frac{dA_2}{dt} = r_1 A - q_{11} A_2 \quad (28i)$$

We assume solutions of the models as; Let

$$S = S_0 + PS_1 + P^2 S_2 + \ldots \quad (29)$$

$$V = V_0 + PV_1 + P^2 V_2 + \ldots \quad (30)$$

$$H = h_0 + Ph_1 + P^2 h_2 + \ldots \quad (31)$$

$$E = e_0 + Pe_1 + P^2 e_2 + \ldots \quad (32)$$

$$\mathrm{I} = i_0 + pi_1 + p^2 i_2 + \ldots \quad (33)$$

$$A = a_0 + Pa_1 + P^2 a_2 + \ldots \quad (34)$$

$$\mathrm{I}_2 = n_0 + Pn_1 + P^2 n_2 + \ldots \quad (35)$$

$$A_T = m_0 + pm_1 + p^2 m_2 + \ldots \quad (36)$$

$$A_2 = y_0 + py_1 + p^2 y_2 + \ldots \quad (37)$$

We now apply HPM to the models in (28), we have from (28a) as

$\frac{dS}{dt} = \pi + \varepsilon H + \rho V - q_1 S$ Therefore

The linear part is

$$\frac{dS}{dt} = 0 \quad (38)$$

With the non linear part as

$$\pi + \varepsilon H + \rho V - q_1 S = 0 \quad (39)$$

Applying HPM, we then have

$$\begin{aligned} &(1-P)\frac{dS}{dt} + \\ &P\left[\frac{dS}{dt} - (\pi + \varepsilon H + \rho V - q_1 S)\right] = 0 \end{aligned} \quad (40)$$

$$\begin{aligned} &\Rightarrow \frac{dS}{dt} - p\frac{dS}{dt} + p\frac{dS}{dt} - p(\pi + \varepsilon H + \rho V - q_1 S) = 0 \\ &= \frac{dS}{dt} - (p\pi + p\varepsilon H + p\rho V - pq_1 S) = 0 \end{aligned} \quad (41)$$

Substituting (29), (30) and (31) into (41), we have

$$\begin{aligned} &(s_{0.}^1 + ps_1^1 + p^2 s_2^1 + \ldots +) - \\ &(p\pi + p\varepsilon(h_0 + ph_1 + p^2 h_2 + \ldots +) + \\ &p\rho(v_0 + p_1 v_1 + p^2 v_2 + \ldots) \\ &- pq_1(s_0 + ps_1 + s_2 p^2 \ldots)) = 0 \end{aligned}$$

Collecting the co-efficients of the powers of $p's$,we have

$$P^0 : S_0^1 = 0 \quad (42)$$

$$P^1 : S_1^1 - (\pi + \varepsilon h_0 + \rho v_0) + q_1 s_0 = 0 \quad (43)$$

$$P_2 : S_2^1 - \varepsilon h_1 - \rho v_1 + q_1 s_1 = 0 \quad (44)$$

Applying HPM to (28b)

$$\frac{dV}{dt} = \delta_1 c S - q_2 V = 0$$

We have

$$\begin{aligned} &(1-P)\frac{dV}{dt} + P\left[\frac{dV}{dt} - \delta_1 c S + q_2 V\right] = 0 \\ &= \frac{dV}{dt} - p\frac{dV}{dt} + p\frac{dV}{dt} - p\delta_1 c S + pq_2 V = 0 \\ &\Rightarrow \frac{dV}{dt} - p\delta_1 c S + pq_2 V = 0 \end{aligned} \quad (45)$$

Substituting (29) and (30) into (45), we have

$$\begin{aligned} &(V_0^1 + PV_1^1 + P^2 V_2^1 + \ldots) - \\ &P\delta_1 c(S_0 + S_1 P + P^2 S_2 + \ldots +) \\ &+ Pq_2(V_0 + PV_1 + P^2 V_2 + \ldots) = 0 \\ &\Rightarrow V_0^1 + PV_1^1 + P^2 V_2^1 + \ldots \\ &-(P\delta_1 c s_0 + p^2 \delta_1 c s_1 + p^3 \delta_1 c s_2) \\ &+ pq_2 v_0 + p^2 q_2 v_1 + p^3 q_2 v_2 + \ldots = 0 \end{aligned}$$

Collecting the co-efficient of p', s we have

$$P^0 : V_0^1 = 0 \tag{46}$$

$$P^1 : V_1^1 - \delta_1 c s_0 + q_2 v_0 = 0 \tag{47}$$

$$P^2 : V_2^1 - \delta_1 c s_0 + q_2 v_1 = 0 \tag{48}$$

Applying HPM to (4.28c)

$$\frac{dH}{dt} = q_3 s - q_4 H$$

We have

$$(1-P)\frac{dH}{dt} + P\left[\frac{dH}{dt} - q_3 S + q_4 H\right] = 0$$

This gives

$$\Rightarrow \frac{dH}{dt} - p\frac{dH}{dt} + p\frac{dH}{dt} - pq_3 S + pq_4 H = 0$$

$$\Rightarrow \frac{dH}{dt} - pq_3 S + pq_4 H = 0 \tag{49}$$

Substituting (29), (31) into (49), we have

$$(h_0^1 + ph_1^1 + p^2 h_2^1 + \ldots) - pq_3(s_0 + ps_1 + p^2 s_2 + \ldots +)$$
$$+ pq_4(h_0 + ph_1 + p^2 h_2 + \ldots) = 0$$

Collecting the co-efficient of the power of $P's$ we have.

$$P^0 : h_0^1 = 0 \tag{50}$$

$$P^1 : h_1^1 - q_3 s_0 + h_0 q_4 = 0 \tag{51}$$

$$P^2 : h_2^1 - q_3 s_1 + h_1 q_4 = 0 \tag{52}$$

Applying HPM to (28d)

$$\frac{dE}{dt} = q_5 H + q_{13} V + q_6 E \tag{53}$$

This gives

$$(1-P)\frac{dE}{dt} + P\left[\frac{dE}{dt} - q_5 H - q_{13} V + q_6 E\right] = 0$$

$$\Rightarrow \frac{dE}{dt} - Pq_5 H - Pq_{13} V + Pq_6 E = 0 \tag{54}$$

Substituting for (32) in (54) above, we have

$$(e_0^1 + pe_1^1 + p^2 e_2{}^1 + \ldots +) - pq_5(h_0 + ph_1 + p^2 h_2 + \ldots) -$$
$$q_{13} p(v_0 + pv_1 + p^2 v_2 + \ldots)$$
$$+ pq_6(e_0 + pe_1 + p^2 e_2 + \ldots +) = 0$$

$$\Rightarrow e_0^1 + pe_1^1 + p^2 e_2{}^1 + \ldots + - pq_5(h_0 + ph_1 + p^2 h_2 + \ldots)$$
$$- q_{13} p(v_0 + pv_1 + p^2 v_2 + \ldots)$$
$$+ pq_6 e_0 + p^2 q_6 e_1 + p^3 q_6 e_2 + \ldots + = 0$$

Collecting the co-efficient of the powers of p ^{s} s, we have `

$$P^0 : e_0^1 = 0 \tag{55}$$

$$P^1 : e_1^1 - q_5 h_0 - q_{13} v_0 + q_6 e_0 = 0 \tag{56}$$

$$P^2 : e_2^1 - q_5 h_1 - q_{13} v_1 + q_6 e_1 = 0 \tag{57}$$

Applying HPM to (28e)

$$\frac{dI}{dt} = \phi E + (1-\sigma_1)\mathrm{I}_2 - q_7 \mathrm{I} = 0$$

Therefore

$$(1-P)\frac{dI}{dt} + p\left[\frac{dI}{dt} - (\phi E + (1-\sigma_1)\mathrm{I}_2 - q_7 \mathrm{I})\right] = 0$$

open the brackets above; this given

$$\frac{dI}{dt} - P\frac{dI}{dt} + P\frac{dI}{dt} - P\phi E - P(1-\sigma_1)\mathrm{I}_2 + Pq_7 \mathrm{I} \tag{58}$$

Substituting (33), (32) and (35) in (58), we have

$$(e_0^{0,} + pe_1^1 + p^2 e_2^1 + \ldots +) - p\phi(e_0 + pe_1 + p^2 e_2 + \ldots +)$$
$$- p(1-\sigma_1)(n_0 + pn_1 + p^2 n_2 + \ldots +)$$
$$+ pq_7(e_0^0 + pe_1^0 + p^2 e_2^0 + \ldots +) = 0$$
$$\Rightarrow i_0^0 + pi_1^0 + p^2 i_2^1 + \ldots + -(p\phi e_0 + p^2 \phi e_1 + p^3 \phi e_2 + \ldots +)$$
$$- p(1-\sigma_1)n_0 - p^2(1-\sigma_1)n_1 - p^3(1-\sigma)n_2 + \ldots +$$
$$+ pq_2 i_0 + p^2 q_7 i_1 + p^3 q_7 i_2 + \ldots + = 0$$

Collecting the co efficient of the powers of $p's$ we have

$$P^0 : i_0^1 = 0 \tag{59}$$

$$p^1 : i_1^1 - \phi e_0 - (1-\sigma_1)n_0 + q_7 i_0 = 0 \tag{60}$$

$$P^2 : i_2^1 - \phi e_1 - (1-\sigma_1)n_1 + q_7 i_1 = 0 \tag{61}$$

Applying HPM to (28f), we have

$$\frac{dA}{dt} = \eta c_3 \mathrm{I} + (1-\sigma_2)A_T - q_8 A = 0$$

Therefore

$$(1-p)\frac{dA}{dt} + p\left[\frac{dA}{dt} - (\eta c_3 \mathrm{I} + (1-\sigma_2)A_T - q_8 A)\right] = 0$$

$$\Rightarrow \frac{dA}{dt} - P\eta c_3 \mathrm{I} - P(1-\sigma_2)A_T + Pq_8 A = 0 \qquad (62)$$

Substituting (34), (36) (33) in (62), we have

$$a_0^1 + pa_1^1 + p^2 a_2^1 + ... + -p\eta c_3(i_0 + pi_1 + p^2 i_2 + ...+)$$
$$-p(1-\sigma_2)(m_0 + pm_1 + p^2 m_1)$$
$$+pq_8(a_0 + pa_1 + p^2 a_2 + ...+) = 0$$

Collecting the co efficient of the powers of $p's$ we have

$$p^0 : a_0^1 = 0 \qquad (63)$$

$$p^1 : a_1^1 - \eta c_3 i_0 - (1-\sigma_2)m_0 + q_8 a_0 = 0 \qquad (64)$$

$$p^2 : a_2^1 - \eta c_3 i_1 - (1-\sigma_2)m_1 + q_8 a_1 = 0 \qquad (65)$$

Applying HPM to (28g) we have

$$\frac{dI_2}{dt} - \tau c I + q_9 I_2 = 0$$

Therefore, - $(1-p)\frac{dI_2}{dt} + p\left[\frac{dI_2}{dt} - \tau c I + q_9 I_2\right] = 0$

This gives

$$\frac{dI_2}{dt} - P\tau c I + Pq_9 I_2 = 0 \qquad (66)$$

Substituting (35), (33) in (66) we have

$$n_0^1 + pn_1^1 + p^2 n_2^1 + ... + -p\tau c(i_0 + pi_1 + pi_2 + ...+)$$
$$+pq_9(n_0 + pn_1 + p^2 n_2 + ...+) = 0$$

Collecting the co efficient of the power of $p''s$ we have

$$p^0 : n_0^1 = 0 \qquad (67)$$

$$p^1 : n_1^1 - \tau c i_0 + q_9 n_0 = 0 \qquad (68)$$

$$p^2 : n_2^1 - \tau c i_1 + q_9 n_1 = 0 \qquad (69)$$

Applying HPM to (28h) we have

Collecting the co-efficient of powers of p's, we have

$$P^0 : \mathrm{m}_0^1 = 0 \qquad (71)$$

$$\mathrm{P}^1 : \mathrm{m}_1^1 - r_1 c a_0 + q_{10} m_0 = 0 \qquad (72)$$

$$P^2 : m_2^1 - r_1 c a_1 + q_{10} m_1 = 0 \qquad (73)$$

Applying (Hpm) to 28i we have

$$\frac{dA_2}{dt} = r_1 A - q_{11} A_2 = 0$$

$$(1-p)\frac{dA_2}{dt} + p[\frac{dA_2}{dt} - r_1 A + q_{11} A_2] = 0 \qquad (74)$$

$$\frac{dA_2}{dt} - pr_1 A + pq_{11} A_2 = 0$$

substituting (34) and (37) in (74) we have

$$y_0^1 + py_1^1 + p^2 y_2^1 + ... + -pr_1(a_0 + pa_1 + p^2 a_2 + ...+)$$
$$+pq_{11}(y_0 + py_1 + p^2 y_2 + ...+) = 0$$

collecting the co-efficient of the power of $P's$ we the have

$$P^0 : \mathrm{y}_0^1 = 0 \qquad (75)$$

$$P^1 : \mathrm{y}_1^1 - r_1 a_0 + q_{11} y_0 = 0 \qquad (76)$$

$$P^2 : \mathrm{y}_1^1 - r_1 a_1 + q_{11} y_1 = 0 \qquad (77)$$

From (42) we have

$$S_0^1 = 0$$

Integrating, we have $s_0 = c_1$ where c_1 is a constant. but $\mathrm{S}(0) = S_0$.
Therefore

$$c_1 = S_0$$
$$\Rightarrow s_0 = S_0 \qquad (78)$$

From (46), we have
$v_0^1 = 0$
integrating,we have
$v_0 = c_2$ where c_2 is constant.
but $\mathrm{V}(0) = V_0$
Therefore $\mathrm{V}_0 = c_2$
$\Rightarrow v_0 = \mathrm{V}_0$ (79)

From (50), we have
$h_0^1 = 0$
integrating,we have
$h_0 = c_3$

but $h(0)=H_0$

$\Rightarrow H_0=c_3$ (80)

$\Rightarrow h_0=H_0$

From (55),we have

$e_0^1=0$

integrating, we have

$e_0=c_4$

but $E(0)=e_0$

$\Rightarrow e(0)=c_4$ (81)

$\Rightarrow e_0=E_0$

From (59) we have

$i_0^1=0$

integrating we have

$i_0=c_5$

where c_5 is a constant

but $i(0)=I(0)$ (82)

$\Rightarrow i_0=I(0)$

From (63) we have

$a_0^1=0$

Integrating we have

$a_0=c_0$ where c_0 is a constant.

but $A(0)=a_0$

$\Rightarrow A(0)=c_6$

$\Rightarrow a_0=A(0)$

$a_0=A_0$

From(67)we have

$n_0^1=0$

upon integration, we have

$n_0=c_7$ where c_7 is a constant .

but $n_0=N(0)$

$\Rightarrow N_0=c_7$ (84)

$\Rightarrow n_0=N_0$

From (71), we have

$m_0^1=0$

upon integration,we have

$m_0=c_8$

but $M_0=A_T$ *i.e* $A_T(0)=m_0$ (85)

$\Rightarrow A_T(0)=c_8$

$\Rightarrow m_0=A_T$

from (75), *we have*

$y_0^1=0$

integrating, we have

$y_0=c_9$ where c_9 is a constant

$y(0)=\mathrm{A}_{2,0}$

$\Rightarrow \mathrm{A}_{2,0}=c_a$

therefore

$y_0=\mathrm{A}_{2,0}$ (86)

From (43), we have

$s_1^1-(\pi+\varepsilon h_0+q_{12}v_0)+q_1s_0$

This gives

$s_1^1=\pi+\varepsilon h_0+q_{12}v_0-q_1s_0$ (87)

Substituting (80), (79), (78)in (87), we have

$s_1^1=\pi+\varepsilon H_0+q_{12}V_0-q_1S_0$

$\Rightarrow \dfrac{ds_1}{dt}=\pi+\varepsilon H_0+q_{12}V_0-q_1S_0$

$ds_1=(\pi+\varepsilon H_0+q_{12}V_0-q_1S_0)dt$

integrating,we have

$s_1=(\pi+\varepsilon H_0+q_{12}V_0-q_1S_0)t+c_{10}$

$s_1(0)=0\Rightarrow c_{10}=0,$

since, $s(0)=S_0$

Therefore

$s_1=(\pi+\varepsilon H_0+q_{12}V_0-q_1S_0)t$ (88)

From (47), we have

$v_1^1-\delta_1cs_0+q_2v_0=0$ (89)

Substituting (78)and (79)in (89).we have

$v_1^1-\delta_1cS_0+q_2V_0=0$

$\Rightarrow v_1^1=\delta_1cS_0-q_2V_0$

integrating we then have

$v_1=(\delta_1cS_0-q_2V_0)t+c_{11}$

where c_{10} is the constant of integration.

$v_1=(\delta_1cS_0-q_2V_0)t$

but $V(0)=0\Rightarrow c_{11}=0$

$\Rightarrow v_1=(\delta_1cS_0-q_2V_0)t$ (90)

From (51) we solve for h_1

$h_1^1+q_3s_0-q_4h_0=0$ (91)

Substitutes (78) and (80) in (90) we have

$$h_1^1 = q_4 H_0 - q_3 S_0$$
$$\frac{dh_1}{dt} = q_4 H_0 - q_3 S_0$$

Integrating we have

$$h_1 = (q_4 H_0 - q_3 S_0)t + c_{12}$$

where c_{12} is the constant of integration
But

$$h_1(0) = 0, since, h(0) = h_0 \Rightarrow \quad c_{12} = 0, \text{we then have}$$

$$h_1 = (q_4 H_0 - q_3 S_0)t \qquad (92)$$

And from (56)

$$e_1^1 = q_5 h_0 + q_{13} v_0 - q_6 e_0 \qquad (93)$$

Substitute (81) in (93), we then have

$$e_1^1 = q_5 H_0 + q_{13} V_0 - q_6 E_0$$
$$\Rightarrow \frac{de_1}{dt} = q_5 H_0 + q_{13} V_0 - q_6 E_0$$
$$\Rightarrow de_1 = (q_5 H_0 + q_{13} V_0 - q_6 E_0)dt$$

Integrating

$$e_1 = (q_5 k + q_{13} V_0 - q_6 E_0)t + c_{13}$$
$$\text{where } c_{13} \text{is the constant of integration and } e_1(0) = 0$$
$$since, e_1(0) = e_{1,0}$$
$$therefore\ c_{13} = 0$$
$$\Rightarrow e_1 = (q_5 H_0 + q_{13} V_0 - q_6 E_0)t \qquad (94)$$

From (60)

$$i_1^1 = \phi e_0 + (1-\sigma_1)n_0 - q_7 i_0$$
$$\frac{di_1}{dt} = \phi e_0 + (1-\sigma_1)n_0 - q_7 i_0 \qquad (95)$$

Substituting (81),(84) and (82) in (95) we have

$$\frac{di_1}{dt} = \phi E_0 + (1-\sigma_1)N_0 - q_7 I_0$$
$$\text{Integrating we have}$$
$$i_1 = (\phi E_0 + (1-\sigma_{10})N_0 - q_7 I_0)t + c_{14}$$
$$i_1(0) = 0 \Rightarrow \quad c_{14} = 0$$
$$since, i_1(0) = i_{1,0}$$

Therefore

$$i_1 = (\phi E_0 + (1-\sigma_1)N_0 - q_7 I_0)t. \qquad (96)$$

From (64) we have

$$a_1^1 = \eta c_3 i_0 + (1-\sigma_2)m_0 + q_8 a_0 \qquad (97)$$

$$\text{substituting (82), (85) and (83) in (97) we have}$$
$$a_1^1 = \eta c_3 I_0 + (1-\sigma_2)A_{T0} + q_8 A_0$$

$$\text{Integrating,we have}$$
$$a_1 = (\eta c_3 I_0 + (1-\sigma_2)A_{T0} + q_8 A_0$$
$$a_1(0) = 0 \Rightarrow \quad c_{15} = 0$$
$$since, a_1(0) = a_{1,0}$$

This implies that

$$a_1 = (\eta c_3 I_0 + (1-\sigma_2)A_{T0} + q_8 A_0)t \qquad (98)$$

$$\text{And from (68) we have}$$

$$n_1^1 = \tau c i_0 - q_9 n_0$$
$$\Rightarrow \frac{dn_1}{dt} = \tau c i_0 - q_9 n_0 \qquad (99)$$

Substituting (82) and (84) in (99) we have

$$\frac{dn_1}{dt} = \tau c I_0 - q_9 N_0$$

$$\text{integrating we have}$$

$$n_1 = (\tau c I_0 - q_9 N_0)t + c_{16}$$
$$A_1(0) = 0 \Rightarrow c_{16} = 0$$
$$since, A_1(0) = A_{1,0}$$
$$\therefore n_1 = (\tau c I_0 - q_9 N_0)t \qquad (100)$$

$$\text{From (4.72), we have}$$

$$m_1^1 = r_1 c a_0 + q_{10} m_0$$

$$\text{Substituting (83) and (85) in (100) we have}$$

$$m_1^1 = r_1 c A_0 + q_{10} A_{T0}$$
$$\text{integrating we have}$$
$$m_1 = (r_1 c A_0 + q_{10} A_{T0})t + c_{17}$$
$$m_1(0) = 0 \Rightarrow c_{17} = 0$$
$$since, m_1(0) = A_{T0}$$

$$\Rightarrow m_1 = (r_1 c A_0 + q_{10} A_{T0})t \quad (101)$$

From (76) we have

$$y_1^1 = r_1 a_0 - q_{11} y_0 \quad (102)$$

Substituting (83) and (4.86) in (102) we have

$$y_1^1 = r_1 A_0 - q_{11} A_{2,0}$$

Integrating we have

$$y_1 = (r_1 A_0 - q_{11} A_{2,0})t + c_{18}$$

$$y_1(0) = 0 \Rightarrow c_{18} = 0$$

$$since, y_1(0) = A_{2,0}$$

$$\therefore y_1 = (r_1 A_0 - q_{11} A_{2,0})t \quad (103)$$

We have from (44)

$$s_2^1 - \varepsilon h_1 - q_{12} v_1 + q_1 s_1 = 0$$

$$\Rightarrow \frac{ds_2}{dt} = \varepsilon h_1 + q_{12} v_1 - q_1 s_1 \quad (104)$$

Substituting (90), (92), and (88) in (104) we have

$$\frac{ds_2}{dt} = -q_1(\pi + \varepsilon H_0 + q_{12} V_0 - q_1 S_0)t$$
$$+ q_{12}(\delta_1 c S_0 - q_2 V_0)t + \varepsilon(q_4 H_0 - q_3 S_0)t$$
$$\frac{ds_2}{dt} = \varepsilon(q_4 H_0 - q_3 S_0)t + q_{12}(\delta_1 c S_0 - q_2 V_0)t$$
$$-q_1(\pi - \varepsilon H_0 + q_{12} V_0 - q_1 S_0)t$$
$$\Rightarrow (\varepsilon q_4 H_0 - \varepsilon q_4 S_0 + q_{12}\delta_1 c S_0 - q_2 q_{12} V_0$$
$$-q_1 \pi + \varepsilon q_1 H_0 - q_1 q_{12} V_0 + q_1^2 S_0)t$$
$$\Rightarrow (\varepsilon q_4 H_0 + \varepsilon q_1 H_0 - \varepsilon q_4 S_0 + q_{12}\delta_1 c S_0$$
$$+ q_1^2 S_0 - q_2 q_{12} V_0 - q_1 q_{12} V_0 - q_1 \pi)t$$
$$\Rightarrow (\varepsilon(q_4 + q_1) H_0 - (\varepsilon q_4 + q_{12}\delta_1 c - q_1^2) S_0$$
$$-(q_2 + q_1) q_{12} V_0 - q_1 \pi)t$$
$$\int ds_2 = \int \begin{matrix}(\varepsilon(q_4 + q_1)H_0 - \\ (\varepsilon q_4 + q_{12}\delta_{1c} - q_1^2)S_0 - (q_2 + q_1)q_{12}V_0 - q_1\pi))tdt\end{matrix}$$

Integrating we have

$$s_2 = (\varepsilon(q_1 + q_4)H_0 - (\varepsilon q_4 + q_{12}\delta_1 c - q_1^2)S_0$$
$$-(q_1 + q_2)q_{12}V_0 - q_1\pi)\frac{t^2}{2} + c_{19}$$

where c_{19} is the constant.

Applying the initial conditions, we have

$$s_2(0) = 0$$

therefore $c_{19} = 0$

$$\Rightarrow s_2 = (\varepsilon(q_1 + q_4)H_0 - (\varepsilon q_4 + q_{12}\delta_{1c} - q_1^2)S_0$$
$$-(q_1 + q_2)q_{12}V_0 - q_1\pi)\frac{t^2}{2} \quad (104)$$

but

$$S = s_0 + ps_1 + p^2 s_2 + ...$$

Substituting (78), (88) and (104) in (29)

$$\Rightarrow S = S_0 + p(\pi + \varepsilon H_0 + q_{12}V_0 - q_1 S_0)t$$
$$+ p^2(\varepsilon(q_1 + q_4)H_0$$
$$-(\varepsilon q_4 + q_{12}\delta_{1c} - q_1^2)S_0 - (q_1 + q_2)q_{12}V_0 - q_1\pi)\frac{t^2}{2} + ...$$

Setting $p = 0$, we have

$$S = S_0$$

And setting $p = 1$, we have

$$S(t) = S_0 + (\pi + \varepsilon H_0 + q_{12}V_0 - q_1 S_0)t + (\varepsilon(q_1 + q_4)H_0 -$$
$$(\varepsilon q_4 + q_{12}\delta_{1c} - q_1^2)S_0 - (q_1 + q_2)q_{12}V_0 - q\pi)\frac{t^2}{2} + \quad (105)$$

From (48) give as

$$v_2^1 - \delta_1 c s_1 + q_2 v_1 = 0$$
$$\Rightarrow v_2^1 = \delta_1 c s_1 - q_2 v_1 = \quad (106)$$

Substituting (88) and (90) in (106) we have

$$v_2^1 = \delta_1 c(\pi + \varepsilon H_0 + q_{12} V_0$$
$$-\delta_1 S_0)t - q_2(\delta_1 c S_0 - q_2 V_0)t$$
$$= (\delta_1 c\pi + \delta_1 c\varepsilon H_0 + \delta_1 c q_{12} V_0$$
$$-\delta_1 c q_1 S_0 - q_2 \delta_1 c S_0 + q_2^2 V_0)t$$
$$= (\delta_1 c\pi + \delta_1 c\varepsilon H_0 +$$
$$(\delta_1 c q_{12} + q_2^2)V_0 - \delta_1 c(q_1 + q_2)S_0)t$$
$$= (\delta_1 c(\pi + \varepsilon H_0) + (\delta_1 c q_{12} + q_2^2)V_0$$
$$-\delta_1 c(q_1 + q_2)S_0)t$$
$$\frac{dv_2}{dt} = (\delta_1 c((\pi + \varepsilon H_0) - (q_1 + q_2)S_0)$$
$$+(\delta_1 c q_{12} + q_2^2)V_0)t$$

Integrating and applying the initial conditions, we have

$$V_2(t)=(\delta_1 c(\pi+\varepsilon H_0)-(q_1+q_2)S_0 +(\delta_1 c q_{12}+q_2^2)V_0)\frac{t^2}{2} \quad (107)$$

Substituting (79), (90) and (107) in (30) we have from (30) as

$$V=v_0+pv_1+p^2v_2+...$$
$$\Rightarrow V=V_0+p(\delta_1 cs_0-q_2V_0)t+$$
$$p^2((\delta_1 c(\pi+\varepsilon H_0)-(q_1+q_2)S_0+(\delta_1 cq_{12}+q_2^2)V_0)\frac{t^2}{2}$$

setting $p=0$ we have

$V=V_0$ *and*

setting $p=1$ we have

$$V=V_0+(\delta_1 cs_0-q_2V_0)t+((\delta_1 c(\pi+\varepsilon H_0) -(q_1+q_2)S_0+(\delta_1 cq_{12}+q_2^2)V_0)\frac{t^2}{2} \quad (108)$$

Substitute (88) and (92) in (109) we will have

$$h_2^1=q_4(q_4H_0-q_3S_0)t-q_3(\pi+\varepsilon H_0+q_{12}V_0-q_1S_0)t$$

From (52),we haave

$$h_2^1+q_3s_1-q_4h_1=0$$
$$\Rightarrow h_2^1=q_4h_1-q_3s_1 \quad (109)$$

open the brackets and simplify

$$=(q_4^2H_0-q_4q_3S_0-q_3\pi-q_3\varepsilon H_0+q_3q_{12}V_0+q_3q_1S_0)t$$
$$=((q_4^2-\varepsilon q_3)H_0-q_3(q_4-q_1)S_0+q_3q_{12}V_0-q_3\pi)t$$
$$h_2^1=((q_4^2-\varepsilon q_3)H_0-q_3(q_4-q_1)S_0+q_3(q_{12}V_0-\pi))t$$

Integrating we have

$$h_2=((q_4^2-\varepsilon q_3)H_0-q_3(q_4-q_1)S_0 +q_3(q_{12}V_0-\pi))\frac{t^2}{2}+C_{20}$$

where C_{20} is the constant of integration.
applying the initial conditions we have

$h_2(0)=0\Rightarrow C_{20}=0$

hence

$$h_2(t)=((q_4^2-\varepsilon q_3)H_0- q_3(q_4-q_1)S_0+q_3(q_{12}V_0-\pi))\frac{t^2}{2}$$
$$\Rightarrow h_2(t)=((q_4^2-\varepsilon q_3)H_0 -q_3(q_4-q_1)S_0+q_3(q_{12}V_0-\pi))\frac{t^2}{2} \quad (110)$$

Substituting (110), (92) and (80) in (30), we have

$$H=h_0+ph_1+p^2h_2+...+$$
$$H=H_0+p(q_4H_0-q_3S_0)t+p^2((q_4^2-\varepsilon q_3)H_0 -q_3(q_4-q_1)S_0-q_{12}V_0-\pi)\frac{t^2}{2}$$

setting $p=0$ we have

$H=H_0$, and

setting $p=1$ we have

$$H(t)=H_0+(q_4H_0-q_3S_0)t+((q_4^2-\varepsilon q_3)H_0 -q_3(q_4-q_1)S_0-q_{12}V_0-\pi)\frac{t^2}{2} \quad (111)$$

From (57)

$$e_2^1=q_5h_1-q_{13}V_1+q_6E_1+ \quad (112)$$

substitute (94) in (112)we have

$$e_2'=q_5(q_4H_0-q_3S_0)t+q_{13}(\delta_1 cS_0-q_2V_0) -q_6(q_5H_0+q_{13}V_0-q_6E_0)t$$
$$e_2'=(q_5(q_4H_0-q_3S_0)+q_{13}(\delta_1 cS_0-q_2V_0 -q_6(q_5H_0+q_{13}V_0-q_6E_0))t$$
$$e_2'=(q_5q_4H_0-q_5q_3S_0+q_{13}\delta_1 cS_0-q_{13}q_2V_0 -q_6q_5H_0+q_{13}q_6V_0+q_6^2E_0)(113)$$
$$e_2'=(q_5(q_4-q_6)H_0-(q_3q_5-q_{13}\delta_1 c)S_0 -q_{13}(q_2+q_6)V_0+q_6^2E_0)t$$

integrating

$$e_2=(q_5(q_4-q_6)H_0-(q_3q_5-q_{13}\delta_1 c)S_0 -q_{13}(q_2+q_6)V_0+q_6^2E_0)\frac{t^2}{2}+C_{21}(114)$$

where C_{21} is the constant of integration,
applying inital condition we have

$C_{21}=0$

$$\Rightarrow e_2=(q_5(q_4-q_6)H_0-(q_3q_5-q_{13}\delta_1 c)S_0 -q_{13}(q_2+q_6)V_0+q_6^2E_0)\frac{t^2}{2}.$$

but $e_2=E_2$

$$\Rightarrow E_2=(q_5(q_4-q_6)H_0-(q_3q_5-q_{13}\delta_1 c)S_0 -q_{13}(q_2+q_6)V_0+q_6^2E_0)\frac{t^2}{2} \quad (115)$$

Substitute (113), (94) and (81) in (32) we have

$$E = E_0 + p(q_5 H_0 + q_{12} V_0 - q_6 E_0)t$$
$$+p^2(q_5(q_4 - q_6)H_0 - \tag{116}$$
$$(q_3 q_5 - q_{13}\delta_1 c)S_0 - q_{13}(q_2 + q_6)V_0 + q_6^2 E_0))\frac{t^2}{2}$$

setting $p = 0$ we have

$E = E_0$

setting $p = 1$ we have (117)

$$E = E_0 + (q_5 H_0 + q_{12} V_0 - q_6 E_0)t +$$
$$\left(\begin{matrix} q_5(q_4 - q_6)H_0 - \\ (q_3 q_5 \\ -q_{13}\delta_1 c)S_0 - q_{13}(q_2 + q_6)V_0 + q_6^2 E_0 \end{matrix}\right))\frac{t^2}{2}.$$

From (61) we have that

$$i_2' - \phi e_1 - (1-\sigma_1)n_1 + q_7 i_1 = 0 \tag{118}$$

substituting (94), (96) and (100) in (118) we have

$$i_2' = \phi(q_5 H_0 + q_{13} V_0 - q_6 E_0)t + (1-\sigma_1)(\tau c I_0 - q_9 N_0)t$$
$$-q_7(\phi E_0 + (1-\sigma_1)N_0 - q_7 I_0)t$$
$$= (\phi q_5 H_0 + q_{13} V_0 - \phi q_6 E_0 + (1-\sigma_1)\tau c I_0 -$$
$$(1-\sigma_1)q_9 N_0 - q_7 \phi E_0 - q_7(1-\sigma_1)N_0 - q_7^2 I_0)t$$
$$= (\phi q_5 H_0 + q_{13} V_0 - \phi(q_6 + q_7)E_0 +$$
$$((1-\sigma_1)\tau c + q_7^2)I_0 - ((1-\sigma_1)(q_9 + q_7))N_0)t$$
$$\frac{di_2'}{dt} = (\phi q_5 H_0 + q_{13} V_0 - \phi(q_6 + q_7)E_0$$
$$+((1-\sigma_1)\tau c + q_7^2)I_0 - ((1-\sigma_1)(q_9 + q_7))N_0)t$$

integrating we have

$$i_1 = (\phi q_5 H_0 + q_{13} V_0 - \phi(q_6 + q_7)E_0 +$$
$$((1-\sigma_1)\tau c + q_7^2)I_0 - (1-\sigma_1)(q_9 + q_7)N_0)\frac{t^2}{2} + C_{22}$$

where C_{22} is the constant of integration.

Applying initial condition

$i_2(0) = 0$

$\Rightarrow C_{22} = 0$

Therefore

$$i_1 = (\phi q_5 H_0 + q_{13} V_0 - \phi(q_6 + q_7)E_0 +$$
$$((1-\sigma_1)\tau c + q_7^2)I_0 - (1-\sigma_1)(q_9 + q_7)N_0)\frac{t^2}{2} \tag{119}$$

but

setting $p = 0$ we have

$I(t) = I_0$ which is trivial.

setting $p = 1$ we have

$$I(t) = I_0 + (\phi E_0 + (1-\sigma_1)N_0 - q_7 I_0)t +$$
$$(\phi q_5 H_0 + q_{13} V_0 - (q_6 + q_7)E_0) + ((1-\sigma_1)\tau c + q_7^2)I_0 \tag{120}$$
$$-(1-\sigma_1)(q_9 + q_7)N_0)\frac{t^2}{2}$$

And from (65) we solve for a_2, this implies that

$$a_2^1 = \eta_{C_3} i_1 + (1-\sigma_2)m_1 - q_8 a_1 \tag{121}$$

Substitutes (98), (96) and (101) in (121)

$$a_2^1 = \eta_{C_3}(\phi E_0 + (1-\sigma_1)N_0 - q_7 I_0)t +$$
$$((1-\sigma_2)(r_1 c A_0 + q_{10} A_{T_0}))t$$
$$-q_8(\eta_{C_3} I_0 + (1-\sigma_3)A_{T_0} + q_8 A_0)t$$
$$a_2^1 = (\eta_{C_3}(\phi E_0 + (1-\sigma_1)N_0 - q_7 I_0) +$$
$$((1-\sigma_2)(r_1 c A_0 + q_{10} A_{T_0}))$$
$$-q_8(\eta_{C_3} I_0 + (1-\sigma_3)A_{T_0} + q_8 A_0))t$$
$$= (\eta_{C_3}\phi E_0 + \eta_{C_3}(1-\sigma_1)N_0 - \eta_{C_3} q_7 I_0 +$$
$$(1-\sigma_2)r_1 c A_0 + (1-\sigma_2)q_{10} A_{T_0}$$
$$-q_8 \eta_{C_3} I_0 + q_8(1-\sigma_3)A_{T_0} + q_8^2 A_0)t$$
$$= (\eta_{C_3}\phi E_0 + \eta_{C_3}(1-\sigma_1)N_0) - \eta_{C_3}(q_7 + q_8)I_0$$
$$+((1-\sigma_2)r_1 c - q_8^2)A_0 + ((1-\sigma_2)q_{10} - q_8(1-\sigma_3)A_{T_0})t$$

Integrating, we have

$$a_2^1 = (\eta_{C_3}(\phi E_0 + \eta_{C_3}(1-\sigma_1)N_0) - \eta_{C_3}(q_7 + q_8)I_0$$
$$+((1-\sigma_2)r_1 c - q_8^2)A_0 + ((1-\sigma_2)q_{10}$$
$$-q_8(1-\sigma_3)A_{T_0})\frac{t^2}{2} + C_{23}$$

where C_{23} is the constant of integration, applying the initial condition $a_2(0) = 0$, *we have* $C_{23} = 0$ therefore

$$a_2(t) = (\eta_{C_3}(\phi E_0 + \eta_{C_3}(1-\sigma_1)N_0) -$$
$$\eta_{C_3}(q_7 + q_8)I_0 + ((1-\sigma_2)r_1 c - q_8^2)A_0 \tag{122}$$
$$+((1-\sigma_2)q_{10} - q_8(1-\sigma_3)A_{T_0})\frac{t^2}{2}$$

Substitute (83), (98) and (122) into (34) we have

$$A = A_0 + p(\eta_{C_3} I_0 + (1-\sigma_3)A_{T_0} + q_8 A_0)t$$
$$+ p^2(\eta_{C_3}(\phi E_0 + (1-\sigma_1)N_0)$$
$$-\eta_{C_3}(q_7+q_8)I_0 + ((1-\sigma_2)r_1 c - q_8^2)A_0$$
$$+((1-\sigma_2)q_{10} - q_8(1-\sigma_3)A_{T_0})\frac{t^2}{2}$$

Setting $p = 0$ we have

$$A = A_0$$

Setting $p = 1$ we have

$$A(t) = A_0 + (\eta_{C_3} I_0 + (1-\sigma_3)A_{T_0} + q_8 A_0)t$$
$$+(\eta_{C_3}(\phi E_0 + (1-\sigma_1)N_0)$$
$$-\eta_{C_3}(q_7+q_8)I_0 + ((1-\sigma_2)r_1 c - q_8^2)A_0 \tag{123}$$
$$+((1-\sigma_2)q_{10} - q_8(1-\sigma_3)A_{T_0})\frac{t^2}{2}$$

From (69) we have

$$n_2' = \tau c i_1 - q_9 n_1 \tag{124}$$

Substituting (96) and (100) into (124), we have

$$n_2' = \tau c(\phi E_0 + (1-\sigma_1)N_0 - q_7 I_0)t$$
$$-q_9(\tau c I_0 - q_9 N_0)t$$
$$= (\tau c\phi E_0 + \tau c(1-\sigma_1)N_0 - q_7\tau c I_0$$
$$-q_9\tau c I_0 + q_9^2 N_0)t$$
$$= (\tau c\phi E_0 + \tau c((1-\sigma_1)+q_9^2)N_0$$
$$-\tau c(q_7+q_9)I_0)t$$
$$\Rightarrow n_2' = (\tau c(\phi E_0 + \tau c((1-\sigma_1)+q_9^2)N_0$$
$$-\tau c(q_7+q_9)I_0)t$$

integrating and applying the iniial condition

$n_2(0) = 0,$ *we have*

$$n_2 = (\tau c\phi E_0 + \tau c((1-\sigma_1)+q_9^2)N_0$$
$$-(q_7+q_9)I_0)\frac{t^2}{2} + c_{24}$$

$n_2(0) = 0 \Rightarrow c_{24} = 0$ *therefore*

$$n_2(t) = (\tau c\phi E_0 + \tau c((1-\sigma_1)$$
$$+q_9^2)N_0 - (q_7+q_9)I_0)\frac{t^2}{2} \tag{124}$$

substitute (84), (100) and (124) in (35) we have

$$I_2 = N_0 + p(\tau c I_0 + q_9 N_0)t$$
$$+p^2(\tau c(\phi E_0 + \tau c((1-\sigma_1)+q_9^2)N_0 - (q_7+q_9)I_0)\frac{t^2}{2} + \ldots$$

setting $p = 0$

$$I_2 = N_0$$

setting $p = 1$ we have

$$I_2 = N_0 + (\tau c I_0 + q_9 N_0)t + (\tau c(\phi E_0$$
$$+\tau c((1-\sigma_1)+q_9^2)N_0 - (q_7+q_9)I_0)\frac{t^2}{2} + \ldots$$

we note that $N_0 = I_{2,0}$ as stated earlier

$$I_2(t) = I_{2,0} + (\tau c I_0 + q_9 I_{2,0})t + (\tau c(\phi E_0$$
$$+\tau c((1-\sigma_1)+q_9^2)I_{2,0} - (q_7+q_9)I_0)\frac{t^2}{2} + \ldots \tag{125}$$

From (73) we have

$$m_2' = r_1 c a_1 - q_{10} m_1 \tag{126a}$$

substituting (98) and (101) in (126) we have

$$m_2' = r_1 c(\eta c_3 + (1-\sigma_3)A_{T0}$$
$$+q_8 A_0)t - q_{10}(r_1 c A_0 + q_{10} A_{T0})t$$
$$= (r_1 c\eta c_3 I_0 + r_1 c(1-\sigma_3)A_{T0}$$
$$+r_1 c q_8 A_0 - q_{10} r_1 c A_0 + q_{10}^2 A_{T0})t$$
$$= (r_1 c\eta c_3 I_0 + (r_1 c(1-\sigma_3) + q_{10}^2)A_{T0}$$
$$+r_1 c(q_8 - q_{10})A_0)t$$
$$= ((r_1 c(\eta c_3 I_0 + (q_8 - q_{10})A_0)$$
$$+(r_1 c(1-\sigma_3) - q_{10}^2)A_{T0})t$$
$$\frac{dm_2}{dt} = (r_1 c(\eta c_3 I_0 + (q_8 - q_{10})A_0)$$
$$+(r_1 c(1-\sigma_3) - q_{10}^2)A_{T0})t$$

integrating we have

$$m_2 = (r_1 c(\eta c_3 I_0 + (q_8 - q_{10})A_0)$$
$$+(r_1 c(1-\sigma_3) - q_{10}^2)A_{T0})\frac{t^2}{2} + c_{25}$$

where c_{25} is the constant of integration, applying the initial condition $m_2(0) = 0$
we have $c_{25} = 0$ *therefore*

$$m_2 = (r_1 c(\eta c_3 I_0 + (q_8 - q_{10})A_0)$$
$$+(r_1 c(1-\sigma_3) - q_{10}^2)A_{T0})\frac{t^2}{2} \tag{126b}$$

substituting (85) (101) and (126) into (36) we have

$$A_T = A_{T0} + p(r_1 c A_0 + q_{10} A_{T0})t +$$
$$p^2(r_1 c(\eta c_3 I_0 + (q_8 - q_{10})A_0) + (r_1 c(1-\sigma_3) - q_{10}^2)A_{T0})\frac{t^2}{2}$$

setting $p=0$

$\Rightarrow A_T = A_{T0}$

setting $p=1$ $we\ have$

$$A_T(t) = A_{T0} + (r_1 c A_0 + q_{10} A_{T0})t + (r_1 c(\eta c_3 I_0 + (q_8 - q_{10})A_0) + (r_1 c(1-\sigma_3) - q_{10}^2) A_{T0}) \frac{t^2}{2} + .. \quad (127)$$

And from (77), we have

$$y_2' = r_1 a_1 - q_{11} y_1 \quad (128)$$

substituting (98), and (103) into (128) we have

$$\Rightarrow y_2' = r_1(\eta c_3 I_0 + (1-\sigma_3)A_{T0} + q_8 A_0)t$$
$$-q_{11}(r_1 A_0 - q_{11} A_{2,0})t$$
$$= (r_1 \eta c_3 I_0 + r_1(1-\sigma_3)A_{T0} + r_1 q_8 A_0$$
$$-q_{11} r_1 A_0 + q_{11}^2 A_{2,0})t$$
$$= (r_1(\eta c_3 I_0 + (1-\sigma_3)A_{T0})$$
$$+r_1(q_8 - q_{11})A_0 + q_{11}^2 A_{2,0})t$$
$$y_2' = (r_1(\eta c_3 I_0 + r_1(1-\sigma_3)A_{T0})$$
$$+(q_8 - q_{11})A_0) + q_{11}^2 A_{2,0})t$$

integrating

$$y_2 = (r_1(\eta c_3 I_0 + (1-\sigma_3)A_{T0} + (q_8 - q_{11})A_0) + q_{11}^2 A_{2,0})\frac{t^2}{2} + c_{26}$$

where c_{26} is the constant of integration,

applying the intia condition

$this\ implies\ that\ c_{26} = 0$

$therefore$

$$y_2 = (r_1(\eta c_3 I_0 + (1-\sigma_3)A_{T0} + (q_8 - q_{11})A_0) + q_{11}^2 A_{2,0})\frac{t^2}{2} \quad (129)$$

Substituting (86), (101) and (129) in (37) we have

$$A_2 = A_{2,0} + p(r_1 A_0 - q_{11} A_{2,0})t + p^2(r_1(\eta c_3 I_0 + (1-\sigma_3)A_{T0} + (q_8 - q_{11})A_0 + q_{11}^2 A_{2,0})\frac{t^2}{2} + ... + \quad (130)$$

setting $p=0$ $in\,(130)$ $we\,have$

$\Rightarrow A_2 = A_{2,0}$

setting $p=1$ $we\ have$

$$A_2(t) = A_{2,0} + (r_1 A_0 - q_{11} A_{2,0}) + (r_1 (\eta c_3 I_0 + (1-\sigma_3)A_{T0} + (q_8 - q_{11})A_0 + q_{11}^2 A_{2,0})\frac{t^2}{2} \quad (131)$$

Therefore (105), (108), (111), (114), (120), (123), (125), (127) and (131) are the solution of our models using the Homotopy perturbation method (HPM).

4. CONCLUSION

The non-linear deterministic compartmental models with controls were solved analytically using the Homotopy perturbation method. The solutions of the models show a series of solution in form of power series. Homotopy perturbation method is and elegant method and a good approach to solve any non-linear, linear partial differential models analytically.

REFERENCES

[1] He JH. (2003). Homotopy perturbation method for new nonlinear analytical technique. Applied Mathematics and Computation 135: 73-79.

[2] He JH. (2006). Homotopy perturbation method for solving boundary value problem. Physics Letters A 350: 87–88.

[3] Taghipour R. (2011). Application of Homtopy perturbation method on some linear and nonlinear parabolic equations IJRRAS 6(1): 55-59.

[4] Sepkowitz KA. (2001). AIDS the first 20 years. N. Engl J. Med. 344(23).

[5] WHO (World Health Organization (2007). WHO case definition of HIV for surveillance and revised clinical staging and immunological classification of HIV related disease in Adults and children. Geneval pp. 6-16.

[6] Vogel M, Schwarze-Zander C, Wasmuth JC, Spengler U, Sauerbruch T, Rockstroh JK. (2010). The treatment of patient with HIV. Deutsches Arzteblatt International 107(28-29).

[7] Evian C. (2006). Primary HIV/AIDS care: a practical guide for primary health care personal in a clinical and supportive setting. Houghton (South Africa): Jacana P. 29.

[8] Elliot T. (2012). Lecture Notes: Medical Microbiology and Infection. John Wiley and Sons. p. 273.

[9] Blankson JN. (2010). Control of HIV-1 replication in elite suppressors. Discovery Medicine 9(46).

[10] Walker BD. (2007). Elites control of HIV infection implication for vaccine and treatment. International AIDS Society USA 15(4).

[11] Holmes CB, Losina E, Walenskey RP. (2003). Review of human immunodeficiency virus type-1 related opportunistic infections in Sub-Sahara Africa clin. Infect Dis 36(5).

[12] Sestak K. (2005). Chronic diarrhea and AIDS insights into studies with non-human primate curr. HIV Res 3(3).

[13] Murray ED, Buttner N, Brice BH. (2012). Depression and psychosis in Neurological practice. Philadeplina PA Elsevien /Saunders p. 101.