

## Differential Transformation Method(DTM) for Approximate Solution of Ordinary Differential Equation(ODE)

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### ABSTRACT

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This paper presents the differential transformation method to evaluate approximate solution of ordinary differential equation along with its application in electrical circuit problem in various time domain. The method has been well established with six test problems and comparison has been made between analytical and numerical results to obtain convergence to a fair degree of accuracy. The results obtained suggest that the present technique can be treated as an efficient and alternative method for solving initial and boundary value problems.

## 1. INTRODUCTION

In the first part of introduction some basic rules and properties on differential transformation method [2-6] have been reported. Section-2 contains numerical as well as analytical solution for homogeneous and non-homogenous ordinary differential equations of various order at different grid points. These are displayed in Table-1, Table-2 and in Table-3. The application of differential transformation method has been illustrated with various time domain  $t$  in section-3. These results have been represented in Table-4, Table-5 and in Table-6. The last section-5 bears conclusion.

### 1.1 Rules of DTM

$Y(k)$  represents the differential transformation of the  $k$ th derivative of the function  $y(x)$  which is defined as

$$Y(k) = \frac{1}{k!} \left[ \frac{d^k y(x-x_0)}{dx^k} \right] \quad (1)$$

The inverse differential transformation of  $Y(k)$  is defined as

$$y(x) = \sum_{k=0}^{\infty} Y(k)(x-x_0)^k \quad (2)$$

But for practical purposes equation (2) is represented as a finite series as follows:

$$y(x) = \sum_{k=0}^n Y(k)(x-x_0)^k \quad (3)$$

Assuming the centre of the above series at zero and combining the results of equation (1) and equation (3) the function  $(x)$  can be expressed as

$$y(x) = \sum_{k=0}^n \frac{1}{k!} \left[ \frac{d^k y(x)}{dx^k} \right] x^k \quad (4)$$

Equation (4) suggests that DTM is a finite approximation of Taylor's series expansion where sum of the terms beyond  $k = n$  is negligibly small and hence discarded. The following

properties of DTM are derived from the definitions in equations (1) and equation (2).

### 1.2 Properties of DTM

Linearity:

If  $y(x) = af(x) \pm bg(x)$  then  $Y(k) = aF(k) \pm bG(k)$  where  $a$  and  $b$  are constants.

Product Rule:

If  $y(x) = f(x)g(x)$  then  $Y(k) = \sum_{k_1=0}^n F(k_1)G(k-k_1)$

Transformation of Polynomial:

If  $y(x) = x^n$  then  $Y(k) = \delta(k-n) = f(x) = \begin{cases} 1, & k = n \\ 0, & k \neq n \end{cases}$

Transformation of  $m$ th order differential:

If  $y(x) = \frac{d^m g(x)}{dx^m}$  then  $Y(k) = \frac{(k+m)!}{k!} Y(k+m)$

Transformation of exponential function:

If  $y(x) = e^{\lambda x}$  then  $Y(k) = \frac{\lambda^k}{k!}$ , where  $\lambda$  is a constant.

Transformation of trigonometric function:

If  $y(x) = \begin{cases} \sin(\omega x + \alpha) \\ \cos(\omega x + \alpha) \end{cases}$  then

$$Y(k) = \begin{cases} \frac{\omega^k}{k!} \sin\left(\frac{k\pi}{2} + \alpha\right) \\ \frac{\omega^k}{k!} \cos\left(\frac{k\pi}{2} + \alpha\right) \end{cases} \text{ where}$$

$\omega$  and  $\alpha$  are constants.

## 2. NUMERICAL EXAMPLES:

The following examples of IVP establish the effectiveness of DTM at select grid points.

Example 1:

$y''' - y'' - y' + y = 0, y(0) = 2, y'(0) = 1, y''(0) = 0.$

The exact solution of the above problem is

$$y(x) = (2 - x)e^x$$

From the initial values we have

$$Y(0) = 2, Y(1) = 1, Y(2) = 0.$$

Successive application of DTM gives the recursive relation

$$Y(k + 3) = \frac{1}{(k+1)(k+2)(k+3)} ((k + 1)(k + 2)Y(k + 2) + (k + 1)Y(k) - Y(0))$$

Using the above relation and initial values

$$Y(3) = -\frac{1}{6}, Y(4) = -\frac{1}{12}, Y(5) = -\frac{1}{30}$$

Using equation (3) the solution has been obtained as

$$y(x) = 2 + x - \frac{1}{6}x^3 - \frac{1}{12}x^4 - \frac{1}{40}x^5 - \frac{1}{180}x^6 - \frac{1}{1008}x^7 - \frac{1}{6720}x^8 - \frac{1}{51840}x^9 - \dots$$

Example 2:

$$y'' + 2y' + y = e^{-x}, y(0) = -1, y'(0) = 1.$$

The exact solution to the above problem is

$$y(x) = \left(\frac{1}{2}x^2 - 1\right)e^{-x}$$

Applying DTM to the given problem we have

And

$$Y(0) = -1, Y(1) = 1$$

Using the recursive relation and inverse DTM the following solution to the given IVP is

$$y(x) = -1 + x - \frac{1}{3}x^3 + \frac{5}{24}x^4 - \frac{3}{40}x^5 + \frac{7}{360}x^6 - \frac{1}{252}x^7 + \frac{3}{4480}x^8 - \frac{1}{10368}x^9 - \dots$$

Example 3:

$$y'' + 4y' + 5y = e^{-t} \sin 2t, y(0) = 0, y'(0) = 1.$$

Exact solution is

$$y(t) = \frac{2}{5}e^{-2t} \sin t - \frac{1}{5}e^{-2t} \cos t + \frac{1}{5}e^{-t}(\cos t + 2 \sin t)$$

Here  $Y(0) = 0, Y(1) = 1$  and application of DTM yields the recursive relation

$$Y(k + 2) = \frac{1}{(k+1)(k+2)} \left[ \sum_{r=0}^k \frac{(-1)^{k-r}}{(k-r)!k!} \cos\left(\frac{k\pi}{2}\right) - 4(k + 1)Y(k + 1) - 5Y(k) \right]$$

By applying inverse DTM, solution of the IVP is as follows

$$y(t) = t - \frac{3}{2}t^2 + t^3 - \frac{3}{8}t^4 + \frac{1}{15}t^5 + \frac{1}{80}t^6 - \frac{1}{70}t^7 + \frac{27}{4480}t^8 - \frac{31}{18144}t^9 + \dots$$

### 3. APPLICATION OF DIFFERENTIAL TRANSFORMATION METHOD (DTM)

In this section application of DTM on electrical circuits(RLC) has been considered. Zhou[1] is the pioneer of the adopted method who efficiently implemented it in solving the electrical circuit problems governing Kirchhoff's voltage laws. The exact solution to the given problems can be obtained in polynomial form by using Laplace Transformation or power series method. By the present method an alternative approximate solution is obtained by using following results. By the basic properties and definitions of Kirchhoff's Voltage law(KVL).

$$LI' + RI + \frac{1}{C} \int I dt = E(t) \tag{5}$$

where  $E(t)$  is the electromotive force,  $E_0 = \text{constant}$ ,

$L = \text{Inductance}$ ,  $R = \text{Resistance}$

$C = \text{Capacitance}$  and  $I(t) = \text{The instantaneous current}$   
The integral equation(5) after differentiation becomes

$$I'' + \frac{R}{L}I' + \frac{1}{CL}I = \frac{E'(t)}{L} \tag{6}$$

The equation(6) is a non homogeneous second order differentialequation [8].

Three examples have been selected governing KVL.

Example 4:

$$I'' + 4I' + 3I = 4.5 \sin 2t \text{ with } I(0) = 0 = I'(0)$$

The exact solution to the problem is:

$$I(t) = -\frac{9}{26}e^{-3t} + \frac{117}{130}e^{-t} - \frac{72}{130}\cos 2t - \frac{9}{130}\sin 2t$$

Applying DTM with  $Y(0) = Y(1) = 0$  and the recursive relation

$$Y(k + 2) = \frac{1}{(k + 1)(k + 2)} (-4(k + 1)Y(k + 1) - 3Y(k) + 4.5 \frac{2^k}{k!} \sin \frac{k\pi}{2})$$

the given IVP has the solution

$$I(t) = -\frac{3}{2}t^3 - \frac{3}{2}t^4 + \frac{27}{40}t^5 - \frac{3}{10}t^6 + \frac{17}{112}t^7 - \frac{67}{1120}t^8 + \frac{251}{13440}t^9 + \dots$$

Example 5:

Solution for the current  $I(t)$  in an RLC circuit with  $R=3$  ohms,  $L=.5$  henry,  $C=.08$  farad and  $E = 12\cos 5t$  assuming zero current and charge at initial time.

The required IVP is

$$I'' + 6I' + 25I = -120 \sin 5t, I(0) = 0, I'(0) = 0$$

Exact solution:

$$I(t) = \sin 4t \left( \frac{3}{2} \cos 9t - \frac{1}{2} \sin 9t + \frac{3}{2} \cos t - \frac{9}{2} \sin t \right) - \cos 4t \left( \frac{1}{2} \cos 9t + \frac{3}{2} \sin 9t - \frac{9}{2} \cos t - \frac{3}{2} \sin t \right) - 4e^{-3t} \cos 4t - 3e^{-3t} \sin 4t$$

Applying DTM, we have  $Y(0) = 0, Y(1) = 0$  and the recursive relation

$$Y(k+2) = \frac{1}{(k+1)(k+2)} (-6(k+1)Y(k+1) - 25Y(k) - 120 \frac{5^k}{k!} \sin \frac{k\pi}{2})$$

Application of inverse DTM gives the polynomial solution as

$$I(t) = -100t^3 + 150t^4 + 70t^5 - 195t^6 + \frac{715}{14}t^7 + \frac{195}{4}t^8 - \frac{2635}{108}t^9 + \frac{79}{72}t^{10} \dots \dots \dots$$

Example 6:  
Solution for the current  $I(t)$  in an R-L-C circuit with  $R = 80 \text{ ohms}, L = 10 \text{ henry}, C = .004 \text{ farad}$

and  $E = 240.5 \sin t$  assuming zero current and charge at initial time:

The required IVP is

$$I'' + 8I' + 25I = 24.05 \cos t, I(0) = 0, I'(0) = 0$$

Exact solution:

Applying DTM we have  $Y(0) = 0, Y(1) = 0$  and the recursive relation

$$Y(k+2) = \frac{1}{(k+1)(k+2)} (-8(k+1)Y(k+1) - 25Y(k) + 24.05 \cos \frac{k\pi}{2})$$

Application of inverse DTM yields the polynomial solution

$$I(t) = \frac{481}{40}t^2 - \frac{481}{15}t^3 + \frac{9139}{240}t^4 - \frac{6253}{300}t^5 - \frac{6253}{1600}t^6 + \dots \dots \dots$$

The following tables show the exact value, approximate value and absolute error in the solution of given IVPs by executing DTM.

**Table 1.** Example 1

x	Exact Value	Approximate Value	Absolute Error
0.1	2.099824744343730	2.099824744343730	0.000000000000000
0.2	2.198524964688300	2.198524964688310	4.884981308350689e-15
0.3	2.294759972879200	2.294759972879610	4.107825191113079e-13
0.4	2.386919516226030	2.386919516235850	9.819256518994735e-12
0.5	2.473081906050190	2.473081906165600	1.154130124803032e-10
0.6	2.550966320546710	2.550966321412570	8.658598282806906e-10
0.7	2.617878519711620	2.617878524476930	4.765317918042911e-09
0.8	2.670649114190960	2.670649135096770	2.090581485703069e-08
0.9	2.705563422272640	2.705563499406090	7.713345295812246e-08
1	2.718281828459040	2.718282076719570	2.482605299114482e-07

**Table 2.** Example 2

x	Exact Value	Approximate Value	Absolute Error
0.1	-0.900313230945780	-0.900313230945790	9.992007221626400E-15
0.2	-0.802356138016422	-0.802356138026808	1.038602537306580E-11
0.3	-0.707481400751040	-0.707481401349163	5.981229955764880E-10
0.4	-0.616694442352788	-0.616694452960846	1.060805798758220E-08
0.5	-0.530714327248554	-0.530714425922912	9.867435790678050E-08
0.6	-0.450025541597102	-0.450026151828571	6.102314689671040E-07
0.7	-0.374921904362514	-0.374924751770211	2.847407697015300E-06
0.8	-0.305543695599711	-0.30554506587654	1.081098794297740E-05
0.9	-0.241908947545656	-0.241944014314341	3.506676868500480E-05
1	-0.183939720585721	-0.184040178571429	1.004579857079850E-04

**Table 3.** Example 3

x	Exact Value	Approximate Value	Absolute Error
0.1	0.001356464598500	0.001356464598485	1.499993705633695e-14
0.2	0.009798598745439	0.009798598715429	3.001000142832666e-11
0.3	0.029801158441399	0.029801155907170	2.534228996792232e-09
0.4	0.063497061223301	0.063497002617905	5.860539599444881e-08
0.5	0.111140193030166	0.111139526367187	6.666629790014200e-07
0.6	0.171495395053878	0.171490552827429	4.842226449003340e-06
0.7	0.242178918565363	0.242153108741396	2.580982396699350e-05
0.8	0.319964492938543	0.319854795629714	1.096973088290376e-04
0.9	0.401064032741230	0.400671801838366	3.922309028639992e-04
1.0	0.481387476388215	0.480163690476191	0.001223785912024

**Table 4.** Example 4

x	Exact Value	Approximate Value	Absolute Error
0.1	0.085963177796689	0.085963177796690	9.992007221626400E-16
0.2	0.147421965065099	0.147421965066131	1.032007812540310E-12
0.3	0.189130851998201	0.189130852086716	8.851500088447040E-11
0.4	0.215113997716010	0.215113999795245	2.079235006391400E-09
0.5	0.228750064528217	0.228750088538653	2.401043600053220E-08
0.6	0.232853257577284	0.232853434512000	1.769347159896730E-07
0.7	0.229748790415279	0.229749746699376	9.562840970200880E-07
0.8	0.221341795521528	0.221345914634384	4.119112855999190E-06
0.9	0.209179272281065	0.209194191246877	1.491896581198850E-05
1	0.194505064836663	0.194505064836663	4.712872594700430E-05

**Table 5.** Example 5

x	Exact Value	Approximate Value	Absolute Error
0.1	-0.084489429654472	-0.084489429645569	8.902989456771593e-12
0.2	-0.549313882391200	-0.549313865210582	1.718061803934745e-08
0.3	-1.443162311236885	-1.443160940833929	1.370402956002437e-06
0.4	-2.532605788606551	-2.532576547487832	2.924111871882573e-05
0.5	-3.441829862462138	-3.441530903811178	2.989586509598574e-04
0.6	-3.807368479298738	-3.805476693942859	1.891785355879e-03
0.7	-3.407366376454890	-3.398922232355096	8.444144099794e-03
0.8	-2.236434699645677	-2.207983015686777	2.8451683958900e-02
0.9	-0.512895321161074	-0.438877301576769	7.4018019584305e-02
1	1.377857673138121	1.520502645502649	1.42644972364528e-01

**Table 6.** Example 6

x	Exact Value	Approximate Value	Absolute Error
0.1	0.091780451967744	0.091780451970833	3.089001276990189e-12
0.2	0.278654281953035	0.278654287231048	5.278013026632777e-09
0.3	0.474228441488846	0.474228816512148	3.750233019683691e-07
0.4	0.636632986671345	0.636640130869728	7.144198382991185e-06
0.5	0.751102449034015	0.751167469125576	6.502009156095223e-05
0.6	0.817595653557294	0.817957675548001	3.620219907070066e-04
0.7	0.842765777105223	0.844146651241225	1.380874136002e-03
0.8	0.835237962554497	0.838929065323003	3.691102768506e-03
0.9	0.803168295545608	0.809094077290491	5.925781744883e-03
1	0.753238351345508	0.750535232032627	2.703119312881e-03

**4. CONCLUSION**

The numerical analysis of results between analytical method and DTM shows high degree of accuracy. The method followed is effective and computation is less time consuming. So the prescribed method suggests a better approach for the solution of higher order IVPs.

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