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Fourier-Bessel transform method for finding vertical stress fields in axisymmetric elasticity problems of elastic half space involving circular foundation areas

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ABSTRACT

In this work, the Fourier-Bessel transformation method was used to determine the vertical stress fields in axisymmetric elasticity problems of elastic half space involving circular foundation areas subject to uniformly distributed loads. A stress-based formulation of the elasticity problem was adopted. The biharmonic stress compatibility equation was solved using the variable separable technique to obtain a general solution for the bounded stress-functions as Fourier-Bessel integrals. Egorov expressions for the vertical stress fields defined in terms of harmonic functions were used to obtain the vertical stress fields. The load distribution was similarly transformed by the Fourier-Bessel transformation. Enforcement of the boundary condition of the equilibrium of the internal vertical stress at the $z = 0$ plane and the applied load yielded the unknown parameter of the bounded Fourier-Bessel transform integral, and thus, the full determination of the bounded stress function $\Omega(r, z)$. The vertical stress fields $\sigma_{zz}(r, z)$ were obtained from the bounded stress potential function using Egorov expressions for $\sigma_{zz}(r, z)$. Evaluation of the integration problem yielded mathematical expressions for the vertical stresses at any point in the elastic half space. The vertical stresses at any point under the center of the circular foundation were also determined, and tabulated. The mathematical expressions for vertical stresses obtained using Fourier-Bessel transform method were identical with solutions in the technical literature.

1. INTRODUCTION

The determination of vertical stress distribution in elastic half space due to distributed load on the surface is a problem of the classical mathematical theory of elasticity [1-5]. In such problems, the elastic half space material can be considered isotropic or non-isotropic, and homogeneous or non homogeneous. Elastic half space problems involving non-homogeneous, non-isotropic materials are very difficult to solve, and in many cases, mathematical solutions are non-existent [1-13]. In this work, the elastic half space material is assumed to be isotropic and homogeneous. Elasticity problems of elastic half space are extensively found in the elastic stress and deformation analysis and design of structural footings and foundations structures. Axisymmetric, elasticity problems of elastic half space involve elasticity problems that produce a circular symmetry of the state of stress about a vertical axis, which is usually the vertical axis of symmetry of the applied load.

In general, axisymmetrical elasticity problems are governed by the simultaneous requirements of the three differential equations of equilibrium, the six strain-displacement equations, and the six equations expressing stresses in terms of strains. These systems of fifteen equations are also required to satisfy the compatibility equations, and the traction and deformation boundary conditions [1-13]. Rigorous mathematical solutions of axisymmetric elasticity problems involve mathematically advanced and intensive analysis and are quite often unwieldy.

Three basic methods commonly used in the reformulation of elasticity problems can lead to simplifications in the governing equations to be solved. The three methods are the displacement method, the stress method and the mixed (hybrid) method [1-13].

The displacement methods involve a reformulation of the system of fifteen governing equations involving fifteen unknowns such that only Cartesian components of the displacement become the only unknown primary variables; and the other unknowns – six stress components and six strain components – are eliminated from the governing equations. Consequently, the governing equations are reduced in number from fifteen to three coupled equations which can be solved to obtain the three unknown displacements. The displacement formulation was presented by Navier, and Lamé and the resulting equations called the Navier-Lamé displacement equations.

In stress formulation/method, the system of fifteen governing equations involving stresses, strains and Cartesian components of displacement as the unknowns are reformulated to eliminate strains and displacements, and have the six Cauchy stresses as the unknown primary variables. This offers the benefit of a reduction in the number of governing equations to six, for three dimensional (3D) elasticity problems. Stress methods were presented by Beltrami, Michell, Airy, Maxwell, Morera, and others. In the mixed method, the governing equations are re-written such that the unknown primary variables are some components of stress, and some components of displacement. The mixed method is not commonly found in the technical literature.

The stress based method was adopted in this work. The simplifications in the analytical and mathematical rigours offered by the reformulation of the general 3D elasticity problem have motivated research on the derivations of stress and displacement functions that apriori satisfy the governing system of partial differential equations of stress-based displacement methods [4, 5, 7, 9]. Such derived stress and displacement functions further simplify the solutions of elasticity problems to the determination of suitable stress and displacement functions that satisfy the boundary conditions of the particular problem considered. This thus reduces the dimensionality of the general problem of the three dimensional mathematical theory of elasticity.

Airy, Morera, Maxwell, Michell, Love, Boussinesq, Ike, Nowacki, and Egorov have derived stress-functions of the space coordinate variables that automatically satisfy the differential equations of equilibrium, and the strain compatibility equations. Elasticity problems of semi-infinite soil media under boundary loads have been studied using displacement-based and stress-based methodologies variously by Onah et al [14], Ike et al [15], Nwoji et al [16], Ike et al [17], Onah et al [18], Nwoji et al [19], Onah et al [20], and Ike [21]. Ike [22] used the Hankel transform method to obtain general solutions for axisymmetric elasticity problems of elastic half space. Ike [23] used the Hankel transform method to solve axisymmetric elasticity problem of cirucular foundation on semi-infinite elastic half space.

2. RESEARCH AIM AND OBJECTIVES

The research aim is to use the Fourier-Bessel transform method to determine the vertical stress fields in axisymmetric elasticity problems of the elastic half space involving circular foundation areas. The specific objectives are:

(i) to determine suitable bounded stress functions that satisfy the stress compatibility equation.
(ii) to apply the Fourier-Bessel transformation to the bounded stress functions in order to obtain the vertical stress field in the Fourier-Bessel transform space.
(iii) to enforce the boundary conditions for equilibrium of internal and external forces, and obtain the unknown functions in the Fourier-Bessel transformation for uniformly distributed load.
(iv) to apply the inversion formula and obtain the vertical stresses in the physical domain space variables for the case of uniformly distributed load on circular foundation.

3. STATEMENT OF RESEARCH PROBLEM

The problem considered is a linear elastic semi-infinite, homogeneous soil mass, idealized as an elastic half space with $-\infty \le x \le \infty$, $-\infty \le y \le \infty$, $0 \le z \le \infty$ where the x, y and z are the three dimensional Cartesian coordinates and the z-axis points downwards as shown in Figure 1.

An axisymmetrical load distribution $p(r)$ is applied on the xy plane over a circular foundation area of radius R where O is the center of the loaded area. Body forces are neglected and infinitesimal deformation assumptions of elasticity theory are used. The problem is to use the Fourier-Bessel transform method to determine the stress fields for typical cases of axisymmetrical loads.

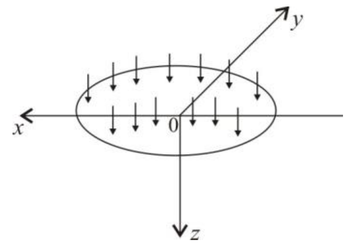

Figure 1. Uniformly distributed load on circular foundation on an elastic half space region and the 3D Cartesian coordinates

4. THEORETICAL FRAMEWORK / GOVERNING EQUATIONS

Axisymmetric elasticity problems of circular foundations of radius $r = R$ subject to axisymmetric load distribution $p(r)$ were formulated in terms of stress functions $\Omega(r, z)$ as the biharmonic problem on the semi-infinite space:

$$\nabla^4\Omega(r, z) = \nabla^2\nabla^2\Omega(r, z) = 0 \quad (1)$$
$$0 \le r \le \infty, \quad 0 \le z \le \infty$$

∇^2 is the Laplacian, while ∇^4 is the biharmonic operator, expressed in terms of the Laplacian as follows:

$$\nabla^4 = \nabla^2\nabla^2 = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\right)^2 \quad (2)$$

Egorov defined the axisymmetrical stress potential (harmonic) function $\Omega(r, z)$ in terms of the axisymmetrical cylindrical coordinates (r, z) as:

$$\sigma_{zz}(r, z) = \frac{\partial}{\partial z}\Omega(r, z) - z\frac{\partial^2}{\partial z^2}\Omega(r, z) \quad (3)$$

The research problem was solved subject to the boundary conditions:

$$\sigma_{zz}(r, z = 0) = -p \qquad 0 \le r \le R \quad (4)$$

$$\sigma_{zz}(r, z = 0) = 0 \qquad r > R \quad (5)$$

$$\tau_{rz}(r, z = 0) = 0 \qquad 0 \le r \le \infty \quad (6)$$

where r is the radial coordinate, z is the depth (vertical) coordinate, $\sigma_{zz}(r, z)$ is the vertical stress field at any arbitrary point with coordinates (r, z) in the elastic half space; τ_{rz} is the shear stress field at any arbitrary point in the elastic half space.

5. METHODOLOGY

The axisymmetrical elasticity problem of a circular foundation carrying axisymmetric load distribution on a semi-infinite elastic space then reduces to the problem of finding suitable bounded stress potential functions $\Omega(r,z)$ that satisfy the boundary conditions of equilibrium of internal and applied forces. We seek a stress function $\Omega(r,z)$ that satisfy the Laplace's equation in axisymmetric cylindrical coordinates. Thus, we seek $\Omega(r,z)$ such that

$$\nabla^2\Omega(r,z) = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\right)\Omega(r,z) = 0 \quad (7)$$

Let $\Omega(r,z)$ be assumed in variable-separable form as

$$\Omega(r,z) = f(r)g(z) \quad (8)$$

Then, the Equation (7) decomposes to ordinary differential equations (ODE) in $f(r)$ and $g(z)$ as follows:

$$r^2 f''(r) + rf'(r) + c_0 r^2 f(r) = 0 \quad (9)$$

$$\ddot{g}(z) - c_0 g(z) = 0 \quad (10)$$

where the primes denote derivatives with respect to r, and the dots denote derivatives with respect to z; and c_0 is the separation constant. The general solution for $c_0 = 1$ is

$$\Omega(r,z) = J_0(r)\left(a_1\exp(-z) + a_2\exp(z)\right) \quad (11)$$

where a_1 and a_2 are integration constants.

For $c_0 \neq 1$,

$$\Omega(r,z) = J_0(c_0 r)\left(c_1\exp(-c_0 z) + c_2\exp(c_0 z)\right) \quad (12)$$

where c_1 and c_2 are integration constants.

The bounded solution is

$$\Omega(r,z) = c_1 J_0(c_0 r)\exp(-c_0 z) \quad (13)$$

or,

$$\Omega(r,z) = a_1 J_0(r)\exp(-z) \quad \text{for } c_0 = 1, \quad (14)$$

where $J_0(r)$ is the Bessel function of the first kind of zero order.

By the linearity and superposition principles, the general bounded solution is

$$\Omega(r,z) = c_1 J_0(\alpha r)\exp(-\alpha z) \quad (15)$$

where α is a constant.

A suitable stress potential function $\Omega(r,z)$ can thus be defined as the Fourier-Bessel integral

$$\Omega(r,z) = \int_0^\infty C(\beta)J_0(\beta r)\exp(-\beta z)d\beta \quad (16)$$

where β is the Fourier-Bessel transform parameter, $C(\beta)$ is an unknown function, which we seek to determine such that $\Omega(r,z)$ would satisfy the particular problem of axisymmetric elasticity. In general $C(\beta)$ and this unknown function depends upon the Fourier-Bessel's transform parameters, β. $J_0(\beta r)$ is the Bessel's function of the first kind with order equal to zero.

The axisymmetric load distribution $p(r)$ on the circular foundation area of radius R on the xy coordinate ($z = 0$) plane can be expressed using the Fourier-Bessel transform as:

$$\bar{p}(\beta) = \int_0^\infty rp(r)J_0(\beta r)dr \quad (17)$$

For uniformly distributed load of intensity p_0 over the circular foundation area, $p(r) = \begin{cases} p_0 & \text{for } 0 < r < R \\ 0 & \text{for } r > R \end{cases}$,

$$\bar{p}(\beta) = \int_0^\infty rp_0 J_0(\beta r)dr = \left\{\int_0^R rp_0 J_0(\beta r)dr + \int_R^\infty r_0 J_0(\beta r)dr\right\} \quad (18)$$

$$\bar{p}(\beta) = p_0\int_0^R rJ_0(\beta r)dr = \frac{p_0 R}{\beta}J_1(\beta R) \quad (19)$$

By Fourier-Bessel transform inversion,

$$p(r) = \int_0^\infty \bar{p}(\beta)J_0(\beta r)\beta\, d\beta \quad (20)$$

$$p(r) = \int_0^\infty p_0 R J_1(\beta R)J_0(\beta r)d\beta \quad (21)$$

The vertical stress distribution on the xy ($z = 0$) plane is given by

$$\sigma_{zz}(r, z=0) = \frac{\partial}{\partial z}\Omega(r,z) \quad (22)$$

$$\sigma_{zz}(r, z=0) = \frac{\partial}{\partial z}\int_0^\infty C(\beta)J_0(\beta r)\exp(-\beta z)d\beta \quad (23)$$

$$\sigma_{zz}(r,0) = -\int_0^\infty \beta C(\beta)J_0(\beta r)d\beta \quad (24)$$

By the requirement of equilibrium of internal and external (applied) stresses, we have for the z coordinate direction,

$$\sigma_{zz}(r, z=0) + p(r) = 0 \quad (25)$$

$$\sigma_{zz}(r, z=0) = -p(r) \quad (26)$$

$$-\int_0^\infty \beta C(\beta)J_0(\beta r)d\beta = -\int_0^\infty p_0 R J_1(\beta R)J_0(\beta r)d\beta \quad (27)$$

Hence,

$$C(\beta) = \frac{p_0 R}{\beta}J_1(\beta R) \quad (28)$$

6. RESULTS

The biharmonic stress function $\Omega(r,z)$ is thus determined as:

$$\Omega(r,z) = \int_0^\infty p_0 R \frac{J_1(\beta R)}{\beta} J_0(\beta r)\exp(-\beta z)d\beta \quad (29)$$

$$\Omega(r,z) = p_0 R \int_0^\infty \frac{J_1(\beta R)}{\beta} J_0(\beta r)\exp(-\beta z)d\beta \quad (30)$$

6.1 Vertical stress fields

The vertical stress field is determined from the biharmonic stress function $\Omega(r,z)$ from Equation (6); where by differentiation,

$$\frac{\partial \Omega}{\partial z} = p_0 R \int_0^\infty -J_1(\beta R)J_0(\beta r)\exp(-\beta z)d\beta \quad (31)$$

$$\frac{\partial^2 \Omega}{\partial z^2} = p_0 R \int_0^\infty \beta J_1(\beta R)J_0(\beta r)\exp(-\beta z)d\beta \quad (32)$$

Thus, the vertical stress field is obtained as:

$$\sigma_{zz}(r,z) = -p_0 R \int_0^\infty J_1(\beta R)J_0(\beta r)\exp(-\beta z)d\beta + 2p_0 R \int_0^\infty \beta J_1(\beta R)J_0(\beta r)\exp(-\beta z)d\beta \quad (33)$$

Simplifying, the vertical stress field becomes:

$$\sigma_{zz} = -p_0 R \left\{ \int_0^\infty J_1(\beta R)J_0(\beta r)\exp(-\beta z)d\beta - \int_0^\infty J_1(\beta R)J_0(\beta r)\cdot \beta z \exp(-\beta z)d\beta \right\} \quad (34)$$

In general, $\sigma_{zz}(r,z)$ is given by:

$$\sigma_{zz}(r,z) = -p_0 R(I_1 + I_2) \quad (35)$$

where $I_1 = \int_0^\infty J_1(\beta R)J_0(\beta r)\exp(-\beta z)d\beta$ (36)

$$I_2 = \int_0^\infty J_1(\beta R)J_0(\beta r)\cdot \beta z \exp(-\beta z)d\beta \quad (37)$$

Solutions for the integration problem presented in Equations (34) and (35) as the products of Bessel functions and exponential functions were presented by Egorov. The solution to the integration problem gives the vertical stress field at any point in the semi-infinite elastic half space as:

$$\sigma_{zz}(r,z) = p_0 \left\{ G - \frac{n}{\pi\sqrt{n^2+(1+\alpha)^2}} \left[\frac{n^2-1+\alpha^2}{n^2+(1-\alpha)^2} E(k) + \frac{1-\alpha}{1+\alpha}\Pi_0(k,m) \right] \right\} \quad (38)$$

where $n = z/R$ (39)

$$\alpha = r/R \quad (40)$$

$$k^2 = \frac{4\alpha}{n^2+(1+\alpha)^2} \quad (41)$$

$$m = \frac{-4\alpha}{(1+\alpha)^2} \quad (42)$$

$$G = \begin{cases} 1 & r < R \quad (43) \\ \frac{1}{2} & r = R \quad (44) \\ 0 & r > R \quad (45) \end{cases}$$

$E(k)$ is the complete elliptic integral of the second kind with a modulus of k and parameter, m. $\Pi_0(k,m)$ is the complete elliptic integral of the third kind with a modulus of k and parameter, m.

In general, the vertical stress distribution at any arbitrary point (r, z) with respect to the origin of an axisymmetric cylindrical coordinate system in an elastic half space due to the uniformly distributed load of intensity p_0 applied over a circular area of radius R is given by

$$\sigma_{zz}(r,z) = p_0(I_{c_1} + I_{c_2}) \quad (46)$$

where I_{c_1} and I_{c_2} are functions of the dimensionless factors z/R and r/z.

For vertical normal stresses under the center of the circular foundation area, under uniformly distributed load of intensity p_0, $r = 0$, for points under the center, and

$$\sigma_{zz}(r=0,z) = p_0 \left\{ G - \frac{n}{\pi\sqrt{(1+n^2)}} \left[\frac{n^2-1}{n^2+1} E(0) + \Pi_0(0,0) \right] \right\} \quad (47)$$

$$\sigma_{zz}(r=0,z) = p_0 \left\{ 1 - \frac{n^3}{\sqrt{(1+n^2)^3}} \right\} \quad (48)$$

$$\sigma_{zz}(r=0,z) = p_0 \left\{ 1 - \left(\frac{n^2}{1+n^2} \right)^{3/2} \right\} \quad (49)$$

$$\sigma_{zz}(r=0,z) = p_0 \left\{ 1 - \left(\frac{1}{1/n^2 + 1} \right)^{3/2} \right\} \quad (50)$$

$$\sigma_{zz}(r=0,z) = p_0 \left\{ 1 - \left(1 + \frac{1}{n^2} \right)^{-3/2} \right\} \quad (51)$$

$$\sigma_{zz}(r=0,z) = p_0\left\{1-(1+n^{-2})^{-3/2}\right\} \quad (52)$$

$$\sigma_{zz}(r=0,z) = p_0\left\{1-(1+(R/z)^2)^{-3/2}\right\} \quad (53)$$

This result can also be obtained by substitution of $r = 0$ into Equation (34) to obtain

$$\sigma_{zz}(r=0,z) = -p_0 R\left\{\int_0^\infty J_1(\beta R)J_0(0)\exp(-\beta z)d\beta + \int_0^\infty J_1(\beta R)J_0(0)\cdot\beta z\exp(-\beta z)d\beta\right\} \quad (54)$$

$$\sigma_{zz}(r=0,z) = -p_0 R\int_0^\infty (1+\beta z)J_1(\beta R)\exp(-\beta z)d\beta \quad (55)$$

$$\sigma_{zz}(r=0,z) = -p_0\left\{-1+z^3(z^2+R^2)^{-3/2}\right\} \quad (56)$$

$$\sigma_{zz}(r=0,z) = p_0\left(1-\frac{z^3}{(z^2+R^2)^{3/2}}\right) \quad (57)$$

$$\sigma_{zz}(r=0,z) = p_0\left(1-\frac{(z^2)^{3/2}}{(z^2+R^2)^{3/2}}\right) = p_0\left\{1-\left(\frac{(z^2)}{(z^2+R^2)}\right)^{3/2}\right\} \quad (58)$$

$$\sigma_{zz}(r=0,z) = p_0\left(1-\left(\frac{1}{(1+(R/z)^2)}\right)^{3/2}\right) = p_0\left(1-\left(1+\left(R/z\right)^2\right)^{-3/2}\right) \quad (59)$$

$$\sigma_{zz}(r=0,z) = p_0\, I_c\left(R/z\right) \quad (60)$$

$$I_c\left(R/z\right) = \left(1-\left(1+\left(R/z\right)^2\right)^{-3/2}\right) \quad (61)$$

$I_c\left(\frac{R}{z}\right)$ is the dimensionless influence coefficient for finding vertical stress distribution at any depth z under the center of a circular foundation area of radius R carrying uniformly distributed load p_0.

7. DISCUSSION

In this work, the Fourier-Bessel transform method has been successfully used to solve for the vertical stress fields in a three dimensional axisymmetric problem of elasticity involving a circular foundation subject to uniformly distributed load applied on the xy Cartesian coordinate plane. The circular foundation was assumed to be on an elastic half space defined by $0 \le r \le \infty$, $0 \le z \le \infty$. The semi-infinite soil medium was assumed homogeneous, isotropic and linear elastic. Stress function approach was adopted to reduce the mathematically exerting problem of solving a system of fifteen partial differential equations to one of finding suitable stress functions of the cylindrical polar coordinates that satisfied the boundary conditions, as well as the bounded requirements of the problem. The method of separation of variables was used to determine the general solution for the biharmonic stress function governing the axisymmetric elasticity problem as the Fourier-Bessel integral given by Equation (16). The Fourier-Bessel integral was obtained in general for bounded solutions in terms of an unknown function $C(\beta)$ which was found to depend upon the Fourier-Bessel transform parameter β. The unknown function $C(\beta)$ was found by invoking the requirements of equilibrium of the vertical stresses and the applied load on the surface $z = 0$, for the case of uniformly distributed load as Equation (28). This yielded a complete determination of the bounded biharmonic stress function $\Omega(r,z)$ as Equation (30). Egorov expression given by Equation (3) was used to obtain the vertical stress field in general as Equation (34). Equation (34) was obtained in terms of integrals I_1 and I_2, which were defined as Equations (36) and (37). Evaluation of the integration problem gave the general solution for vertical stress at any point (r, z) in a semi-infinite linear elastic, isotropic, homogeneous soil as Equation (38). Equation (38) can be represented in terms of functions I_{c_1} and I_{c_2} which are found to be functions of dimensionless factors z/R and r/z. The functions $I_{c_1} I_{c_1}$ and I_{c_2} were evaluated and represented in tabular form as Tables 1, 2, 3, and 4. The problem of vertical stress determination under the center of uniformly loaded circular foundations was also considered as a simplification of the problem solved by putting $r = 0$ in the expression obtained – Equation 38 – to yield the vertical stress field under the center of uniformly loaded circular foundations as Equation (47), which simplified to Equation (53). Equation (53) was similarly found to be a function of the dimensionless influence factor $I_c(R/z)$ which depended upon the circular foundation radius R and the depth, z. I_c was found as Equation (61), and was calculated and tabulated as Table 5. It is observed that the solutions obtained agreed with solutions from the technical literature, and are identical with solutions obtained by Ike [23] using the Hankel transform method.

8. CONCLUSIONS

From the study, the following conclusions can be made:

(i) The Fourier-Bessel transform method is an effective mathematical technique for solving the axisymmetric three dimensional elasticity problem of circular foundation area subject to a uniformly distributed load where the foundation is on a linear elastic, isotropic homogeneous, semi-infinite soil media.

(ii) The stress function formulation of the axisymmetric problem simplified the 3D elasticity problem of solving a system of fifteen governing equations to one of finding suitable stress functions that satisfied the biharmonic stress compatibility equation as well as the demands of equilibrium of internal and external forces and the bounded requirements of the solution.

(iii) Mathematical expressions obtained as solutions for the vertical stress fields under the center of the circular foundation for the case of uniformly distributed loads were found to be symmetrical functions with respect to the vertical axis of symmetry ($r = 0$) of the problem – which is the vertical axis under the center of the circular foundation. This is in agreement with the symmetrical nature of the problem considered and the symmetrical nature of the applied uniformly distributed load about the vertical axis of symmetry ($r = 0$).

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APPENDIX

Vertical stress field due to uniformly distributed load over a circular area in soil idealised as elastic half space.

At any arbitrary point (r, z) in the elastic soil, the increase in the vertical stress (σ_z) at any point located at a depth z at any radial distance r from the center of the uniformly loaded area can be given in terms of the non-dimensional coefficients I_{c_1} and I_{c_2} as:

$$\sigma_z = p(I_{c_1} + I_{c_2})$$

where I_{c_1} and I_{c_2} are functions of z/R and r/R

Table 1. Vertical stress influence coefficients due to uniformly distributed load over a circular area of radius *R* in semi-infinite linear elastic (elastic half space) soil

Variation of I_{c_1} with *z*/*R* and *r*/*R*

z/*R* \ *r*/*R*	**0**	**0.2**	**0.4**	**0.6**	**0.8**	**1**	**1.2**	**1.5**	**2**
0	1.0	1.0	1.0	1.0	1.0	0.5	0	0	0
0.1	0.90050	0.89748	0.88679	0.86126	0.78797	0.43015	0.09645	0.02787	0.00856
0.2	0.80388	0.79824	0.77884	0.73483	0.63014	0.38269	0.15433	0.05251	0.01680
0.3	0.71265	0.70518	0.68316	0.62690	0.52081	0.34375	0.17964	0.07199	0.02440
0.4	0.62861	0.62015	0.59241	0.53767	0.44329	0.31048	0.18709	0.08593	0.03118
0.5	0.55279	0.54403	0.51622	0.46448	0.38390	0.28156	0.18556	0.09499	0.03701
0.6	0.48550	0.47691	0.45078	0.40427	0.33676	0.25588	0.17952	0.10010	
0.7	0.42654	0.41874	0.39491	0.35428	0.29833	0.21727	0.17124	0.10228	0.04558
0.8	0.37531	0.36832	0.34729	0.31243	0.26581	0.21297	0.16206	0.10236	
0.9	0.33104	0.32492	0.30669	0.27707	0.23832	0.19488	0.15253	0.10094	
1	0.29289	0.28763	0.27005	0.24697	0.21468	0.17868	0.14329	0.09849	0.05185
1.2	0.23178	0.22795	0.21662	0.19890	0.17626	0.15101	0.12570	0.09192	0.05260
1.5	0.16795	0.16552	0.15877	0.14804	0.13436	0.11892	0.10296	0.08048	0.05116
2	0.10557	0.10453	0.10140	0.09647	0.09011	0.08269	0.07471	0.06275	0.04496
2.5	0.07152	0.07008	0.06947	0.06698	0.06373	0.05974	0.05555	0.04880	0.03787
3	0.05132	0.05101	0.05022	0.04886	0.04707	0.04487	0.04241	0.03839	0.03150
4	0.02986	0.02976	0.02907	0.02802	0.02832	0.02749	0.02651	0.02490	0.02193
5	0.01942	0.01938				0.01835			0.01573
6	0.01361					0.01307			0.01168
7	0.01005					0.00976			0.00894
8	0.00772					0.00755			0.00703
9	0.00612					0.00600			0.00566
10								0.00477	0.00465

Table 2. Variation of I_{c_1} with *z*/*R* and *r*/*R* (*continued*)

z/*R* \ *r*/*R*	**3**	**4**	**5**	**6**	**7**	**8**	**10**	**12**	**14**
0	0	0	0	0	0	0	0	0	0
0.1	0.00211	0.00084	0.00042						
0.2	0.00419	0.00167	0.00063	0.00048	0.00030	0.00020			
0.3	0.00622	0.00250							
0.5	0.01013	0.00407	0.00209	0.00118	0.00071	0.00053	0.00025	0.00014	0.00009
1	0.01742	0.00761	0.00393	0.00236	0.00143	0.00097	0.00050	0.00029	0.00018
1.2	0.01935	0.00871	0.00459	0.00269	0.00171	0.00115			
1.5	0.02142	0.01013	0.00548	0.00325	0.00210	0.00141	0.00073	0.00043	0.00027
2	0.02221	0.01160	0.00659	0.00399	0.00264	0.00180	0.00094	0.00056	0.00036
2.5	0.02143	0.01221	0.00732	0.00463	0.00308	0.00214	0.00115	0.00068	0.00043
3	0.01980	0.01220	0.00770	0.00505	0.00346	0.00242	0.00132	0.00079	0.00051
4	0.01592	0.01109	0.00768	0.00536	0.00354	0.00282	0.00160	0.00099	0.00065
5	0.01249	0.00949	0.00708	0.00527	0.00394	0.00298	0.00179	0.00113	0.00075
6	0.00983	0.00795	0.00628	0.00492	0.00384	0.00299	0.00188	0.00124	0.00084
7	0.00784	0.00661	0.00548	0.00443	0.00360	0.00291	0.00193	0.00130	0.00091
8	0.00635	0.00554	0.00472	0.00398	0.00332	0.00276	0.00189	0.00134	0.00094
9	0.00520	0.00466	0.00409	0.00353	0.00301	0.00256	0.00184	0.00133	0.00096
10	0.00438	0.00397	0.00352	0.00326	0.00273	0.00241			

Table 3. Variation of I_{c_2} with *z*/*R* and *r*/*R*

z/*R* \ *r*/*R*	**0**	**0.2**	**0.4**	**0.6**	**0.8**	**1**	**1.2**	**1.5**	**2**
0	0	0	0	0	0	0	0	0	0
0.1	0.09852	0.10140	0.11138	0.13424	0.18796	0.05388	–0.07899	–0.02672	–0.00845
0.2	0.18857	0.19306	0.20272	0.23524	0.25983	0.08513	–0.07759	–0.04448	–0.01593
0.3	0.26362	0.26787	0.28018	0.29483	0.27257	0.10757	–0.04316	–0.04999	–0.02166
0.4	0.32016	0.32259	0.32748	0.32273	0.26925	0.12404	–0.00766	–0.04535	–0.02522
0.5	0.35777	0.35752	0.35323	0.33106	0.26236	0.13591	0.02165	–0.03455	–0.02651
0.6	0.37831	0.37531	0.36308	0.32822	0.25411	0.14440	0.04457	–0.02101	
0.7	0.38487	0.37962	0.36972	0.31029	0.24638	0.14986	0.06209	–0.00702	–0.02329
0.8	0.38091	0.37408	0.35133	0.30699	0.23779	0.15292	0.07530	0.00614	
0.9	0.36962	0.36275	0.33734	0.29299	0.22891	0.15404	0.08507	0.01795	
1	0.35355	0.34553	0.32075	0.27819	0.21978	0.15355	0.09210	0.02814	–0.01005

1.2	0.31485	0.30730	0.28481	0.24836	0.20113	0.14915	0.10002	0.04378	0.00023
1.5	0.25602	0.25025	0.23338	0.20694	0.17368	0.13732	0.10193	0.05745	0.01385
2	0.17889	0.18144	0.16644	0.15198	0.13375	0.11331	0.09254	0.06371	0.02836
2.5	0.12807	0.12633	0.12126	0.11327	0.10298	0.09130	0.07869	0.06022	0.03429
3	0.09487	0.09394	0.09099	0.08635	0.08033	0.07325	0.06551	0.05354	0.03511
4	0.05707	0.05666	0.05562	0.05383	0.05145	0.04773	0.04532	0.03995	0.03066
5	0.03772	0.03760				0.03384			0.02474
6	0.02666					0.02468			0.01968
7	0.01980					0.01868			0.01577
8	0.01526					0.01459			0.01279
9	0.01212					0.1170			0.01054
10								0.00924	0.00879

Table 4. Variation of I_{c_2} with z/R and r/R (*continued*)

z/R \ r/R	**3**	**4**	**5**	**6**	**7**	**8**	**10**	**12**	**14**
0	0	0	0	0	0	0	0	0	0
0.1	–0.00210	–0.00084	–0.00042						
0.2	–0.00412	–0.00166	–0.00083	–0.00024	–0.00015	–0.00010			
0.3	–0.00599	–0.00245							
0.5	–0.00991	–0.00388	–0.00199	–0.00116	–0.00073	–0.00049	–0.00025	–0.00014	–0.00009
1	–0.01115	–0.00608	–0.00344	–0.00210	–0.00135	–0.00092	–0.00048	–0.00028	–0.00018
1.2	–0.00995	–0.00632	–0.00378	–0.00236	–0.00156	–0.00107			
1.5	–0.00669	–0.00600	–0.00401	–0.00265	–0.00181	–0.00126	–0.00068	–0.00040	–0.00026
2	0.00028	–0.00410	–0.00371	–0.00278	–0.00202	–0.00148	–0.00084	–0.00050	–0.00033
2.5	0.00661	–0.00130	–0.00271	–0.00250	–0.00201	–0.00156	–0.00094	–0.00059	–0.00039
3	0.01112	0.00157	–0.00134	–0.00192	–0.00179	–0.00151	–0.00099	–0.00065	–0.00046
4	0.01515	0.00595	0.00155	–0.00029	–0.00094	–0.00109	–0.00094	–0.00068	–0.00050
5	0.01522	0.00810	0.00371	0.00132	0.00013	–0.00043	–0.00070	–0.00061	–0.00049
6	0.01380	0.00867	0.00496	0.00254	0.00110	0.00028	–0.00037	–0.00047	–0.00045
7	0.01204	0.00842	0.00547	0.00332	0.00185	0.00093	–0.00002	–0.00029	–0.00037
8	0.01034	0.00779	0.00554	0.00372	0.00236	0.00141	0.00035	–0.00008	–0.00025
9	0.00888	0.00705	0.00533	0.00386	0.00265	0.00178	0.00066	0.00012	–0.00012
10	0.00764	0.00631	0.00501	0.00382	0.00281	0.00199			

Table 5. Vertical normal stress distribution influence coefficient $I_c(R/z)$ under the center of uniformly loaded circular foundation area of radius R.

$$I_c\left(R/z\right) = 1 - \left(1 + \left(R/z\right)^2\right)^{-3/2}$$

R*/*z	***I*c**	***R*/*z***	***I*c**	***R*/*z***	***I*c**	***R*/*z***	***I*c**
0.00	0.00000	0.30	0.12126	0.60	0.36949	0.90	0.58934
0.01	0.00015	0.31	0.12859	0.61	0.37781	0.91	0.59542
0.02	0.00060	0.32	0.13605	0.62	0.38609	0.92	0.60142
0.03	0.00135	0.33	0.14363	0.63	0.39431	0.93	0.60734
0.04	0.00240	0.34	0.15133	0.64	0.40247	0.94	0.61317
0.05	0.00374	0.35	0.15915	0.65	0.41058	0.95	0.61892
0.06	0.00538	0.36	0.16706	0.66	0.41863	0.96	0.62459
0.07	0.00731	0.37	0.17507	0.67	0.42662	0.97	0.63018
0.08	0.00952	0.38	0.18317	0.68	0.43454	0.98	0.63568
0.09	0.01203	0.39	0.19134	0.69	0.44240	0.99	0.64110
0.10	0.01481	0.40	0.19959	0.70	0.45018	1.00	0.64645
0.11	0.01788	0.41	0.20790	0.71	0.45789	1.01	0.65171
0.12	0.02122	0.42	0.21627	0.72	0.46553	1.02	0.65690
0.13	0.02483	0.43	0.22469	0.73	0.47310	1.03	0.66200
0.14	0.02870	0.44	0.23315	0.74	0.48059	1.04	0.66703
0.15	0.03283	0.45	0.24165	0.75	0.48800	1.05	0.67198
0.16	0.03721	0.46	0.25017	0.76	0.49533	1.06	0.67686
0.17	0.04184	0.47	0.25872	0.77	0.50259	1.07	0.68166
0.18	0.04670	0.48	0.26729	0.78	0.50976	1.08	0.68639
0.19	0.05181	0.49	0.27587	0.79	0.51685	1.09	0.69104
0.20	0.05713	0.50	0.28446	0.80	0.52386	1.10	0.69562
0.21	0.06268	0.51	0.29304	0.81	0.53079	1.11	0.70013
0.22	0.06844	0.52	0.30162	0.82	0.53763	1.12	0.70457
0.23	0.07441	0.53	0.31019	0.83	0.54439	1.13	0.70894

0.24	0.08057	0.54	0.31875	0.84	0.55106	1.14	0.71342
0.25	0.08692	0.55	0.32728	0.85	0.55766	1.15	0.71747
0.26	0.09346	0.56	0.33579	0.86	0.56416	1.16	0.72163
0.27	0.10017	0.57	0.34427	0.87	0.57058	1.17	0.72573
0.28	0.10704	0.58	0.35272	0.88	0.57692	1.18	0.72976
0.29	0.11408	0.59	0.36112	0.89	0.58317	1.19	0.73373
1.20	0.73763	1.60	0.85112	2.00	0.91056	5.00	0.99246
1.21	0.74147	1.61	0.85312	2.02	0.91267	5.20	0.99327
1.22	0.74525	1.62	0.85507	2.04	0.91472	5.40	0.99396
1.23	0.74896	1.63	0.85700	2.06	0.91672	5.60	0.99457
1.24	0.75262	1.64	0.85890	2.08	0.91865	5.80	0.99510
1.25	0.75622	1.65	0.86077	2.10	0.92053		
1.26	0.75976	1.66	0.86260	2.15	0.92499	6.00	0.99556
1.27	0.76324	1.67	0.86441	2.20	0.92914	6.50	0.99648
1.28	0.76666	1.68	0.86619	2.25	0.93301		
1.29	0.77003	1.69	0.86794	2.30 2.35	0.93661 0.93997		
1.30	0.77334	1.70	0.86966	2.40	0.94310	7.00	0.99717
1.31	0.77660	1.71	0.87136	2.45	0.94603	7.50	0.99769
1.32	0.77981	1.72	0.87302	2.50	0.94877		
1.33	0.78296	1.73	0.87467	2.55	0.95134	8.00	0.99809
1.34	0.78606	1.74	0.87628	2.60	0.95374		
1.35	0.78911	1.75	0.87787	2.65	0.95599	9.00	0.99865
1.36	0.79211	1.76	0.87944	2.70	0.95810		
1.37	0.79507	1.77	0.88098	2.75	0.96009	10.00	0.99901
1.38	0.79797	1.78	0.88250	2.80	0.96195		
1.39	0.80083	1.79	0.88399	2.85	0.96371	12.00	0.99943
1.40	0.80364	1.80	0.88546	2.90	0.96536	14.00	0.99964
1.41	0.80640	1.81	0.88691	2.95	0.96691		
1.42	0.80912	1.82	0.88833			16.00	0.99976
1.43	0.81179	1.83	0.88974	3.00	0.96838		
1.44	0.81442	1.84	0.89112	3.10	0.97106	18.00	0.99983
1.45	0.81701	1.85	0.89248	3.20	0.97346		
1.46	0.81955	1.86	0.89382	3.30	0.97561	20.00	0.99988
1.47	0.82206	1.87	0.89514	3.40	0.97753		
1.48	0.82452	1.88	0.89643	3.50	0.97927	25.00	0.99994
1.49	0.82694	1.89	0.89771	3.60	0.98083	30.00	0.99996
1.50	0.82932	1.90	0.89897	3.70	0.98224		
1.51	0.83167	1.91	0.90021	3.80	0.98352	40.00	0.99998
1.52	0.83397	1.92	0.90143	3.90	0.98468		
1.53	0.83624	1.93	0.90263			50.00	0.99999
1.54	0.83847	1.94	0.90382	4.00	0.98573		
1.55	0.84067	1.95	0.90498	4.20	0.98757	100.00	1.00000
1.56	0.84283	1.96	0.90613	4.40	0.98911		
1.57	0.84495	1.97	0.90726	4.60	0.99041	∞	1.00000
1.58	0.84704	1.98	0.90838	4.80	0.99152		
1.59	0.84910	1.99	0.90948				

NOMENCLATURE

x, y, z	three dimensional Cartesian coordinates
3D	three dimensional
r, z	axisymmetrical cylindrical polar coordinate
r	radial coordinate (m)
R	radius of circular foundation (m)
$\Omega(r,z)$	axisymmetrical stress potential function in terms of the axisymmetrical cylindrical coordinates (r, z)
$\sigma_{zz}(r,z)$	vertical stress field (distribution) in axisymmetrical (cylindrical) polar coordinates (kN/m^2)
$\tau_{rz}(r, z)$	shear stress field (distribution) in cylindrical polar coordinates (kN/m^2)
$p(r)$	axisymmetrical load distribution (kN/m^2)
$f(r)$	function of r
$g(z)$	function of z
c_0	separation constant
$J_0(\beta r)$	Bessel function of the first kind with order equal to zero, and parameter, β
$J_0(r)$	Bessel function of the first kind of zero order
c_1	integration constant
$C(\beta)$	unknown function in the Fourier – Bessel integral
β	Fourier – Bessel transform parameter
$J_1(\beta R)$	Bessel function of the first kind with order equal to one and parameter, β
$E(k)$	complete elliptic integral of the second kind with a modulus of k
$\Pi_0(k,m)$	complete elliptic integral of the third kind with a modulus of k and parameter m
α	non-dimensional (dimensionless) radial coordinate
p_0	intensity of uniformly distributed load applied on the circular foundation area

I_{c_1} vertical stress influence coefficients in Tables 1 and 2

I_{c_2} vertical stress influence coefficients in Tables 3 and 4

G non-dimensional integration value (parameter) whose value depends on r

$I_c\left(\frac{R}{z}\right)$ dimensionless influence coefficient for finding vertical stress distribution of any depth z under the centre of a circular foundation area of radius R carrying uniformly distributed load, p_0

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}$$

∇^2 Laplace partial differential operator in axisymmetric cylindrical coordinates

∇^4 biharmonic (partial differential) operator in axisymmetric cylindrical coordinates

$$\nabla^4 = \nabla^2\nabla^2 = (\nabla^2)^2 = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\right)^2$$

MATHEMATICAL SYMBOLS

$\frac{\partial}{\partial z}$ partial derivative with respect to z

$\frac{\partial^2}{\partial z^2}$ second partial derivative with respect to z

$\int_0^\infty (\ )\,d\beta$ integration with respect to β between the limits 0 and ∞

$\exp(-\beta z)$ exponential function

$f'(r)$ first derivative of $f(r)$ with respect to r

$f''(r)$ second derivative of $f(r)$ with respect to r

$<$ less than

$>$ greater than

$\frac{\partial}{\partial r}$ partial derivative with respect to r

$\frac{\partial^2}{\partial r^2}$ second partial derivative with respect to r