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Flexural analysis of rectangular kirchhoff plate on winkler foundation using galerkin-vlasov variational method

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ABSTRACT

In this work, the Galerkin-Vlasov variational method was implemented for the problem of bending of rectangular thin plate resting on Winkler foundation with simply supported edges at x = 0, x = a, y = 0, y = b. The Galerkin integral statement was written for the problem for the case of arbitrary distributed load and solutions obtained for deflections, bending and twisting moments. Solutions were then obtained for point loads, sinusoidal loads, uniform loads and linearly distributed loads. It was found that the foundation modulus reduces the maximum deflections, as well as the maximum bending moments which occur at the center for uniformly distributed loads. Solutions obtained were found to be the same as those obtained using the Navier double trigonometric series method.

1. INTRODUCTION

Structures resting on elastic foundations are commonly used in engineering applications to model foundation structures like mats or rafts, airport runway pavements, rigid pavements, column footings and combined footings. Structures that are two dimensional are modeled as plates. Plates are described using Kirchhoff theory which is for thin plates for which $\frac{1}{100} \leq h/b \leq \frac{1}{20}$, Mindlin [1] theory, Reissner [2,3] theory which apply for moderately thick plates for which $\frac{1}{20} \leq h/b \leq \frac{1}{3}$. Other plate theories suitable for shear deformation plates have been proposed by Shimpi [4], Reddy [5], and Levinson [6]. Thick plates have been described using the mathematical theory of elasticity. Modelling and description of the interaction of the foundation on the structure/plate has always been problematic and difficult. However, mathematical descriptions of the soil reactive pressure distribution on the plate have been proposed by Winkler [7] who gave a one parameter model of the soil reaction. Others who proposed two parameter elastic foundation models include Pasternak [8], Hetenyi [9], Filonenko Borodich [10], and Kerr [11]. Soil structure interaction models have also been studied by Gorbunov-Posadov [12], Caselunghe and Eriksson [13], Ghaitani et al [14], and Rajpurohit et al [15]. Mathematically, Winkler [7] one parameter elastic foundation model gives the soil reaction $q_s(x, y)$ at any point as directly proportional to the deflection $w(x, y)$ at the point. Thus,

$$q_s(x, y) = kw(x, y) = p_s(x, y) \quad (1)$$

where k is the Winkler foundation modulus or coefficient of subgrade modulus.

Filonenko-Borodich [10], Hetenyi [9] and Pasternak [8] presented the expressions for two parameter elastic foundation models as:

$$q_s(x, y) = kw(x, y) - G\nabla^2 w(x, y) = p_s(x, y) \quad (2)$$

$$q_s(x, y) = kw(x, y) - T\nabla^2 w(x, y) = p_s(x, y) \quad (3)$$

where k and T are the Filonenko-Borodich elastic foundation parameters, and ∇^2 is the Laplacian; k and G are the Pasternak foundation parameters

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \quad (4)$$

Hetenyi [7] presented the two parameter elastic foundation model as:

$$q_s(x, y) = kw(x, y) + D\nabla^4 w(x, y) = p_s(x, y) \quad (5)$$

where D is the plate modulus, and k and D are the Hetenyi foundation parameters ∇^4 is the biharmonic operator

$$\nabla^4 = \nabla^2\nabla^2 = \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4} \quad (6)$$

In general, plate on elastic foundation problems are boundary value problems (BVP) defined as differential equations which are to be solved subject to certain loading and restraint (geometric) boundary conditions. Methods for solving BVP have been employed in the technical literature for solving such problems. Problem of plates on elastic foundation have been solved in the literature using a variety of methods such as: Navier method, Levy method, and numerical/approximate methods such as Finite Element Method, Finite Difference Method, Finite Strip Method, Ritz variational method. Mama et al [16] used the finite Fourier sine transform method to solve the problem of rectangular

Kirchhoff plate on Winkler foundation for the case of simply supported edges and transverse distributed loads on the plate domain. They obtained exact solutions identical to the solutions obtained using Navier's double trigonometric series method.

Ike [17] extended the previous work done by Mama et al [16] in that new particular types of distributed transverse load namely, bisinusoidal distribution and linear distribution over the entire plate domain were considered and solved using the Fourier sine transform method. In addition, numerical problems were considered and solved by Ike [17] for simply supported Kirchhoff plates resting on Winkler foundations for varying values of the non-dimensional Winkler parameter; for cases of uniformly distributed transverse load on the plate domain.

Other researchers who have worked on the plate on elastic foundation problem are: Althobaiti and Prikazchikov [18]; Zhong, Zhao and Hu [19]; Li, Zhong and Li [20]; Li, Zhong and Tian [21]; Li et al. [22]; Zhang, Shi and Wang [23]; Agarana, Gbadeyan and Ajayi [24]; Are, Idowu and Gbadeyin [25]; Agarana and Gbadeyin [26]; Tahuoneh and Yas [27]; and Ye et al. [28].

2. RESEARCH AIM AND OBJECTIVES

The aim of this research is to use the Galerkin-Vlasov variational method to obtain solutions to the flexural problem of simply supported Kirchhoff plate resting on Winkler foundation for the case of transversely applied distributed loads. The specific objectives are:

(i) to obtain the Galerkin-Vlasov variational integral statement of the problem of simply supported Kirchhoff plate on Winkler foundation for the case of arbitrary distribution of transverse loads.

(ii) to transform the governing partial differential equation of equilibrium of simply supported Kirchhoff plate on Winkler foundation under distributed transverse load to an algebraic problem using the Galerkin-Vlasov variational method.

(iii) to solve the Galerkin-Vlasov integral statement to obtain general solution for the deflection for any distributed transverse load, as well as the corresponding internal force resultants.

(iv) to solve the resulting algebraic equation to obtain the solution for a generalised distribution of transverse load on the Kirchhoff plate on Winkler foundation problem.

(v) to obtain solutions for deflections and internal forces for particular types of transverse load distributions, namely:

(a) point load P_0 applied at a known point (x_0, y_0) on the plate domain

(b) bisinusoidal distributed load over the entire plate domain

(c) uniformly distributed load over the entire plate domain

(d) linearly distributed load over the plate domain.

2.1 Theoretical framework

The classical thin plate theory or Kirchhoff plate theory was adopted for this research as the theoretical framework for the plate problem. It is a linear infinitesimal theory developed for plates whose thickness, h, to governing span, a, ratio (h/a) are less than 0.10.

The fundamental assumptions, otherwise called the Kirchhoff's hypotheses are:

(i) Points on the plate lying initially on a normal to the middle plane remain on the normal to the middle plane even after bending deformations, implying that shear deformations are neglected.

(ii) The normal and shear stresses in the direction transversal to the plate are so small that they can be neglected without significant errors. Thus, $\sigma_{zz} = 0,\ \tau_{xz} = \tau_{yz} = 0$ where σ_{zz} is the normal stress in vertical direction, τ_{xz} and τ_{yz} are shear stresses.

(iii) There is no deformation in the middle surface of the plate, which remains neutral during flexural deformation.

(iv) The plate material is homogeneous, linear elastic and isotropic.

2.2 Displacement field

The theory assumes that the displacement field could be completely defined using the transverse displacement of the middle surface $w(x, y, z = 0) = w(x, y)$ as:

$$u = -z\frac{\partial w}{\partial x}(x, y) \tag{7}$$

$$v = -z\frac{\partial w}{\partial y}(x, y) \tag{8}$$

$$w = w(x, y) \tag{9}$$

where u, v are the inplane displacements, and z is the coordinate variable in the thickness direction.

2.3 Strain-displacement relations

The strain-displacement relations of small-displacement or infinitesimal strain elasticity are used to obtain the strain displacement relations, as follows:

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} = -z\frac{\partial^2 w}{\partial x^2} \tag{10}$$

$$\varepsilon_{yy} = \frac{\partial v}{\partial y} = -z\frac{\partial^2 w}{\partial y^2} \tag{11}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = -2z\frac{\partial^2 w}{\partial x \partial y} \tag{12}$$

$$\varepsilon_{zz} = \frac{\partial w}{\partial z} = 0 \tag{13}$$

2.4 Stress-strain laws

The stress strain laws are simplified by the assumption in Equation (1) and (2) to the following:

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{Bmatrix} = \frac{E}{1-\mu^2} \begin{pmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & G\left(\frac{1-\mu^2}{E}\right) \end{pmatrix} \begin{Bmatrix} -z\frac{\partial^2 w}{\partial x^2} \\ -z\frac{\partial^2 w}{\partial y^2} \\ -2z\frac{\partial^2 w}{\partial x \partial y} \end{Bmatrix} \tag{14}$$

where μ is the Poisson's ratio, E is the Young's modulus of elasticity, and G is the shear modulus.

2.5 Internal stress resultants

The internal stress resultants M_{xx}, M_{yy} and M_{xy}, and Q_x, and Q_y, where M_{xx}, M_{yy} are the bending moments, M_{xy} is the twisting moment, and Q_x and Q_y are the shear forces are given by:

$$M_{xx} = -\int_{-h/2}^{h/2} \sigma_{xx} z dz = \int_{-h/2}^{h/2} \frac{Ez^2}{1-\mu^2}\left(\frac{\partial^2 w}{\partial x^2} + \mu \frac{\partial^2 w}{\partial y^2}\right) dz \tag{15}$$

$$M_{xx} = \int_{-h/2}^{h/2} \frac{Ez^2 dz}{1-\mu^2}\left(\frac{\partial^2 w}{\partial x^2} + \mu \frac{\partial^2 w}{\partial y^2}\right) \tag{16}$$

$$M_{xx} = D\left(\frac{\partial^2 w}{\partial x^2} + \mu \frac{\partial^2 w}{\partial y^2}\right) \tag{17}$$

$$\text{where } D = \int_{-h/2}^{h/2} \frac{Ez^2 dz}{1-\mu^2} = \frac{Eh^3}{12(1-\mu^2)} \tag{18}$$

D is called the flexural rigidity of the plate, and h is the plate thickness.

$$M_{yy} = -\int_{-h/2}^{h/2} \sigma_{yy} z dz = D\left(\frac{\partial^2 w}{\partial y^2} + \mu \frac{\partial^2 w}{\partial x^2}\right) \tag{19}$$

$$M_{xy} = -\int_{-h/2}^{h/2} \tau_{xy} z dz = D(1-\mu)\frac{\partial^2 w}{\partial x \partial y} \tag{20}$$

$$Q_x = \int_{-h/2}^{h/2} \tau_{xz} dz \tag{21}$$

$$Q_y = \int_{-h/2}^{h/2} \tau_{yz} dz \tag{22}$$

Some conventions adopt a negative sign for the bending and twisting moments given by Equations (17), (19) and (20).

2.6 Differential equations of equilibrium

The differential equations of equilibrium in terms of the force resultants are the three equations:

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + q(x,y) - p_s(x,y) = 0 \tag{23}$$

$$\frac{\partial M_{xx}}{\partial x} + \frac{\partial M_{xy}}{\partial y} + Q_x = 0 \tag{24}$$

$$\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_{yy}}{\partial y} + Q_y = 0 \tag{25}$$

These three equations could be solved simultaneously to obtain one equation namely

$$\frac{\partial^2 M_{xx}}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_{yy}}{\partial y^2} - q + p_s = 0 \tag{26}$$

Substitution of the force resultants yields

$$\frac{\partial^2}{\partial x^2}\left\{D\left(\frac{\partial^2 w}{\partial x^2} + \mu\frac{\partial^2 w}{\partial y^2}\right)\right\} + \frac{\partial^2}{\partial y^2}\left\{D\left(\frac{\partial^2 w}{\partial y^2} + \mu\frac{\partial^2 w}{\partial x^2}\right)\right\} + 2\frac{\partial^2}{\partial x \partial y}\left\{D(1-\mu)\frac{\partial^2 w}{\partial x \partial y}\right\} - q + p_s = 0 \tag{27}$$

Simplification yields

$$D\left(\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4}\right) + p_s = q \tag{28}$$

For the Winkler foundation, $p_s = kw$ and

$$D\left(\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4}\right) + kw = q \tag{29}$$

$$D\nabla^2\nabla^2 w + kw = q \tag{30}$$

$$D\nabla^4 w + kw = q \tag{31}$$

3. METHODOLOGY

The Galerkin method is an approximate method used in obtaining approximate solutions to boundary value problems and integral equations. It converts a continuous operator problem to a discrete problem. In principle, it is equivalent to the application of the method of variation of parameters to a function space, and thus converts the equation to a weak formulation. In solving a differential equation of the general form

$$Lw(x,y) = p(x,y) \tag{32}$$

where L is the differential operator, x, y the independent variables, $w(x, y)$ the unknown function of x, y and $p(x, y)$ a known function of x, y, the Galerkin method assumes an approximate solution given by a linear combination of basis functions $\varphi_{ij}(x,y)$ in the function space as:

$$\bar{w}(x,y) = \sum_{i=1}^{M}\sum_{j=1}^{N} c_{ij}\varphi_{ij}(x,y) \tag{33}$$

where $i = 1, 2, 3, \ldots, M$; $j = 1, 2, 3, \ldots, N$.

The basis functions $\varphi_{ij}(x,y)$ satisfy the boundary conditions and c_{ij} are unknown constants (parameters). The Galerkin's weighted integral statement for the minimization of the error (residual) function becomes:

$$\iint_{R^2}\left(\sum\sum c_{ij}L\varphi_{ij}(x,y)-p(x,y)\right)\varphi_{kr}\,dxdy=0 \quad (34)$$

where R^2 is the two dimensional domain of integration.
or,

$$\sum_{i=1}^{M}\sum_{j=1}^{N}c_{ij}\iint_{R^2}(L\varphi_{ij})\varphi_{kr}\,dxdy=\iint_{R^2}p(x,y)\varphi_{kr}\,dxdy \quad (35)$$

If $\varphi_{ij}(x,y)$ are orthogonal functions, then Galerkin-Vlasov integral statements simplify to become:

$$\sum_{i}^{M}\sum_{j}^{N}c_{ij}\iint_{R^2}(L\varphi_{ij})\varphi_{ij}\,dxdy=\iint_{R^2}p(x,y)\varphi_{ij}\,dxdy \quad (36)$$

The merits of the Galerkin's method include:

(i) its rapid convergence to the exact solution when the right basis functions that satisfy all the boundary conditions are used.

(ii) its universal applicability since it does not require the total potential energy functional.

(iii) the solution of complex problems is simplified to the evaluation of certain definite integrals which can be performed numerically when analytical integrations are impossible or difficult.

(iv) its close connection to the Rayleigh Ritz method and its use in the derivation of finite element characteristic matrices.

However, the method has the shortcoming that it:

(a) sometimes involves the arduous task of definite integrations

(b) its accuracy is influenced by the order of approximating polynomial functions, and the choice of trial functions.

4. RESULTS

4.1 Galerkin-Vlasov method of rectangular Kirchhoff plate on Winkler foundation

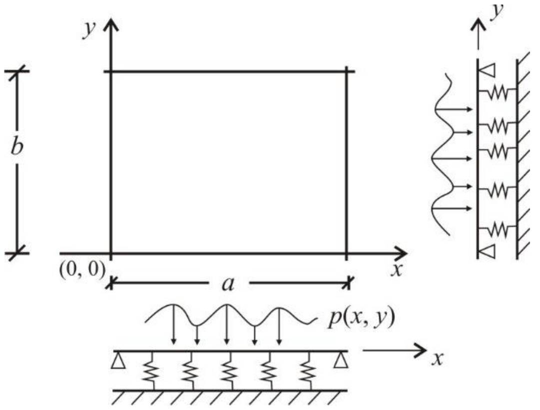

Figure 1. Simply supported Kirchhoff plate resting on a Winkler foundation, and carrying distributed transverse load

A rectangular simply supported Kirchhoff plate for which $0.01 < h/b < 0.05$ and length a and width b, where $a \geq b$, h is the plate thickness resting on a Winkler foundation as shown in Figure 1 was considered.

The edges $x = 0$, $x = a$, $y = 0$, $y = b$ are simply supported and the plate is subject to a distributed transverse load given generally by $p(x, y)$. The governing partial differential equation of equilibrium of the Kirchhoff plate on Winkler foundation problem is given for homogenous isotropic plates and static loads by the fourth order equation:

$$D\nabla^4 w(x,y)+kw(x,y)=p(x,y) \quad (37)$$

for $0\leq x\leq a,\ \ 0\leq y\leq b$

where ∇^4 is the biharmonic operator, given as Equation (6), k is the modulus of subgrade reaction or Winkler foundation constant, h is the plate thickness, μ is the Poisson's ratio, E is the Young's modulus of elasticity, D is the plate flexural rigidity and $w(x, y)$ is the transverse deflection of the plate middle surface, x and y are the inplane Cartesian coordinate variables. In operator form, the PDE is expressed as:

$$(D\nabla^4+k)w(x,y)-p(x,y)=0 \quad (38)$$

The geometric and force boundary conditions at the simply supported edges are:

$$w(x=0,y)=w(x=a,y)=0 \quad (39)$$

$$\frac{\partial^2 w}{\partial x^2}(x=0,y)=\frac{\partial^2 w}{\partial x^2}(x=a,y)=0 \quad (40)$$

$$w(x,y=0)=w(x,y=b)=0 \quad (41)$$

$$\frac{\partial^2 w}{\partial y^2}(x,y=0)=\frac{\partial^2 w}{\partial y^2}(x,y=b)=0 \quad (42)$$

By the Vlasov method, a trial displacement function that satisfies the boundary conditions is

$$\bar{w}(x,y)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}w_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}=\sum_{m}^{\infty}\sum_{n}^{\infty}w_{mn}\varphi_{mn}(x,y) \quad (43)$$

where w_{mn} are generalised displacement parameters, and

$$\varphi_{mn}=\sin\frac{m\pi x}{a}\sin\frac{n\pi x}{b} \quad (44)$$

are the basis functions.

The Galerkin-Vlasov variational integral is then:

$$\int_0^b\int_0^a\left\{(D\nabla^4+k)\sum_{m}^{\infty}\sum_{n}^{\infty}w_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}-p(x,y)\right\}\sin\frac{m'\pi x}{a}\sin\frac{n'\pi y}{b}\,dxdy=0 \quad (45)$$

$$\sum_{m}^{\infty}\sum_{n}^{\infty}w_{mn}\int_0^b\int_0^a(D\nabla^4+k)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\sin\frac{m'\pi x}{a}\sin\frac{n'\pi y}{b}\,dxdy$$

$$= \int_0^b \int_0^a p(x,y) \sin\frac{m'\pi x}{a} \sin\frac{n'\pi y}{b}\, dxdy \quad (46)$$

$$\sum_m^\infty \sum_n^\infty w_{mn} \int_0^b \int_0^a \left[D\left(\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 \right)^2 + k \right]$$

$$\times \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \sin\frac{m'\pi x}{a} \sin\frac{n'\pi y}{b}\, dxdy$$

$$= \int_0^b \int_0^a p(x,y) \sin\frac{m'\pi x}{a} \sin\frac{n'\pi y}{b}\, dxdy \quad (47)$$

The basis functions are orthogonal functions, and Equation (47) simplifies to:

$$\sum_{m=1}^\infty \sum_{n=1}^\infty w_{mn} \left[D\left(\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 \right)^2 + k \right] \int_0^b \int_0^a \sin^2\frac{m\pi x}{a} \sin^2\frac{n\pi y}{b}\, dxdy$$

$$= \sum_m^\infty \sum_n^\infty \int_0^b \int_0^a p_{mn} \sin^2\frac{m\pi x}{a} \sin^2\frac{n\pi y}{b}\, dxdy \quad (48)$$

where $p(x, y)$ has been expressed using the Fourier sine series representation as

$$p(x,y) = \sum_{m=1}^\infty \sum_{n=1}^\infty p_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \quad (49)$$

where p_{mn} is the Fourier sine series coefficient of the distributed load $p(x, y)$.

Let

$$I_{(mn)} = \int_0^b \int_0^a \sin^2\frac{m\pi x}{a} \sin^2\frac{n\pi y}{b}\, dxdy \quad (50)$$

Then

$$\sum_m^\infty \sum_n^\infty w_{mn} \left[D\left(\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 \right)^2 + k \right] I_{(mn)} = \sum_m^\infty \sum_n^\infty p_{mn} I_{(mn)} \quad (51)$$

Thus,

$$w_{mn} = \frac{p_{mn}}{D\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 \right]^2 + k} \quad (52)$$

Then,

$$w(x,y) = \sum_{m=1}^\infty \sum_{n=1}^\infty \frac{p_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}}{D\left(\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 \right)^2 + k} \quad (53)$$

$$w(x,y) = \sum_m^\infty \sum_n^\infty \frac{p_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}}{D\left[\left(\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 \right)^2 + \frac{k}{D} \right]} \quad (54)$$

The bending moment distributions are obtained from Equations (17), (19) and (20) for the general case of arbitrary distribution of $p(x, y)$ as:

$$M_{xx} = \sum_m^\infty \sum_n^\infty \frac{\left(\left(\frac{m\pi}{a}\right)^2 + \mu\left(\frac{n\pi}{b}\right)^2 \right) p_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}}{\left[\left(\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 \right)^2 + \frac{k}{D} \right]} \quad (55)$$

$$M_{yy} = \sum_m^\infty \sum_n^\infty \frac{\left(\left(\frac{n\pi}{b}\right)^2 + \mu\left(\frac{m\pi}{a}\right)^2 \right) p_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}}{\left[\left(\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 \right)^2 + \frac{k}{D} \right]} \quad (56)$$

$$M_{xy} = -(1-\mu) \sum_m^\infty \sum_n^\infty \frac{p_{mn} \left(\frac{m\pi}{a} \frac{n\pi}{b} \right) \cos\frac{m\pi x}{a} \cos\frac{n\pi y}{b}}{\left[\left(\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 \right)^2 + \frac{k}{D} \right]} \quad (57)$$

Galerkin-Vlasov solutions for point load P_0 at (x_0, y_0) $0 \le x_0 \le a,\ \ 0 \le y \le b$

For point load P_0 applied at (x_0, y_0), the Fourier sine series coefficient p_{mn} is

$$p_{mn} = \frac{4}{ab} \int_0^b \int_0^a p(x,y) \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}\, dxdy \quad (58)$$

$$p_{mn} = \frac{4}{ab} \int_0^b \int_0^a P_0 \delta(x = x_0, y = y_0) \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}\, dxdy \quad (59)$$

$$p_{mn} = \frac{4P_0}{ab} \sin\frac{m\pi x_0}{a} \sin\frac{n\pi y_0}{b} \quad (60)$$

Then the solutions for $w(x, y)$, M_{xx} and M_{yy} become:

$$w(x,y) = \frac{4P_0}{ab} \sum_m^\infty \sum_n^\infty \frac{\sin\frac{m\pi x_0}{a} \sin\frac{n\pi y_0}{b} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}}{D\left[\left(\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 \right)^2 + \frac{k}{D} \right]} \quad (61)$$

$$M_{xx} = \frac{4P_0}{ab} \sum_m^\infty \sum_n^\infty \frac{\left(\left(\frac{m\pi}{a}\right)^2 + \mu\left(\frac{n\pi}{b}\right)^2 \right) \sin\frac{m\pi x_0}{a} \sin\frac{m\pi x}{a} \sin\frac{n\pi y_0}{b} \sin\frac{n\pi y}{b}}{\left[\left(\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 \right)^2 + \frac{k}{D} \right]} \quad (62)$$

$$M_{yy} = \frac{4P_0}{ab} \sum_m^\infty \sum_n^\infty \frac{\left(\left(\frac{n\pi}{b}\right)^2 + \mu\left(\frac{m\pi}{a}\right)^2 \right) \sin\frac{m\pi x_0}{a} \sin\frac{m\pi x}{a} \sin\frac{n\pi y_0}{b} \sin\frac{n\pi y}{b}}{\left[\left(\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 \right)^2 + \frac{k}{D} \right]} \quad (63)$$

For m = 1, 3, 5, … n = 1, 3, 5, …

For square plates, where $x_0 = a/2,\ y_0 = b/2$,

$$w(x,y) = \frac{4P_0}{ab}\sum_m^\infty\sum_n^\infty \frac{\sin^2\frac{m\pi}{2}\sin^2\frac{n\pi}{2}}{D\left[\pi^4\left(\frac{m^2}{a^2}+\frac{n^2}{a^2}\right)^2+\frac{k}{D}\right]} \quad (64)$$

$$w(x,y) = \frac{4P_0a^4}{ab}\sum_m^\infty\sum_n^\infty \frac{\sin^2\frac{m\pi}{2}\sin^2\frac{n\pi}{2}}{D\left[\pi^4(m^2+n^2)^2+\frac{ka^4}{D}\right]} \quad (65)$$

$$w(x,y) = \frac{4P_0a^3}{b}\sum_m^\infty\sum_n^\infty \frac{\sin^2\frac{m\pi}{2}\sin^2\frac{n\pi}{2}}{D\left[\pi^4(m^2+n^2)^2+K\right]} \quad (66)$$

where $K = ka^4/D$

$$M_{xx_{\max}} = \frac{4P_0}{ab}\sum_m^\infty\sum_n^\infty \frac{\pi^2\left(\left(\frac{m}{a}\right)^2+\mu\left(\frac{n}{a}\right)^2\right)\sin^2\frac{m\pi}{2}\sin^2\frac{n\pi}{2}}{\pi^4\left(\frac{m^2}{a^2}+\frac{n^2}{a^2}\right)+\frac{k}{D}} \quad (67)$$

$$M_{xx_{\max}} = \frac{4P_0}{ab}\sum_m^\infty\sum_n^\infty \frac{\frac{\pi^2}{a^2}(m^2+\mu n^2)\sin^2\frac{m\pi}{2}\sin^2\frac{n\pi}{2}}{\left[\frac{\pi^4}{a^4}(m^2+n^2)^2+\frac{k}{D}\right]} \quad (68)$$

$$M_{xx_{\max}} = \frac{4P_0}{ab}\sum_m^\infty\sum_n^\infty \frac{\pi^2a^2(m^2+\mu n^2)\sin^2\frac{m\pi}{2}\sin^2\frac{n\pi}{2}}{\pi^4(m^2+n^2)^2+\frac{ka^4}{D}} \quad (69)$$

$$M_{xx_{\max}} = \frac{4P_0a}{b}\sum_m^\infty\sum_n^\infty \frac{\pi^2(m^2+\mu n^2)\sin^2\frac{m\pi}{2}\sin^2\frac{n\pi}{2}}{\pi^4(m^2+n^2)^2+\frac{ka^4}{D}} = M_{yy_{\max}} \quad (70)$$

Galerkin-Vlasov solution for sinusoidal load $p(x,y) = p_0\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}$.

Here, the Fourier sine series coefficient p_{mn} is

$$p_{mn} = \frac{4}{ab}\int_0^b\int_0^a p_0\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\,dxdy \quad (71)$$

$$p_{mn} = \begin{cases} p_0 & m=1, n=1 \\ 0 & m\neq 1, n\neq 1 \end{cases} \quad (72)$$

Then solutions become:

$$w(x,y) = \frac{p_0\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}}{D\left[\left(\left(\frac{\pi}{a}\right)^2+\left(\frac{\pi}{b}\right)^2\right)^2+\frac{k}{D}\right]} \quad (73)$$

For square thin plates, $a = b$, and we have:

$$w(x,y) = \frac{p_0\sin\frac{\pi x}{a}\sin\frac{\pi y}{a}}{D\left(\left(\frac{\pi^2}{a^2}+\frac{\pi^2}{a^2}\right)^2+\frac{k}{D}\right)} = \frac{p_0\sin\frac{\pi x}{a}\sin\frac{\pi y}{a}}{D\left(\frac{4\pi^4}{a^4}+\frac{k}{D}\right)} \quad (74)$$

$$w(x,y) = \frac{a^4p_0\sin\frac{\pi x}{a}\sin\frac{\pi y}{a}}{D\left(4\pi^4+\frac{ka^4}{D}\right)} = \frac{\sin\frac{\pi x}{a}\sin\frac{\pi y}{a}}{\left(4\pi^4+\frac{ka^4}{D}\right)}\frac{p_0a^4}{D} \quad (75)$$

$$w\left(x = a/2,\ y = b/2\right) = \left(\frac{1}{4\pi^4+K}\right)\frac{p_0a^4}{D} \quad (76)$$

The bending moments are found from the moment displacement relations as:

$$M_{xx} = \frac{\left(\left(\frac{\pi}{a}\right)^2+\mu\left(\frac{\pi}{b}\right)^2\right)p_0\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}}{\left[\left(\left(\frac{\pi}{a}\right)^2+\left(\frac{\pi}{b}\right)^2\right)^2+\frac{k}{D}\right]} \quad (77)$$

$$M_{yy} = \frac{\left(\left(\frac{\pi}{b}\right)^2+\mu\left(\frac{\pi}{a}\right)^2\right)p_0\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}}{\left[\left(\left(\frac{\pi}{a}\right)^2+\left(\frac{\pi}{b}\right)^2\right)^2+\frac{k}{D}\right]} \quad (78)$$

For square plates,

$$M_{xx} = \frac{\frac{\pi^2}{a^2}(1+\mu)p_0\sin\frac{\pi x}{a}\sin\frac{\pi y}{a}}{\left(\frac{4\pi^4}{a^4}+\frac{k}{D}\right)} \quad (79)$$

$$M_{xx} = \frac{\pi^2a^2(1+\mu)p_0\sin\frac{\pi x}{a}\sin\frac{\pi y}{a}}{\left(4\pi^4+\frac{ka^4}{D}\right)} \quad (80)$$

$$M_{xx_{\max}} = \left(\frac{\pi^2(1+\mu)}{4\pi^4+K}\right)p_0a^2 \quad (81)$$

$$M_{yy_{\max}} = \left(\frac{\pi^2(1+\mu)}{4\pi^4+K}\right)p_0a^2 \quad (82)$$

Galerkin-Vlasov solutions for uniformly distributed load $p_0(x, y) = p_0$

The Fourier sine series coefficient for the load p_{mn} is

$$p_{mn} = \frac{4}{ab}\int_0^b\int_0^a p_0\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\,dxdy \quad (83)$$

$$p_{mn} = \frac{4p_0}{ab}\left[\frac{-a}{m\pi}\cos\frac{m\pi x}{a}\right]_0^a\left[\frac{-b}{n\pi}\cos\frac{n\pi y}{b}\right]_0^b \quad (84)$$

$$p_{mn} = \frac{16p_0}{mn\pi^2} \quad (85)$$

$$w(x,y) = \frac{16p_0}{\pi^2}\sum_m^\infty\sum_n^\infty \frac{\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}}{D\left[\pi^4\left(\left(\frac{m}{a}\right)^2+\left(\frac{n}{b}\right)^2\right)^2+\frac{k}{D}\right]mn} \quad (86)$$

$$w_{\max} = \frac{16p_0}{\pi^2 D}\sum_m^\infty\sum_n^\infty \frac{\sin\frac{m\pi}{2}\sin\frac{n\pi}{2}}{\left[\pi^4\left(\left(\frac{m}{a}\right)^2+\left(\frac{n}{b}\right)^2\right)^2+\frac{k}{D}\right]mn} \quad (87)$$

$$w_{\max} = \frac{16p_0a^4}{\pi^2 D}\sum_m^\infty\sum_n^\infty \left\{\frac{\sin\frac{m\pi}{2}\sin\frac{n\pi}{2}}{mn\left[\pi^4\left(m^2+n\frac{a^2}{b^2}\right)^2+\frac{ka^4}{D}\right]}\right\} \quad (88)$$

$$M_{xx_{\max}} = \frac{16p_0}{\pi^2}\sum_m^\infty\sum_n^\infty \left\{\frac{\left(\left(\frac{m\pi}{a}\right)^2+\mu\left(\frac{n\pi}{b}\right)^2\right)\sin\frac{m\pi}{2}\sin\frac{n\pi}{2}}{\left[\left(\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right)^2+\frac{k}{D}\right]mn}\right\} \quad (89)$$

$$M_{xx_{\max}} = 16p_0a^2\sum_m^\infty\sum_n^\infty \left\{\frac{\left(m^2+\mu(nr)^2\right)\sin\frac{m\pi}{2}\sin\frac{n\pi}{2}}{mn\left[\pi^4\left(m^2+(nr)^2\right)^2+\frac{ka^4}{D}\right]}\right\} \quad (90)$$

where $r = a/b$

$$M_{yy_{\max}} = 16p_0a^2\sum_m^\infty\sum_n^\infty \left\{\frac{\left((nr)^2+\mu m^2\right)\sin\frac{m\pi}{2}\sin\frac{n\pi}{2}}{mn\left[\pi^4\left(m^2+(nr)^2\right)^2+\frac{ka^4}{D}\right]}\right\} \quad (91)$$

For square plates, $r = 1$,

$$w_{\max} = \frac{16p_0a^4}{D\pi^2}\sum_m^\infty\sum_n^\infty \left\{\frac{\sin\frac{m\pi}{2}\sin\frac{n\pi}{2}}{mn\left[\pi^4(m^2+n)^2+\frac{ka^4}{D}\right]}\right\} \quad (92)$$

$$M_{xx_{\max}} = M_{xx}\left(\frac{a}{2},\frac{b}{2}\right) = 16p_0a^2\sum_m^\infty\sum_n^\infty \left\{\frac{(m^2+\mu n^2)\sin\frac{m\pi}{2}\sin\frac{n\pi}{2}}{mn\left[\pi^4(m^2+n^2)^2+\frac{ka^4}{D}\right]}\right\} = M_{yy_{\max}} \quad (93)$$

$$M_{xy} = \frac{-(1-\mu)16p_0}{\pi^2}\sum_m^\infty\sum_n^\infty \frac{\frac{\pi^2 mn}{ab}\left(\cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b}\right)}{mn\left[\pi^4\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2+\frac{k}{D}\right]} \quad (94)$$

$$M_{xy} = \frac{-(1-\mu)16p_0}{ab}\sum_m^\infty\sum_n^\infty \frac{\cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b}}{\left[\pi^4\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2+\frac{k}{D}\right]} \quad (95)$$

$$M_{xy} = \frac{-16p_0(1-\mu)a^4}{ab}\sum_m^\infty\sum_n^\infty \left\{\frac{1}{\pi^4(m^2+n^2r^2)^2+\frac{ka^4}{D}}\right\} \quad (96)$$

$$M_{xy_{\max}} = \frac{-16p_0(1-\mu)a^3}{b}\sum_m^\infty\sum_n^\infty \left\{\frac{1}{\pi^4(m^2+n^2r^2)^2+\frac{ka^4}{D}}\right\} \quad (97)$$

For square plates,

$$M_{xy_{\max}} = -16p_0(1-\mu)a^2\sum_m^\infty\sum_n^\infty \left\{\frac{1}{\pi^4(m^2+n^2)^2+\frac{ka^4}{D}}\right\} \quad (98)$$

Galerkin-Vlasov solutions for linearly distributed load $p = p_0 x/a$, $(0 \le x \le a,\ 0 \le y \le b)$.

In this case, the Fourier sine series coefficient is

$$p_{mn} = \frac{4}{ab}\int_0^b\int_0^a \frac{p_0 x}{a}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\,dxdy \quad (99)$$

$$p_{mn} = \frac{8p_0\cos m\pi}{mn\pi^2} \quad (100)$$

Then,

$$w(x,y) = \frac{8p_0}{\pi^2 D}\sum_m^\infty\sum_n^\infty \frac{\cos m\pi\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}}{mn\left[\left(\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right)^2+\frac{k}{D}\right]} \quad (101)$$

$$w(x,y) = \frac{8p_0}{\pi^2 D}\sum_m^\infty\sum_n^\infty \frac{\cos m\pi\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}}{mn\left[\pi^4\left(\left(\frac{m}{a}\right)^2+\left(\frac{n}{b}\right)^2\right)^2+\frac{k}{D}\right]} \quad (102)$$

$$w(x,y) = \frac{8p_0a^4}{\pi^2 D}\sum_m^\infty\sum_n^\infty \frac{\cos m\pi\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}}{mn\left[\pi^4(m^2+n^2r^2)^2+\frac{ka^4}{D}\right]} \quad (103)$$

$$w_c = \frac{8p_0a^4}{\pi^2 D}\sum_m^\infty\sum_n^\infty \frac{\cos m\pi \sin\frac{m\pi}{2}\sin\frac{n\pi}{2}}{mn\left[\pi^4(m^2+n^2r^2)^2+\frac{ka^4}{D}\right]} \quad (104)$$

w_c is the center deflection.

$$M_{xx} = \frac{8p_0}{\pi^2}\sum_m^\infty\sum_n^\infty \frac{\left(\left(\frac{m\pi}{a}\right)^2+\mu\left(\frac{n\pi}{b}\right)^2\right)\cos m\pi \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}}{mn\left[\left(\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right)^2+\frac{k}{D}\right]} \quad (105)$$

$$M_{xx} = \frac{8p_0}{\pi^2}\sum_m^\infty\sum_n^\infty \left\{\frac{\pi^2\left(\left(\frac{m}{a}\right)^2+\mu\left(\frac{n}{b}\right)^2\right)\cos m\pi \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}}{\left[\pi^4\left(\left(\frac{m}{a}\right)^2+\left(\frac{n}{b}\right)^2\right)^2+\frac{k}{D}\right]mn}\right\} \quad (106)$$

$$M_{xx} = \frac{8p_0a^2}{\pi^2}\sum_m^\infty\sum_n^\infty \left\{\frac{\pi^2\left(m^2+\mu(nr)^2\right)\cos m\pi \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}}{mn\left[\pi^4\left(m^2+(nr)^2\right)^2+\frac{ka^4}{D}\right]}\right\} \quad (107)$$

At the center, the bending moment is:

$$M_{xx_c} = \frac{8p_0a^2}{\pi^2}\sum_m^\infty\sum_n^\infty \left\{\frac{\pi^2\left(m^2+\mu(nr)^2\right)\cos m\pi \sin\frac{m\pi}{2}\sin\frac{n\pi}{2}}{mn\left[\pi^4\left(m^2+(nr)^2\right)^2+\frac{ka^4}{D}\right]}\right\} \quad (108)$$

$$M_{xx_c} = p_0a^2\cdot\frac{8}{\pi^2}\sum_m^\infty\sum_n^\infty \left\{\frac{\pi^2\left(m^2+\mu(nr)^2\right)\cos m\pi \sin\frac{m\pi}{2}\sin\frac{n\pi}{2}}{mn\left[\pi^4\left(m^2+(nr)^2\right)^2+\frac{ka^4}{D}\right]}\right\} \quad (109)$$

$$M_{yy} = \frac{8p_0}{\pi^2}\sum_m^\infty\sum_n^\infty \left\{\frac{\left(\left(\frac{n\pi}{b}\right)^2+\mu\left(\frac{m\pi}{a}\right)^2\right)\cos m\pi \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}}{mn\left[\left(\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right)^2+\frac{k}{D}\right]}\right\} \quad (110)$$

$$M_{yy} = \frac{8p_0a^2}{\pi^2}\sum_m^\infty\sum_n^\infty \frac{\pi^2\left(n^2r^2+\mu m^2\right)\cos m\pi \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}}{mn\left[\pi^4\left(m^2+n^2r^2\right)^2+\frac{ka^4}{D}\right]} \quad (111)$$

$$M_{yy} = \frac{8p_0a^2}{\pi^2}\sum_m^\infty\sum_n^\infty \frac{\pi^2\left(n^2r^2+\mu m^2\right)\cos m\pi \sin\frac{m\pi}{2}\sin\frac{n\pi}{2}}{mn\left[\pi^4\left(m^2+n^2r^2\right)^2+\frac{ka^4}{D}\right]} \quad (112)$$

For square plates, $r = 1$,

$$w_c = \frac{8p_0a^4}{\pi^2 D}\sum_m^\infty\sum_n^\infty \frac{\cos m\pi \sin\frac{m\pi}{2}\sin\frac{n\pi}{2}}{mn\left[\pi^4(m^2+n^2)^2+\frac{ka^4}{D}\right]} \quad (113)$$

$$M_{xx_c} = \frac{8p_0a^2}{\pi^2}\sum_m^\infty\sum_n^\infty \frac{\pi^2\left(m^2+\mu n^2\right)\cos m\pi \sin\frac{m\pi}{2}\sin\frac{n\pi}{2}}{mn\left[\pi^4(m^2+n^2)^2+\frac{ka^4}{D}\right]}$$

$$= M_{yy}\left(x = a/2,\ y = b/2\right) = M_{yy} \quad (114)$$

Table 1. Galerkin solution for deflection and bending moment coefficients for simply supported Kirchhoff plate on Winkler foundation for uniform load on the plate and square plates $\left(a/b = 1\right)$, $K = \left(\frac{ka^4}{D}\right)^{1/4}$

K	$w_{\max}\left(\times 10^{-3}\frac{qa^4}{D}\right)$	$M_{xx_{\max}}\left(\times 10^{-2}qa^2\right)$	$M_{yy_{\max}}\left(\times 10^{-2}qa^2\right)$	$M_{xy_{\max}}\left(\times 10^{-2}qa^2\right)$
1	4.053	4.809	4.809	2.943
3	3.348	3.910	3.910	2.456
5	1.507	1.575	1.575	1.181

Table 2. Galerkin solution for maximum deflections and bending moment coefficients for simply supported square Kirchhoff plate on Winkler foundations for sinusoidal load $p(x,y) = p_0\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}$, $(\mu = 0.30)$

$K = \left(\frac{ka^4}{D}\right)^{1/4}$	$w_{\max}\times 10^{-2}\frac{pa^4}{D}$	$M_{xx}\times pa^2$	$M_{yy}\times pa^2$	$M_{xy}\times pa^2$
1	$2.5599 \simeq 2.56$	0.03285	0.03285	0.001792
3	2.1248	0.02726	0.02726	0.001487
5	0.09856	0.01265	0.01265	0.0006899
0	2.5665	0.03293	0.03293	0.001797

The Galerkin-Vlasov solutions for the maximum deflection and maximum bending moments which occurs at the plate center ($x = a/2$, $y = b/2$) for square Kirchhoff plates resting on Winkler foundations for values of $K = \left(\frac{ka^4}{D}\right)^{1/4}$ equal to $K = 1$, $K = 3$, and $K = 5$ have been calculated and presented in Table 1 for the case of uniformly distributed transverse load p_0 over the entire plate domain. The Galerkin-Vlasov solutions for the maximum deflection and maximum bending and twisting moments for square Kirchhoff plates resting on Winkler foundations for values of K equal to 1, 3, and 5 for the case of sinusoidal load distribution over the plate domain and simply supported edges are shown in Table 2.

5. DISCUSSION

The Galerkin-Vlasov variational method has been implemented successfully in this work to solve the fourth order partial differential equation for the Kirchhoff plate on Winkler foundation problem when the four edges $x = 0$, $x = a$, $y = 0$, and $y = b$ are simply supported and the plate is subjected to transversely applied distributed load. The trial displacement function used is given as Equation (43) and the shape functions satisfy the boundary conditions at the edges. The Galerkin-Vlasov variational integral statement which is a weak formulation of the boundary value problem of Kirchhoff plate on Winkler foundation subject to transverse load distribution $p(x, y)$ is given as Equation (45). The solution for the unknown undetermined displacement parameters of the trial displacement function was found, for a Fourier double sine series representation of the distributed transverse load as Equation (52); which solution is valid for any arbitrary distribution of transverse load expressible by a Fourier sine series. The general expression for the transverse displacement was then found as Equation (53) for the generalised transverse load distribution. The internal bending moment distributions and twisting moments were found using the bending moment-deflection (curvature) equations and twisting moment-deflection (curvature) equations as Equations (55), (56) and (57). The general solutions obtained were then particularized for the cases of

(i) point load P_0 applied at a point (x_0, y_0) in the plate domain where $0 \le x_0 \le a,\ \ 0 \le y_0 \le b$

(ii) sinusoidal load applied over the entire plate surface

(iii) uniformly distributed load of intensity p_0 over the entire plate region

(iv) linearly distributed load $p(x, y) = p_0 x/a$ over the entire plate surface.

The Galerkin-Vlasov solutions obtained for the deflections and bending moments for the case of point load P_0 applied at point (x_0, y_0) are given as Equations (61), (62) and (63). Their maximum values for square plates on Winkler foundations were obtained as Equations (66) and (70). For the case of sinusoidal load, the deflections and moments were obtained as Equations (75), (77), and (78). Their maximum values for square plates on Winkler foundations were found at the plate center as Equations (76) and (81). The Galerkin-Vlasov solutions for the case of uniform transverse load was found for maximum deflections and maximum moments as Equations (88), (89), (90) and (91). The maximum values for deflection and moments for square plate on Winkler foundation were found as Equations (92), (93) and (98). For the case of linearly distributed transverse load on the Kirchhoff plate on Winkler foundation, the Galerkin-Vlasov solutions were obtained for deflections and moments as Equations (103), (107) and (111). Their values at the plate center were obtained as Equations (104), (108) and (112) for rectangular thin plates on Winkler foundations and Equations (113) and (114) for square thin plates on Winkler foundations. Here, the maximum values may not occur at the plate center due to the non symmetrical nature of the load on the plate about the plate center. The Galerkin-Vlasov solutions for square Kirchhoff plate resting on Winkler foundation for the case of uniformly distributed transverse load and simply supported edges shown tabulated in Table 1 for values of $K = \left(ka^4/D\right)^{1/4}$ equal to $K = 1$, $K = 3$, and $K = 5$ show that the maximum deflections and bending and twisting moments at the plate center reduce as the values of K increase. It is observed that the Galerkin-Vlasov variational solutions yielded analytical mathematical closed form solutions which were identical with the solutions obtained using a Navier double Fourier sine series method for the problem. Table 2 represents the Galerkin-Vlasov solutions for maximum deflections and moments for simply supported square Kirchhoff plates under sinusoidal load distribution. Table 2 shows that for sinusoidal distribution of transverse loads, the maximum values of transverse deflection and bending moments occur at the plate center and reduce as the Winkler foundation modulus increases.

6. CONCLUSIONS

The following conclusions can be made from this study:

(i) the Galerkin-Vlasov variational method yielded mathematically closed form solutions to the Kirchhoff plate on Winkler foundation problem, which were exact within the limitations of the Kirchhoff plate theory and the Winkler model used in the problem formulation.

(ii) the Galerkin-Vlasov solutions were exactly identical with the solutions obtained using a Navier trigonometric series method for the same problem.

(iii) the Galerkin-Vlasov solutions obtained were exact because the exact shape functions were used in the displacement trial function.

(iv) the Winkler foundation has the effect of reducing the maximum deflections and bending and twisting moments at the center of the plate.

(v) the use of orthogonality functions in the displacement shape function simplified the definite integration, and reduced the boundary value problem to an algebraic one.

(vi) convergence of the expressions obtained for the displacements were faster than those obtained for the moments.

(vii) convergence of the expressions obtained for the case of point load was very slow due to the singularity property of the point load, and its representation by many terms of the Fourier series.

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