Caputo-Fabrizio Fractional Derivative to Solve the Fractional Model of Energy Supply-Demand System

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ABSTRACT

The aim of this study, is to present the fractional model of energy supply-demand system (ES-DS) based on the Caputo-Fabrizio derivative. For the first time, the existence and uniqueness of solution of the fractional model of ES-DS are proved and it is the main novelty of this paper. Also, we know that the obtained results from mathematical models with fractional order are more accurate than usual models. This model is based on four important functions, energy resources demand (ERD) ε_1, energy resource supply (ERS) ε_2, energy resource import (ESI) ε_3 and renewable energy resources (RER) ε_4. Also, applying the obtained numerical results, we can forecast the rate of these functions for spacial interval of time.

1. INTRODUCTION

Mathematical models play an important role in the analysis, behavior measurement and future prediction of various phenomena. Recently, many mathematical models have been presented in various fields. Yuzbasi et al. [1, 2] applied the exponential Galerkin method and Bessel collocation method for solving model of HIV infection of CD4+T cells. Also, Noeiaghdam and Khoshrouye Ghiasi [3, 4] discussed the homotopy analysis transform method to approximate both CD4+T and CD8+T-cells and Ahmad Naik et al. [5-7] studied the fractional model of HIV infection. Mathematical models of computer viruses were illustrated by Ozturk et al. [8] using Chebyshev polynomials and its fractional form was studied by Singh et al. [9] applying fractional derivative. Moreover the non-linear SIR model of computer viruses was solved by homotopy analysis method, variational iteration method and Adomian decomposition method [10-12]. Guerrero et al. [13] applied the homotopy analysis method and Sikander et al. [14] used the variation of parameter method for solving model of smoking habit and the model of tuberculosis infection is illustrated by Atangana-Baleanu derivative [15] and variable-order fractional derivatives [16]. For more mathematical models you can see ref. [17-20].

Energy is one of applicable fields that plays an important role in the economy of countries and human life. Many criteria and factors can affect the production and supply of energy. Therefore, the security of energy demand is one of the important issues that has been considered in recent years because it has a fundamental role in energy development. Since many factors can affect the ES-DS, this model is a complex and unstable system. But given the importance of energy supply and demand, controlling this system has always been of strategic importance. Therefore, in order to develop

long-term, medium-term and short-term plans, accurate models are needed so that we can take control of the ES-DS with the necessary forecasts. Sun et al. [21] considered the time-delayed feedback control of the energy resource chaotic system. The energy resources demand-supply system and its dynamical analysis was illustrated by Sun and Tian [22]. Also, four-dimensional energy resources system was discussed with adaptive control and synchronization [23], with the model reference control [24] and with linear feedback control [25]. Moreover, supply-demand balancing renewable electricity using storage systems is one of the main challenges of contemporary power engineering. Sidorov et al. [26] considered application of integral equations for electric load leveling problems using storage systems.

Fractional calculus has real applications in different fields of engineering and sciences. Wang et al. [27] applied the fractional calculus for image denoising, Chen et al. [28] combined the application of blockchain technology in fractional calculus model of supply chain financial system, Zhang et al. [29] studied the fractional calculus based modeling of open circuit voltage of lithium-ion batteries for electric vehicles. Furthermore, solving fractional integrals and equations are among interesting problems. Angelis et al. [30] discussed the mean-value approach to solve fractional differential and integral equations, Odibat [31] presented the approximations of fractional integrals and Caputo fractional derivatives and Jahanshahi et al. [32] applied the fractional Gauss-Jacobi quadrature rule for approximating fractional integrals and derivatives. Also, Suleman [33] applied the Elzaki projected differential transform method for fractional order system of linear and nonlinear fractional partial differential equation.

The aim of this study is to present the fractional model of ES-DS using Caputo-Fabrizio fractional derivative. This

model is based on the ERD ε_1, ERS ε_2, ERI ε_3 and RER ε_4. Furthermore, many other parameters are included to present the mentioned model. The fixed-point theorem is used to prove the existence of solution. Also, uniqueness of solution of model is illustrated. Finally, the nonlinear fractional model of ES-DS is approximated and the numerical results are analysed.

2. PRELIMINARIES

In this section, definition of the Caputo-Fabrizio fractional derivative and its extended form are presented. Also non-integer order integral of Caputo type of the function with fractional order is defined. For more information see ref. [34, 35].

Definition 1 *[34] The Caputo-Fabrizio fractional derivative for $g \in H^1(a,b), b > a, \varphi \in [0,1]$ is defined as follows*

$$D_t^\varphi(g(t)) = \frac{\mathcal{W}(\varphi)}{1-\varphi} \int_a^t g'(x) \exp\left[-\varphi \frac{t-x}{1-\varphi}\right] dx, \quad (1)$$

where in order to normalize function $\mathcal{W}(\varphi)$ we have $\mathcal{W}(0) = \mathcal{W}(1) = 1$. For $g \notin H^1(a,b)$ the Caputo-Fabrizio fractional derivative can be defined in the following form

$$D_t^\varphi(g(t)) = \frac{\varphi \mathcal{W}(\varphi)}{1-\varphi} \int_a^t (g(t) - g(x)) \exp\left[-\varphi \frac{t-x}{1-\varphi}\right] dx. \quad (2)$$

We should note that for $\sigma = \frac{1-\varphi}{\varphi} \in [0,\infty], \varphi = \frac{1}{1+\sigma} \in [0,1]$, Eq. (2) can be written as

$$D_t^\varphi(g(t)) = \frac{N(\sigma)}{\sigma} \int_a^t g'(x) \exp\left[\frac{t-x}{\sigma}\right] dx, \quad (3)$$

where, $N(0) = N(\infty) = 1$ and $\lim_{\sigma \to 0} \frac{1}{\sigma} \exp\left[\frac{t-x}{\sigma}\right] = \delta(x - t)$.

Definition 2 *For function g, the φ order fractional integral for $0<\varphi<1$ can be defined as follows*

$$I_t^\varphi(g(t)) = \frac{2(1-\varphi)}{(2-\varphi)\mathcal{W}(\varphi)} g(t) + \frac{2\varphi}{(2-\varphi)\mathcal{W}(\varphi)} \int_0^t g(s) ds, t \geq 0. \quad (4)$$

We should remaind that the new form of φ order Caputo derivative suggested by Nieto and Losada in [35] in the following form

$$D_t^\varphi(g(t)) = \frac{1}{1-\varphi} \int_a^t g'(x) \exp\left[-\varphi \frac{t-x}{1-\varphi}\right] dx, \quad (5)$$

where, $0<\varphi<1$.

3. MODEL DESCRIPTION

For the first time, the 3-D ES-DS was presented by Sun et al. [21, 22]. In these studies, the stability of model and numerical results were discussed. Sun et al. [23-25] extended this model to the four dimensional nonlinear differential equations with unknown parameters as follows

$$\begin{cases} \mathcal{E}'_1 = a_1\mathcal{E}_1(1 - \mathcal{E}_1/\mathcal{W}) - a_2(\mathcal{E}_2 + \mathcal{E}_3) - d_3\mathcal{E}_4, \\ \mathcal{E}'_2 = -z_1\mathcal{E}_2 - z_2\mathcal{E}_3 + z_3\mathcal{E}_1[N - (\mathcal{E}_1 - \mathcal{E}_3)], \\ \mathcal{E}'_3 = s_1\mathcal{E}_3(s_2\mathcal{E}_1 - s_3), \\ \mathcal{E}'_4 = d_1\mathcal{E}_1 - d_2\mathcal{E}_4, \end{cases} \quad (6)$$

where, parameters $a_i, z_i, s_i, d_i, \mathcal{W}, N > 0$, are positive constants, and $N < \mathcal{W}$. The ERD of area A showed by ε_1, the ERS of area B to A showed by ε_2, the energy resource import in area A demonstrated by ε_3 and the renewable energy resources of area A displayed by ε_4. The elasticity factor of A ERD indicated by a_1. The energy supply factor of B expressed by a_2, which affects the energy demand of area A. The energy import coefficient of region A is displayed with a_2, which affects the demand for energy resources of region A. The maximum value of ERD of A showed by $\mathcal{W}$. N is the valve value. The coefficient of the energy supply of region B to A, the energy import of region A and the influence of ERD of A to the rate of energy resources supply of region B showed by z_1, z_2 and z_3 respectively. The velocity factor of energy import of A represented by s_1. The benefit of imported energy for per unit displayed by s_2 and the cost of imported energy exhibited by s_3. d_1 is the influence factor of ERD of region A to the rate of applying RER. d_2 is the influence factor of RER to the rate of applying RER. d_3 is the influence factor of the RER to the energy resources demand of region A. When we apply the following values for mentioned parameters, the system will be in the chaotic state [23, 24].

$$a_1 = 0.09; a_2 = 0.5;$$
$$z_1 = 0.06; z_2 = 0.082; z_3 = 0.07;$$
$$s_1 = 0.2; s_2 = 0.5; s_3 = 0.4;$$
$$d_1 = 0.1; d_2 = 0.06; d_3 = 0.07; \mathcal{W} = 1.8; N = 1.$$

We know that fractional models are so accurate and applicable than usual models. Thus in this study the nonlinear model of energy supply-demand system (6) is generalized to the fractional form as follows

$$\begin{cases} {}^{CF}_0D_t^\varphi\mathcal{E}'_1 = a_1\mathcal{E}_1(1 - \mathcal{E}_1/\mathcal{W}) - a_2(\mathcal{E}_2 + \mathcal{E}_3) - d_3\mathcal{E}_4, \\ {}^{CF}_0D_t^\varphi\mathcal{E}'_2 = -z_1\mathcal{E}_2 - z_2\mathcal{E}_3 + z_3\mathcal{E}_1[N - (\mathcal{E}_1 - \mathcal{E}_3)], \\ {}^{CF}_0D_t^\varphi\mathcal{E}'_3 = s_1\mathcal{E}_3(s_2\mathcal{E}_1 - s_3), \\ {}^{CF}_0D_t^\varphi\mathcal{E}'_4 = d_1\mathcal{E}_1 - d_2\mathcal{E}_4, \end{cases} \quad (7)$$

is illustrated where

$$\mathcal{E}_1(0) = \alpha_1, \mathcal{E}_2(0) = \alpha_2, \mathcal{E}_3(0) = \alpha_3, \mathcal{E}_4(0) = \alpha_4. \quad (8)$$

4. EXISTANCE AND UNIQUENESS OF FRACTIONAL ORDER ES-DS

The existence theorem is presented based on the fixed-point theory. Applying fractional operator for Eq. (7) we have

$$\begin{cases} \mathcal{E}_1(t) - \mathcal{E}_1(0) \\ \quad =_0^{CF} I_t^\varphi [a_1\mathcal{E}_1(1 - \mathcal{E}_1/\mathcal{W}) - a_2(\mathcal{E}_2 + \mathcal{E}_3) - d_3\mathcal{E}_4], \\ \mathcal{E}_2(t) - \mathcal{E}_2(0) \\ \quad =_0^{CF} I_t^\varphi [-z_1\mathcal{E}_2 - z_2\mathcal{E}_3 + z_3\mathcal{E}_1[N - (\mathcal{E}_1 - \mathcal{E}_3)], \\ \mathcal{E}_3(t) - \mathcal{E}_3(0) =_0^{CF} I_t^\varphi [s_1\mathcal{E}_3(s_2\mathcal{E}_1 - s_3)], \\ \mathcal{E}_4(t) - \mathcal{E}_4(0) =_0^{CF} I_t^\varphi [d_1\mathcal{E}_1 - d_2\mathcal{E}_4], \end{cases} \quad (9)$$

and using Definition 2 we can write

$$\mathcal{E}_1(t) - \mathcal{E}_1(0) = \frac{2(1-\varphi)}{(2-\varphi)\mathcal{W}(\varphi)}[a_1\mathcal{E}_1(t)(1 - \mathcal{E}_1(t)/\mathcal{W}) - a_2(\mathcal{E}_2(t) + \mathcal{E}_3(t)) - d_3\mathcal{E}_4(t)]$$

$$+ \frac{2\varphi}{(2-\varphi)\mathcal{W}(\varphi)}\int_0^t [a_1\mathcal{E}_1(y)(1 - \mathcal{E}_1(y)/\mathcal{W}) - a_2(\mathcal{E}_2(y) + \mathcal{E}_3(y)) - d_3\mathcal{E}_4(y)]dy,$$

$$\mathcal{E}_2(t) - \mathcal{E}_2(0) = \frac{2(1-\varphi)}{(2-\varphi)\mathcal{W}(\varphi)}[-z_1\mathcal{E}_2(t) - z_2\mathcal{E}_3(t) + z_3\mathcal{E}_1(t)[N - (\mathcal{E}_1(t) - \mathcal{E}_3(t))]]$$

$$+ \frac{2\varphi}{(2-\varphi)\mathcal{W}(\varphi)}\int_0^t [-z_1\mathcal{E}_2(y) - z_2\mathcal{E}_3(y) + z_3\mathcal{E}_1(y)[N - (\mathcal{E}_1(y) - \mathcal{E}_3(y))]]dy, \quad (10)$$

$$\mathcal{E}_3(t) - \mathcal{E}_3(0) = \frac{2(1-\varphi)}{(2-\varphi)\mathcal{W}(\varphi)}[s_1\mathcal{E}_3(t)(s_2\mathcal{E}_1(t) - s_3)]$$

$$+ \frac{2\varphi}{(2-\varphi)\mathcal{W}(\varphi)}\int_0^t [s_1\mathcal{E}_3(y)(s_2\mathcal{E}_1(y) - s_3)]dy,$$

$$\mathcal{E}_4(t) - \mathcal{E}_4(0) = \frac{2(1-\varphi)}{(2-\varphi)\mathcal{W}(\varphi)}[d_1\mathcal{E}_1(t) - d_2\mathcal{E}_4(t)] + \frac{2\varphi}{(2-\varphi)\mathcal{W}(\varphi)}\int_0^t [d_1\mathcal{E}_1(y) - d_2\mathcal{E}_4(y)]dy.$$

In order to apply the brief form of relations we denote as

$$k_1(t, \mathcal{E}_1) = a_1\mathcal{E}_1(t)(1 - \mathcal{E}_1(t)/\mathcal{W}) - a_2(\mathcal{E}_2(t) + \mathcal{E}_3(t)) - d_3\mathcal{E}_4(t),$$

$$k_2(t, \mathcal{E}_1) = -z_1\mathcal{E}_2(t) - z_2\mathcal{E}_3(t) + z_3\mathcal{E}_1(t)[N - (\mathcal{E}_1(t) - \mathcal{E}_3(t)),$$

$$k_3(t, \mathcal{E}_1) = s_1\mathcal{E}_3(t)(s_2\mathcal{E}_1(t) - s_3), \quad (11)$$

$$k_4(t, \mathcal{E}_1) = d_1\mathcal{E}_1(t) - d_2\mathcal{E}_4(t).$$

Theorem 1 *The Lipchitz condition is satisfied for kernels* k_1, k_2, k_3 *and* k_4 *if*

$$0 \le a_1, z_1, s_1, s_2, s_3, d_2 < 1.$$

Proof: Applying Eq. (11) for function $k_1(t, \varepsilon_1)$ we can write

$$\| k_1(t, \mathcal{E}_1) - k_1(t, \mathcal{E}_1^*) \|$$

$$= \| a_1(\mathcal{E}_1(t) - \mathcal{E}_1^*(t)) $$

$$- \frac{a_1}{\mathcal{W}}(\mathcal{E}_1(t) - \mathcal{E}_1^*(t))^2 \|, \quad (12)$$

and using triangular inequality for Eq. (12) we get

$$\| k_1(t, \mathcal{E}_1) - k_1(t, \mathcal{E}_1^*) \|$$

$$\le \| a_1(\mathcal{E}_1(t) - \mathcal{E}_1^*(t)) \|$$

$$+ \| \frac{a_1}{\mathcal{W}}(\mathcal{E}_1(t) - \mathcal{E}_1^*(t))^2 \| \quad (13)$$

$$\le a_1 \| (\mathcal{E}_1(t) - \mathcal{E}_1^*(t)) \|$$

$$+ \frac{a_1}{\mathcal{W}} \| \mathcal{E}_1(t) - \mathcal{E}_1^*(t) \|^2.$$

We know $0 \le a_1 \le 1$ and $\mathcal{W}$ is large value, greater than 1 so the second term will be so small and we can write

$$\| k_1(t, \mathcal{E}_1) - k_1(t, \mathcal{E}_1^*) \| \le a_1 \| (\mathcal{E}_1(t) - \mathcal{E}_1^*(t)) \|.$$

Similarly, for $k_2(t, \varepsilon_2)$ and $k_3 (t, \varepsilon_3)$ we have

$$\| k_2(t, \mathcal{E}_2) - k_2(t, \mathcal{E}_2^*) \|$$
$$=\| -z_1(\mathcal{E}_2(t) - \mathcal{E}_2^*(t)) \|$$
$$\leq z_1 \| (\mathcal{E}_2(t) - \mathcal{E}_2^*(t)) \|,$$

and

$$\| k_3(t, \mathcal{E}_3) - k_3(t, \mathcal{E}_3^*) \|$$
$$=\| s_1(\mathcal{E}_3(t) - \mathcal{E}_3^*(t))(s_2\mathcal{E}_1(t) - s_3) \|$$
$$\leq s_1 \| \mathcal{E}_3(t) - \mathcal{E}_3^*(t) \| (s_2 \| \mathcal{E}_1(t) \| - s_3)$$
$$\leq s_1(s_2 a - s_3) \| \mathcal{E}_3(t) - \mathcal{E}_3^*(t) \|,$$

where, $\|\varepsilon_1(t)\| \leq a$. And finally for $k_4(t, \varepsilon_4)$ we get

$$\| k_4(t, \mathcal{E}_4) - k_4(t, \mathcal{E}_4^*) \|$$
$$=\| d_2(\mathcal{E}_4(t) - \mathcal{E}_4^*(t)) \|$$
$$\leq d_2 \| (\mathcal{E}_4(t) - \mathcal{E}_4^*(t)) \|.$$

Thus, these relations show that kernels k_1, k_2, k_3 and k_4 satisfy in Lipschitz condition if

$$0 \leq a_1, z_1, s_1, s_2, s_3, d_2 < 1,$$

then they will be contraction. Based on Eq. (10) we can write

$$\mathcal{E}_1(t) = \mathcal{E}_1(0) + \frac{2(1-\varphi)}{(2-\varphi)\mathcal{W}(\varphi)} k_1(t, \mathcal{E}_1) + \frac{2\varphi}{(2-\varphi)\mathcal{W}(\varphi)} \int_0^t k_1(y, \mathcal{E}_1) dy,$$

$$\mathcal{E}_2(t) = \mathcal{E}_2(0) + \frac{2(1-\varphi)}{(2-\varphi)\mathcal{W}(\varphi)} k_2(t, \mathcal{E}_2) + \frac{2\varphi}{(2-\varphi)\mathcal{W}(\varphi)} \int_0^t k_2(y, \mathcal{E}_2) dy,$$

$$\mathcal{E}_3(t) = \mathcal{E}_3(0) + \frac{2(1-\varphi)}{(2-\varphi)\mathcal{W}(\varphi)} k_3(t, \mathcal{E}_3) + \frac{2\varphi}{(2-\varphi)\mathcal{W}(\varphi)} \int_0^t k_3(y, \mathcal{E}_3) dy,$$

$$\mathcal{E}_4(t) = \mathcal{E}_4(0) + \frac{2(1-\varphi)}{(2-\varphi)\mathcal{W}(\varphi)} k_4(t, \mathcal{E}_4) + \frac{2\varphi}{(2-\varphi)\mathcal{W}(\varphi)} \int_0^t k_4(y, \mathcal{E}_4) dy,$$

(14)

and the following successive formulas can be obtained

$$\mathcal{E}_{1,n}(t) = \frac{2(1-\varphi)}{(2-\varphi)\mathcal{W}(\varphi)} k_1(t, \mathcal{E}_{1,n-1}) + \frac{2\varphi}{(2-\varphi)\mathcal{W}(\varphi)} \int_0^t k_1(y, \mathcal{E}_{1,n-1}) dy,$$

$$\mathcal{E}_{2,n}(t) = \frac{2(1-\varphi)}{(2-\varphi)\mathcal{W}(\varphi)} k_2(t, \mathcal{E}_{2,n-1}) + \frac{2\varphi}{(2-\varphi)\mathcal{W}(\varphi)} \int_0^t k_2(y, \mathcal{E}_{2,n-1}) dy,$$

$$\mathcal{E}_{3,n}(t) = \frac{2(1-\varphi)}{(2-\varphi)\mathcal{W}(\varphi)} k_3(t, \mathcal{E}_{3,n-1}) + \frac{2\varphi}{(2-\varphi)\mathcal{W}(\varphi)} \int_0^t k_3(y, \mathcal{E}_{3,n-1}) dy,$$

$$\mathcal{E}_{4,n}(t) = \frac{2(1-\varphi)}{(2-\varphi)\mathcal{W}(\varphi)} k_4(t, \mathcal{E}_{4,n-1}) + \frac{2\varphi}{(2-\varphi)\mathcal{W}(\varphi)} \int_0^t k_4(y, \mathcal{E}_{4,n-1}) dy,$$

with initial conditions

$$\mathcal{E}_{1,0}(t) = \mathcal{E}_1(0),$$
$$\mathcal{E}_{2,0}(t) = \mathcal{E}_2(0),$$
$$\mathcal{E}_{3,0}(t) = \mathcal{E}_3(0),$$
$$\mathcal{E}_{4,0}(t) = \mathcal{E}_4(0).$$

Also, the difference between two successive terms is obtained in the following forms

$$\phi_n(t) = \mathcal{E}_{1,n}(t) - \mathcal{E}_{1,n-1}(t) = \frac{2(1-\varphi)}{(2-\varphi)\mathcal{W}(\varphi)} (k_1(t, \mathcal{E}_{1,n-1}) - k_1(t, \mathcal{E}_{1,n-2})) + \frac{2\varphi}{(2-\varphi)\mathcal{W}(\varphi)} \int_0^t (k_1(y, \mathcal{E}_{1,n-1}) - k_1(y, \mathcal{E}_{1,n-2})) dy,$$

$$\psi_n(t) = \mathcal{E}_{2,n}(t) - \mathcal{E}_{2,n-1}(t) = \frac{2(1-\varphi)}{(2-\varphi)\mathcal{W}(\varphi)} (k_2(t, \mathcal{E}_{2,n-1}) - k_2(t, \mathcal{E}_{2,n-2})) + \frac{2\varphi}{(2-\varphi)\mathcal{W}(\varphi)} \int_0^t (k_2(y, \mathcal{E}_{2,n-1}) - k_2(y, \mathcal{E}_{2,n-2})) dy,$$

$$\xi_n(t) = \mathcal{E}_{3,n}(t) - \mathcal{E}_{3,n-1}(t) = \frac{2(1-\varphi)}{(2-\varphi)\mathcal{W}(\varphi)} (k_3(t, \mathcal{E}_{3,n-1}) - k_3(t, \mathcal{E}_{3,n-2})) + \frac{2\varphi}{(2-\varphi)\mathcal{W}(\varphi)} \int_0^t (k_3(y, \mathcal{E}_{3,n-1}) - k_3(y, \mathcal{E}_{3,n-2})) dy,$$

$$\chi_n(t) = \mathcal{E}_{4,n}(t) - \mathcal{E}_{4,n-1}(t) = \frac{2(1-\varphi)}{(2-\varphi)\mathcal{W}(\varphi)} (k_4(t, \mathcal{E}_{4,n-1}) - k_4(t, \mathcal{E}_{4,n-2})) + \frac{2\varphi}{(2-\varphi)\mathcal{W}(\varphi)} \int_0^t (k_4(y, \mathcal{E}_{4,n-1}) - k_4(y, \mathcal{E}_{4,n-2})) dy,$$

and for n-th terms we can write

$$\mathcal{E}_{1,n}(t) = \sum_{i=0}^n \phi_i(t), \mathcal{E}_{2,n}(t) = \sum_{i=0}^n \psi_i(t),$$

$$\mathcal{E}_{3,n}(t) = \sum_{i=0}^n \xi_i(t), \mathcal{E}_{4,n}(t) = \sum_{i=0}^n \chi_i(t).$$

For $\phi_n(t)$ we have

$$\| \phi_n(t) \| = \| \mathcal{E}_{1,n}(t) - \mathcal{E}_{1,n-1}(t) \| = \left\| \frac{2(1-\varphi)}{(2-\varphi)\mathcal{W}(\varphi)} (k_1(t, \mathcal{E}_{1,n-1}) - k_1(t, \mathcal{E}_{1,n-2})) + \frac{2\varphi}{(2-\varphi)\mathcal{W}(\varphi)} \int_0^t (k_1(y, \mathcal{E}_{1,n-1}) - k_1(y, \mathcal{E}_{1,n-2})) dy \right\|,$$

and using triangular inequality for above equation the following relation can be obtained

$$\| \mathcal{E}_{1,n}(t) - \mathcal{E}_{1,n-1}(t) \| \leq \frac{2(1-\varphi)}{(2-\varphi)\mathcal{W}(\varphi)} \|(k_1(t, \mathcal{E}_{1,n-1}) - k_1(t, \mathcal{E}_{1,n-2}))\|$$

$$+ \frac{2\varphi}{(2-\varphi)\mathcal{W}(\varphi)} \int_0^t \|(k_1(y, \mathcal{E}_{1,n-1}) - k_1(y, \mathcal{E}_{1,n-2}))\| dy.$$

Since kernel coincides to the Lipchitz condition then we can write

$$\| \mathcal{E}_{1,n}(t) - \mathcal{E}_{1,n-1}(t) \| \leq \frac{2(1-\varphi)}{(2-\varphi)\mathcal{W}(\varphi)} a_1 \|\mathcal{E}_{1,n-1} - \mathcal{E}_{1,n-2}\|$$

$$+ \frac{2\varphi}{(2-\varphi)\mathcal{W}(\varphi)} a_1 \int_0^t \|\mathcal{E}_{1,n-1} - \mathcal{E}_{1,n-2}\| dy,$$

and we get

$$\| \phi_n(t) \| \leq \frac{2(1-\varphi)}{(2-\varphi)\mathcal{W}(\varphi)} a_1 \|\phi_{n-1}(t)\| + \frac{2\varphi}{(2-\varphi)\mathcal{W}(\varphi)} a_1 \int_0^t \|\phi_{n-1}(y)\| dy. \tag{15}$$

Repeating this process to $\mathcal{E}_{2,n}(t), \mathcal{E}_{3,n}(t)$ and $\mathcal{E}_{4,n}(t)$ we have

$$\| \psi_n(t) \| \leq \frac{2(1-\varphi)}{(2-\varphi)\mathcal{W}(\varphi)} z_1 \|\psi_{n-1}(t)\| + \frac{2\varphi}{(2-\varphi)\mathcal{W}(\varphi)} z_1 \int_0^t \|\psi_{n-1}(y)\| dy,$$

$$\| \xi_n(t) \| \leq \frac{2(1-\varphi)}{(2-\varphi)\mathcal{W}(\varphi)} s_1(s_2 a - s_3) \|\xi_{n-1}(t)\| + \frac{2\varphi}{(2-\varphi)\mathcal{W}(\varphi)} s_1(s_2 a - s_3) \int_0^t \|\xi_{n-1}(y)\| dy, \tag{16}$$

$$\| \chi_n(t) \| \leq \frac{2(1-\varphi)}{(2-\varphi)\mathcal{W}(\varphi)} d_2 \|\chi_{n-1}(t)\| + \frac{2\varphi}{(2-\varphi)\mathcal{W}(\varphi)} d_2 \int_0^t \|\chi_{n-1}(y)\| dy.$$

So Theorem 1 is proved.

Theorem 2 *The nonlinear fractional energy supply-demand system (7) has exact coupled solutions that we can find t_0 such that*

$$\frac{2(1-\varphi)}{(2-\varphi)\mathcal{W}(\varphi)} a_1 + \frac{2\varphi}{(2-\varphi)\mathcal{W}(\varphi)} a_2 t_0 < 1.$$

Proof: We know that functions $\mathcal{E}_1(t), \mathcal{E}_2(t), \mathcal{E}_3(t)$ and $\mathcal{E}_4(t)$ are bounded and Lipchitz condition is connected for Eqs. (15) and (16). Thus we need to show

$$\| \phi_n(t) \| \leq \| \mathcal{E}_{1,n}(0) \| \left[\frac{2(1-\varphi)}{(2-\varphi)\mathcal{W}(\varphi)} a_1 + \frac{2\varphi}{(2-\varphi)\mathcal{W}(\varphi)} a_1 t\right]^n,$$

$$\| \psi_n(t) \| \leq \| \mathcal{E}_{2,n}(0) \| \left[\frac{2(1-\varphi)}{(2-\varphi)\mathcal{W}(\varphi)} z_1 + \frac{2\varphi}{(2-\varphi)\mathcal{W}(\varphi)} z_1 t\right]^n,$$

$$\| \xi_n(t) \| \leq \| \mathcal{E}_{3,n}(0) \| \left[\frac{2(1-\varphi)}{(2-\varphi)\mathcal{W}(\varphi)} s_1(s_2 a - s_3) + \frac{2\varphi}{(2-\varphi)\mathcal{W}(\varphi)} s_1(s_2 a - s_3)t\right]^n, \tag{17}$$

$$\| \psi_n(t) \| \leq \| \mathcal{E}_{4,n}(0) \| \left[\frac{2(1-\varphi)}{(2-\varphi)\mathcal{W}(\varphi)} d_2 + \frac{2\varphi}{(2-\varphi)\mathcal{W}(\varphi)} d_2 t\right]^n.$$

At first, we assume that

$$\mathcal{E}_1(t) - \mathcal{E}_1(0) = \mathcal{E}_{1,n}(t) - B_n(t),$$

$$\mathcal{E}_2(t) - \mathcal{E}_2(0) = \mathcal{E}_{2,n}(t) - C_n(t),$$

$$\mathcal{E}_3(t) - \mathcal{E}_3(0) = \mathcal{E}_{3,n}(t) - D_n(t),$$

$$\mathcal{E}_4(t) - \mathcal{E}_4(0) = \mathcal{E}_{4,n}(t) - E_n(t).$$

So we can write

$$\| B_n(t) \| = \left\| \frac{2(1-\varphi)}{(2-\varphi)\mathcal{W}(\varphi)} (k(t,\mathcal{E}_1) - k(t,\mathcal{E}_{1,n-1})) + \frac{2\varphi}{(2-\varphi)\mathcal{W}(\varphi)} \int_0^t (k(y,\mathcal{E}_1) - k(y,\mathcal{E}_{1,n-1}))dy \right\|$$

$$\leq \frac{2(1-\varphi)}{(2-\varphi)\mathcal{W}(\varphi)} \|k(t,\mathcal{E}_1) - k(t,\mathcal{E}_{1,n-1})\| + \frac{2\varphi}{(2-\varphi)\mathcal{W}(\varphi)} \int_0^t \|k(y,\mathcal{E}_1) - k(y,\mathcal{E}_{1,n-1})\|dy$$

$$\leq \frac{2(1-\varphi)}{(2-\varphi)\mathcal{W}(\varphi)} a_1 \|\mathcal{E}_1 - \mathcal{E}_{1,n-1}\| + \frac{2\varphi}{(2-\varphi)\mathcal{W}(\varphi)} a_1 \|\mathcal{E}_1 - \mathcal{E}_{1,n-1}\|t.$$

Using this process it obtains

$$\| B_n(t) \| \leq \left(\frac{2(1-\varphi)}{(2-\varphi)\mathcal{W}(\varphi)} + \frac{2\varphi}{(2-\varphi)\mathcal{W}(\varphi)} t \right)^{n+1} a_1^{n+1} a$$

such that at point t_0 we get

$$\| B_n(t) \| \leq \left(\frac{2(1-\varphi)}{(2-\varphi)\mathcal{W}(\varphi)} + \frac{2\varphi}{(2-\varphi)\mathcal{W}(\varphi)} t_0 \right)^{n+1} a_1^{n+1} a$$

taking limit for above equation when $n \to \infty$ then $\| B_n(t) \| \to 0$.

By similar way we get

$$\| C_n(t) \| \to 0, \| D_n(t) \| \to 0, \| E_n(t) \| \to 0.$$

Thus proof of existence is completed.

Now the uniqueness of a system of solutions of Eq. (7) should be studied. Let us there is another system od solutions for Eq. (7) then

$$\mathcal{E}_1(t) - \mathcal{E}_{1,1}(t) = \frac{2(1-\varphi)}{(2-\varphi)\mathcal{W}(\varphi)} (k_1(t,\mathcal{E}_1) - k_1(t,\mathcal{E}_{1,1}))$$
$$+ \frac{2\varphi}{(2-\varphi)\mathcal{W}(\varphi)} \int_0^t (k_1(y,\mathcal{E}_1) - k_1(y,\mathcal{E}_{1,1}))dy,$$

and applying norm we have

$$\| \mathcal{E}_1(t) - \mathcal{E}_{1,1}(t) \| = \frac{2(1-\varphi)}{(2-\varphi)\mathcal{W}(\varphi)} \| k_1(t,\mathcal{E}_1) - k_1(t,\mathcal{E}_{1,1})$$
$$\| + \frac{2\varphi}{(2-\varphi)\mathcal{W}(\varphi)} \int_0^t$$
$$\| k_1(y,\mathcal{E}_1) - k_1(y,\mathcal{E}_{1,1}) \| dy.$$

Then using Lipschitz condition we have

$$\| \mathcal{E}_1(t) - \mathcal{E}_{1,1}(t) \| \leq \frac{2(1-\varphi)}{(2-\varphi)\mathcal{W}(\varphi)} a_1 \| \mathcal{E}_1 - \mathcal{E}_{1,1} \| + \frac{2\varphi}{(2-\varphi)\mathcal{W}(\varphi)} a_1 t$$
$$\| \mathcal{E}_1 - \mathcal{E}_{1,1} \|,$$

and it leads to

$$\| \mathcal{E}_1(t) - \mathcal{E}_{1,1}(t) \| \left(1 - \frac{2(1-\varphi)}{(2-\varphi)\mathcal{W}(\varphi)} a_1 - \frac{2\varphi}{(2-\varphi)\mathcal{W}(\varphi)} a_1 t \right)$$
$$\leq 0.$$

Theorem 3 *The system of equations (7) has a unique solution if*

$$\left(1 - \frac{2(1-\varphi)}{(2-\varphi)\mathcal{W}(\varphi)} a_1 - \frac{2\varphi}{(2-\varphi)\mathcal{W}(\varphi)} a_1 t \right) > 0. \qquad (18)$$

Proof: Assume that Eq. (18) holds then

$$\| \mathcal{E}_1(t) - \mathcal{E}_{1,1}(t) \| \left(1 - \frac{2(1-\varphi)}{(2-\varphi)\mathcal{W}(\varphi)} a_1 - \frac{2\varphi}{(2-\varphi)\mathcal{W}(\varphi)} a_1 t \right)$$
$$\leq 0$$

implies that $\| \mathcal{E}_1(t) - \mathcal{E}_{1,1}(t) \| = 0$. Then we get $\mathcal{E}_1(t) = \mathcal{E}_{1,1}(t)$. For other cases we can write

$$\mathcal{E}_2(t) = \mathcal{E}_{2,1}(t),$$
$$\mathcal{E}_3(t) = \mathcal{E}_{31,1}(t),$$
$$\mathcal{E}_4(t) = \mathcal{E}_{4,1}(t).$$

Thus, the uniqueness of the system of equation (7) is verified.

5. NUMERICAL DISCUSSION

In this section, the numerical solution of nonlinear fractional model 7 is obtained for $n=5$ and $\varphi=1$ based on the mentioned values in Section 3 as follows

$$\mathcal{E}_1(t) = 1 - 0.33t - 0.01235t^2 - 0.001222t^3 + \cdots - 3.862757^{-37}t^{30} - 5.493722^{-39}t^{31},$$

$$\mathcal{E}_2(t) = 1 - 0.072t - 0.01021t^2 - 0.0027571t^3 + \cdots - 2.938463^{-37}t^{30} - 3.962139^{-39}t^{31},$$

$$\mathcal{E}_3(t) = 1 + 0.0199t - 0.0008t^2 - 0.0001986t^3 + \cdots + 3.5277104^{-37}t^{30} + 5.327245^{-39}t^{31},$$

$$\mathcal{E}_4(t) = 1 + 0.04t - 0.0177t^2 - 0.0000576666t^3 + \cdots - 4.1859907^{-19}t^{15} - 1.1534777^{-20}t^{16}.$$
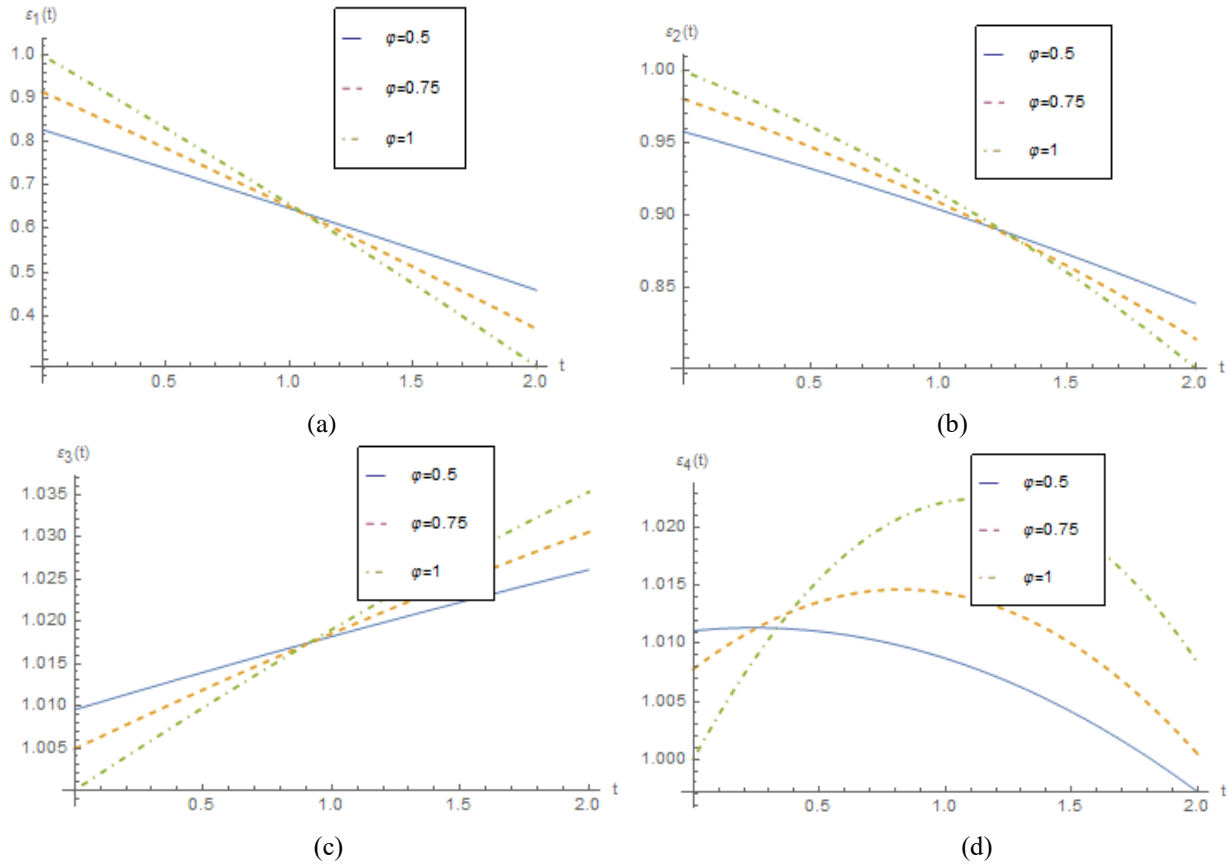
Figure 1. The response of the solutions (a) $\varepsilon_1(t)$, (b) $\varepsilon_2(t)$, (c) $\varepsilon_3(t)$ and (d) $\varepsilon_4(t)$ with respect to t for φ=0.5, 0.75, 1

Also, the approximate solutions for φ=0.5, 0.75, 1 are presented in Figure 1 at time [0, 2]. As mentioned before, ε_1 is ERD of area A, ε_2 is ERS of area B to A, ε_3 is the ERI to area A and finally ε_4 is the RER of area A. From Figure 1(a), we can observe that ERD for φ=0.5 it is decreased with time t and for φ=0.75 and 1 it is more decreased. In (b), ERS is almost in same condition with ERD. Based on (c), ERI to area A is increasing for φ=0.5 and it is more increasing for φ=0.75 and 1 and finally from (d) RER of area A for φ=0.5 is decreasing but for φ=0.75 and 1 and $0\leq t\leq1$ is increasing and for $1\leq t\leq2$ is decreasing.

Figure 2, shows the approximate solutions of $\varepsilon_1(t)$, $\varepsilon_2(t)$, $\varepsilon_3(t)$ and $\varepsilon_4(t)$ for φ=0.5, 0.75, 1. From Figure 2, we can see that when ERD of area A is decreasing and ERS of area B to A is decreasing slowly then the ERI to area A is decreasing very slowly and the RER of area A is fix. The residual errors of $\varepsilon_1(t)$, $\varepsilon_2(t)$, $\varepsilon_3(t)$ and $\varepsilon_4(t)$ for φ=0.5, 0.75, 1 are presented in Table 1.
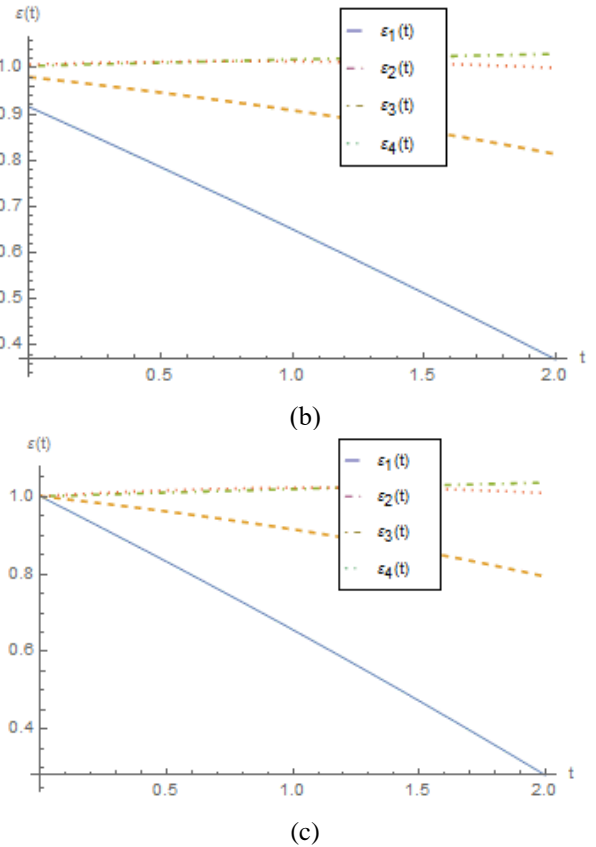
Figure 2. The plots of approximate solutions of $\varepsilon_1(t)$, $\varepsilon_2(t)$, $\varepsilon_3(t)$ and $\varepsilon_4(t)$ for (a) φ=0.5, (b) φ=0.75 and (c) φ=1

Table 1. The residual errors of $\varepsilon_1(t)$, $\varepsilon_2(t)$, $\varepsilon_3(t)$ and $\varepsilon_4(t)$ for $\varphi=0.5, 0.75, 1$

function	t	$\varphi=0.5$	$\varphi=0.75$	$\varphi=1$
$\varepsilon_1(t)$	0.0	0.147886	0.0709089	0
	0.2	0.145562	0.0665751	0.00664704
	0.4	0.143262	0.0622533	0.0133781
	0.6	0.140988	0.0579496	0.0201769
	0.8	0.138742	0.0536703	0.0270255
	1.0	0.136526	0.0494219	0.0339046
	1.2	0.13434	0.0452114	0.0407936
	1.4	0.132186	0.041046	0.0476709
	1.6	0.130066	0.0369329	0.054514
	1.8	0.127981	0.0328796	0.0612996
	2.0	0.125934	0.0288937	0.0680043
$\varepsilon_2(t)$	0.0	0.0224797	0.00859077	1.38778×10^{-17}
	0.2	0.0210296	0.00533641	0.00492826
	0.4	0.0196747	0.00218032	0.00989604
	0.6	0.0184181	0.000865924	0.0148763
	0.7	0.0172635	0.00379033	0.01984
	1.0	0.0162142	0.00658057	0.0247567
	1.2	0.0152738	0.00922404	0.0295944
	1.4	0.0144457	0.0117079	0.0343199
	1.6	0.0137336	0.0140191	0.0388991
	1.8	0.0131409	0.0161446	0.0432973
	2.0	0.0126712	0.018071	0.0474794
$\varepsilon_3(t)$	0.0	0.00611767	0.00267734	0
	0.2	0.00958746	0.00760386	0.0062537
	0.4	0.0130939	0.0126045	0.0126191
	0.6	0.0166373	0.017681	0.019102
	0.7	0.020218	0.0228352	0.0257082
	1.0	0.0238363	0.0280686	0.0324426
	1.2	0.0274924	0.0333827	0.0393097
	1.4	0.0311866	0.0387786	0.0463138
	1.6	0.0349191	0.0442577	0.0534585
	1.8	0.0386902	0.049821	0.0607473
	2.0	0.0424999	0.0554695	0.0681831
$\varepsilon_4(t)$	0.0	0.020052	0.0144559	0
	0.2	0.0182766	0.013136	1.05237×10^{-11}
	0.4	0.0164984	0.0118146	3.40934×10^{-10}
	0.6	0.0147174	0.0104913	2.62041×10^{-9}
	0.7	0.0129333	0.00916531	1.11735×10^{-8}
	1.0	0.0111459	0.0078363	3.44944×10^{-8}
	1.2	0.00935498	0.00650376	8.68037×10^{-8}
	1.4	0.00756039	0.00516722	1.89681×10^{-7}
	1.6	0.00576193	0.00382627	3.73765×10^{-7}
	1.8	0.00395943	0.0024805	6.80508×10^{-7}
	2.0	0.00215273	0.00112953	1.16398×10^{-6}

6. CONCLUSION

Based on the important role of functional calculus in different fields, we focused on the fractional model of nonlinear ES-DS. Using this applicable model we can adjust and control the supply and demand of the energy. Therefore, this model can be exploited for countries with limited energy resources. We applied the Caputo-Fabrizio derivative and its properties to solve the problem. Several theorems were proved to show the existence and uniqueness of solution and it is the main novelty of this study. We used the special data to forecast on the special time and based on them we found the numerical results. In the future researches, we will apply the fractional semi-analytical methods such as the homotopy analysis method and the homotopy perturbation method for solving the fractional model of ES-DS.

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