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Numerical Calculation for Error Formula for Boole's Rule

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ABSTRACT

The main aim of this paper is to calculate the value of integration which Singular Integrands or have Integrands with Singular derivatives at one end (or both ends) of the integration region by deriving errors formula (correction terms) using Composite Boole's rule and Romberg integral to improve the results.

1. INTRODUCTION

Many researchers have worked to find the values Numerical integration [1, 2], and of them work in particular to find of continuous and Improper integrations let us mention just a few Fox [3], Fox and Linda [4], Wu and Liu [5] and Davis and Rabinowitz [6] studied the integrals its integrands singularity but neglects the singularity introduced to derive errors formula (correction terms) by using composite Boole rule [7, 8] to calculate an approximate values of integrals have its integrands have singular derivatives and singular integrals at end of region of integration and we will use Romberg integration (acceleration) [9, 10] to improve the results.

Let ***I*** is integral

$$I = \int_a^b f(x)dx \tag{1}$$

It can calculate in approximation by Composite Boole's rules

$$I = \int_a^b f(x)dx \cong B(h) + E_B \tag{2}$$

where, *B(h)* is composite Boole's rule is defined.

Consider a *f(x)* over interval [*a, b*]

$$B(h) = \frac{2h}{45}\sum_{j=1}^{\frac{n}{4}}(7f_{4k-4} + 32f_{4k-3} + 12f_{4k-2} + 32f_{4k-1} + 7f_{4k}) \tag{3}$$

where, $n=4k$, $k \geq 1$, that is n is restricted to be a multiple of 4, $x_j = a + jh$ $h = \frac{(b-a)}{n}$, and $f_j = f(x_j), j = 0, 1, 2, \dots, n$.

E_B error formula for the Boole rule given by

$$E_B = -\frac{2\,h^6(b-a)}{945}f^6(\eta) + -\frac{2h^8(b-a)}{900}f^8(\eta) \tag{4}$$

where, for some η in $[a,\ b]$ [7]

We can use Romberg Integration will write the rule with the error formulas as follows

$$I - B(h) = A_1h^6 + A_2h^8 + \cdots \tag{5}$$

where, A_i, i=1, 2, 3, … is coefficients [8].

2. MAIN RESULTS

2.1 Integrals for Integrands with Singularity Derivatives Points

Theorem (1):

Let the function *f(x)* is continuous and differentiable at each point of the region of integrals [*a*, *b*], but at least one of the derivatives at the point (*a*), The approximate value of integration can be calculated from the following rule

$$I = \int_a^b f(x)dx \cong B(h) + E_B(h)$$

Such that

$$E_B(h) = [a_1h^7D^6 + a_1h^8D^7 + a_1h^9D^8 + \cdots]f(x_3) + A_1\,h^6 + A_2\,h^8\ + \cdots$$

$A_i, a_i, i = 1,2,3, \dots$ are constants depend on the derivatives of the function *f(x)* on the integral region and *B(h)* Similar to the formula (3), and $x_j = a + jh$,

$j = 0, 1, 2, \dots, n$

Proof: we can write the integral *I* in the form

$$I = \int_a^b f(x)dx = \int_{x_0}^{x_n} f(x)dx = \int_{x_0}^{x_4} f(x)dx + \int_{x_4}^{x_n} f(x)dx \tag{6}$$

For the first integration is defined at each point of the integrations but its derivatives are not defined at the point x_0 in the partial region $[x_0, x_4]$ we using (Taylor's series [11]) for the function *f(x)* about the point x_4 we get

$$f(x) = f(x_4) + (x - x_4)f'(x_4) + \frac{(x - x_4)^2}{2!}f''(x_4) + \frac{(x - x_4)^3}{3!}f'''(x_4) + \frac{(x - x_4)^4}{4!}f^4(x_4) + \frac{(x - x_4)^5}{5!}f^5(x_4) + \frac{(x - x_4)^6}{6!}f^6(x_4) + \cdots \quad (7)$$

We suppose that all partial derivatives to *f(x)* exist at the point (x_4), By taking the integral for formula (7) in the region $[x_0, x_4]$ we obtain

$$I_1 = \int_{x_0}^{x_4} f(x) = 4hf(x_4) - 8h^2 f'(x_4) + \frac{32h^3}{3}f''(x_4) - \frac{32h^4}{3}f'''(x_4) + \frac{128h^5}{15}f^4(x_4) - \frac{4^4h^6}{45}f^5(x_4) + \frac{4^5h^7}{315}f^6(x_4) + \cdots \quad (8)$$

From formula (7) we get

(1) $f(x_0) = f(x_4) - 4hf'(x_4) + 8h^2 f''(x_4) - \frac{32h^3}{3}f'''(x_4) + \frac{32h^4}{3}f^4(x_4) - \frac{128h^5}{15}f^5(x_4) + \frac{4^4h^6}{45}f^6(x_4) + \cdots$

(2) $f(x_0 + h) = f(x_4) - 3hf'(x_4) + \frac{9}{2}h^2 f''(x_4) - \frac{9h^3}{2}f'''(x_4) + \frac{27h^4}{8}f^4(x_4) - \frac{81h^5}{40}f^5(x_4) + \frac{81h^6}{80}f^6(x_4) + \cdots$

(3) $f(x_0 + 2h) = f(x_4) - 2hf'(x_4) + 2h^2 f''(x_4) - \frac{4h^3}{3}f'''(x_4) + \frac{2h^4}{3}f^4(x_4) - \frac{4h^5}{15}f^5(x_4) + \frac{4h^6}{45}f^6(x_4) + \cdots$

(4) $f(x_0 + 3h) = f(x_4) - hf'(x_4) + \frac{1}{2}h^2 f''(x_4) - \frac{h^3}{6}f'''(x_4) + \frac{h^4}{24}f^4(x_4) - \frac{h^5}{120}f^5(x_4) + \frac{h^6}{720}f^6(x_4) + \cdots$

from (8) and (1), (2), (3), (4) and

$f(x_4) = Ef(x_3) = f(x_0 + 3h)$ we get $E = e^{hD}$ where, $Df(x) = f'(x)$. [10]

$$I_1 = \int_{x_0}^{x_4} f(x)dx = \frac{2h}{45}[7f(x_0) + 32f(x_1) + 12f(x_2) + 32f(x_3) + 7f(x_4)] + [\frac{1532}{21}h^7 f^6(x_4) + \cdots]\, Ef(x_3) \ldots \quad (9)$$

$$I_1 = \int_{x_0}^{x_4} f(x)\,dx = \frac{2h}{45}[7f(x_0) + 32f(x_1) + 12f(x_2)] + 32f(x_3) + 7f(x_4)]f(x_3) + [\, a_1h^7D^6 + a_2h^8D^7 + + a_3h^9D^8 + \ldots] \ldots. \quad (10)$$

Such that a_i is constants, for all , $i = 1, 2, 3, \ldots$

And the second integral is continuous and differentiable at each point of the region of integrals $[x_4, x_n]$

$$I_2 = \int_{x_4}^{x_n} f(x)dx = \frac{2h}{45}\sum_{j=2}^{\frac{n}{4}}(7f(x_{4j-4}) + 32f(x_{4j-3}) + 12f(x_{4j-2}) + 32f(x_{4j-1}) + 7f_{(x_{4j})}) + A_1h^6 + A_2h^8 + \cdots \quad (11)$$

Such that A_i is constants, for all , $i = 1,2,3, \ldots$
from formulas (10), (11) we get

$$I = \int_{x_0}^{x_n} f(x)dx = \frac{2h}{45}\sum_{j=1}^{\frac{n}{4}}(7f(x_{4j-4}) + 32f(x_{4j-3}) + 12f(x_{4j-2}) + 32f(x_{4j-1}) + 7f_{(x_{4j})}) + A_1h^6 + A_2h^8 + \cdots$$

Theorem (2)

Let the function *f(x)* is continuous and differentiable at each point of the region of integrals [a, b], at least one of the derivatives at the point (b), The approximate value of integration can be calculated from the following rule.

$$I = \int_a^b f(x)\,dx = B(h) + E_B(h)$$
$$E_B(h) = [c_1h^7D^6 + c_2h^8D^7 + c_3h^9D^8 + \ldots]f(x_{n-1})$$

Such that $A_i, c_i, i = 1,2,3, \ldots$ are constants depend on the derivatives of the function *f(x)* on the integral region and *B(h)* Similar to the formula (3), and $x_j = a + jh$, *j*=0, 1, 2, …

Proof: we can write the integral *I* in the form

$$I = \int_a^b f(x)dx = \int_{x_0}^{x_n} f(x)dx = \int_{x_0}^{x_{n-4}} f(x)dx + \int_{x_{n-4}}^{x_n} f(x)dx \ldots \quad (12)$$

For the first integration is defined at each point of the integrations we get

$$I_1 = \int_{x_0}^{x_{n-4}} f(x)dx = \frac{2h}{45}\sum_{j=1}^{(\frac{n}{4})-4}(7f(x_{4j-4}) + 32f(x_{4j-3}) + 12f(x_{4j-2}) + 32f(x_{4j-1}) + 7f_{(x_{4j})}) + A_1h^6 + A_2h^8 + \cdots \quad (13)$$

Such that A_i is constants, for all, *i*=1, 2, 3, …

And the second integral but its partial derivatives are not defined at the point x_4 in the partial region $[x_4, x_{n-4}]$ we using Taylor's series for the function *f(x)* about the point (x_{n-4})

$$f(x) = f(x_{n-4}) + (x - x_{n-4})f'(x_{n-4}) + \frac{(x - x_{n-4})^2}{2!}f''(x_{n-4}) + \frac{(x - x_{n-4})^3}{3!}f'''(x_{n-4}) \quad (14)$$

$$+\frac{(x-x_{n-4})^4}{4!}f^4(x_{n-4})$$
$$+\frac{(x-x_{n-4})^5}{5!}f^5(x_{n-4})$$
$$+\frac{(x-x_{n-4})^6}{6!}f^6(x_{n-4})+\cdots \quad \ldots$$

We suppose that all derivatives to *f(x)* exist at the point (x_{n-4}), By taking the integral for formula (7) in the region we obtain

$$I_2=\int_{x_0}^{x_4} f(x)\,dx=4h\,f(x_4)+8h^2f'(x_4)+\frac{32h^3}{3}f''(x_4)+\frac{-32h^4}{3}f'''(x_4)+\frac{128h^5}{15}f^4(x_4)+\frac{4^4\,h^6}{45}f^5(x_4)+\frac{4^5h^7}{315}f^6(x_4)+\cdots \quad \ldots \tag{15}$$

From formula (14) we get

(1) $f(x_n)=f(x_{n-4})+4hf'(x_{n-4})+8h^2f''(x_{n-4})+\frac{32h^3}{3}f'''(x_{n-4})+\frac{32h^4}{3}f^4(x_{n-4})+\frac{128h^5}{15}f^5(x_{n-4})+\frac{4^4h^6}{45}f^6(x_{n-4})+\cdots$

(2) $f(x_n-h)=f(x_{n-4})+3hf'(x_{n-4})+\frac{9}{2}h^2f''(x_{n-4})+\frac{9h^3}{2}f'''(x_{n-4})+\frac{27h^4}{8}f^4(x_{n-4})+\frac{81h^5}{40}f^5(x_{n-4})+\frac{81h^6}{80}f^6(x_{n-4})+\cdots$

(3) $f(x_n-2h)=f(x_{n-4})+2hf'(x_{n-4})+2h^2f''(x_{n-4})+\frac{4h^3}{3}f'''(x_{n-4})+\frac{2h^4}{3}f^4(x_{n-4})+\frac{4h^5}{15}f^5(x_{n-4})+\frac{4h^6}{45}f^6(x_{n-4})+\cdots$

(4) $f(x_0-3h)=f(x_{n-4})+hf'(x_{x_{n-4}})+\frac{1}{2}h^2f''(x_{n-4})+\frac{h^3}{6}f'''(x_{n-4})+\frac{h^4}{24}f^4(x_{n-4})+\frac{h^5}{120}f^5(x_{n-4})+\frac{h^6}{720}f^6(x_{n-4})+\cdots$

from (15) and (1), (2), (3), (4) and

$E^{-1}f(x_n)=f(x_{n-1})=f(x_n-h)$ we get $E^{-1}=e^{-hD}$ where $Df(x)=f'(x)$

$$I_2=\int_{x_0}^{x_4} f(x)dx=\frac{2h}{45}[7f(x_0)+32f(x_1)+12f(x_2)+32f(x_3)+7f(x_4)]+\left[\frac{1532}{21}h^7D^6+\cdots\right]E^{-1}f(x_{n-1}) \quad \ldots \tag{16}$$

$$I_2=\int_{x_0}^{x_4} f(x)=\frac{2h}{45}[7f(x_0)+32f(x_1)+12f(x_2)+32f(x_3)+7f(x_4)]+[c_1h^7D^6+c_2h^8D^7+c_3h^9D^8+\cdots]f(x_{n-1})\ldots \tag{17}$$

c_i, i=1, 2, 3, … is constants

from formulas (13), (17) we get

$$I=\int_{x_0}^{x_n} f(x)dx=\frac{2h}{45}\sum_{j=1}^{\frac{n}{4}}(7f(x_{4j-4})+32f(x_{4j-3})+12\,f(x_{4j-2})+32f(x_{4j-1})+7f_{(x_{4j})})+[c_1h^7D^6+c_2h^8D^7+c_3h^9D^8+\cdots]f(x_{n-1})+A_1h^6+A_2h^8+\cdots$$

Corollary: Let the function $f(x)$ is continuous and differentiable at each point of the region of integrals [a, b], but at least one of the derivatives at the point the points (*a, b*), The approximate value of integration can be calculated from the following rul

$$I=\int_a^b f(x)dx\cong B(h)+E_B(h)$$
$$E_B(h)=[b_1h^7D^6+b_2h^8D^7+b_3h^9D^8+\cdots]f(x_3)+[c_1h^7D^6+c_2h^8D^7+c_3h^9D^8+\cdots]f(x_{n-1})+A_1\,h^6+A_2h^8+\cdots$$

Such that A_i, a_i, c_i, , for all, i=1, 2, 3, … are constants depend on the partial derivatives of the function *f(x)* on the integral region and $B(h)$ Similar to the formula (3), and $x_j=a+jh$, j=0, 1, 2, ….

Proof: Can be easily using a theorem (1), (2)

2.2 Integrals with Singular integrands

The function is continuous and derivable in the integration region [a, b] but not defined at the point a. Here we cannot calculate the value of integration using the theorem (2-1), because it uses the value of the integral in the point and to obtain the value of integration using the above theorem, Davies and Rabinowitz [6] and calculating the error formulas of them, as well as if they are not defined in the point or both points are not defined

3. EXAMPLES

(1) $\int_0^1 x^{\frac{3}{2}}\,dx$ which analytical value is 0.4 approximate to 14 decimal places the integrand which has singula in the second derivative $(x_0=0)$ and the formula of the correction terms $(A_1\,h^{2.5}+A_2h^8+A_3h^{10}+A_4h^{12}+\cdots)$ Such that A_i, $i=1,2,3,\ldots$ are constants from the table (1) The value of the integration is correct for six decimal places using a base Boole , while using the Romberg integral method with the above rule, we note that the value equal to the analytical value when (subintervals n=64)

Table 1

n	Bool values	$h^{2.5}$	h^6	h^8	h^{10}
4	0.40030278197718				
8	0.40005360500749	0.40000009743428			
16	0.40000947773754	0.40000000196966	0.40000000045435		
32	0.40000167547077	0.40000000003359	0.40000000000286	0.40000000000109	
64	0.40000029618463	0.40000000000054	0.40000000000001	0.40000000000000	0.40000000000000

(2) $\int_0^1 (1-x)^{\frac{-1}{4}} dx$ which analytical value is 1.33333333333333 approximate to 14 decimal places To calculate the integration numerically note that the integrand at has singular point at the upper end ($x_n = 1$) the error formula is $(t_1 h^{0.75} + t_2 h^8 + t_3 h^{10} + t_4 h^{12} + \cdots)$ Such that t_i, $i =$ 1,2,3, ... are constants from the table (2) The value of the integration is correct for one decimal places using a base Boole, while using the Romberg integral method with the above rule, we note that the value equal to the analytical value when (subintervals *n*=128)

Table 2

n	Bool values	$h^{0.75}$	h^6	h^8	h^{10}	h^{12}
4	1.12123952438139					
8	1.20722309696577	1.33333702591451				
16	1.25834777436493	1.33333342266093	1.33333336546643			
32	1.28874665387441	1.33333333497350	1.33333333358163	1.33333333345660		
64	1.30682193512078	1.33333333336021	1.33333333333461	1.33333333333364	1.33333333333352	
128	1.31756956164199	1.33333333333376	1.33333333333334	1.33333333333333	1.33333333333333	1.33333333333333

(3) $\int_0^1 -x^2 \, ln(x) dx$ which analytical value is 0.111111111111111approximate to 15 decimal places to calculate the integration numerically note that the integrand at has singular point at the lower end ($x_0 = 0$) and the error formulas for this integration is $(d_1 h^3 + d_2 h^6 + d_3 h^8 + d_4 h^{10} + \cdots)$ Such that d_i, $i = 1,2,3, \ldots$ are constants from the table (3) The value of the integration is correct for seven decimal places using a base Boole , while using the Romberg integral method with the above rule, we note that the value equal to the analytical value when (subintervals *n*=128).

Table 3

n	*Bool values*	h^3	h^6	h^8	h^{10}	h^{12}
4	0.111447861867241					
8	0.111153371033969	0.111111300914930				
16	0.111116396808825	0.111111114776662	0.111111102367444			
32	0.111111771877083	0.111111111172548	0.111111110932274	0.111111111068223		
64	0.111111193707714	0.111111111112090	0.111111111108059	0.111111111110850	0.111111111111017	
128	0.111111121435700	0.111111111111126	0.111111111111062	0.111111111111110	0.111111111111111	0.111111111111111

(4) $\int_0^1 \sqrt{x}\, e^x dx$ which analytical value is 0.12556300825518 approximate to 13 decimal places To calculate the integration numerically note that the integrand at has singular point at the lower end ($x_0 = 0$) and the error formulas for this integration is $(j_1 h^{1.5} + j_2 h^{2.5} + j_3 h^{3.5} + j_4 h^{4.5} + j_5 h^{5.5} + j_6 h^6 + j_7 h^{6.5} + \cdots)$ Such that d_i, $i =$ 1,2,3, ... are constants from the table (4) The value of the integration is correct for four decimal places using a base Boole , while using the Romberg integral method with the above rule, we note that the value equal to the analytical value when (subintervals *n*=256).

Table 4

n	Bool values	$h^{1.5}$	$h^{2.5}$	$h^{3.5}$	$h^{4.5}$	$h^{5.5}$	h^6
4	1.2470029863351						
8	1.2525311205590	1.2555545575607					
16	1.2545254859732	1.2556162406373	1.2556294862905				
32	1.2552379232167	1.2556275680836	1.2556300005081	1.2556300503658			
64	1.2554911416387	1.2556296313923	1.2556300744615	1.2556300816319	1.2556300830775		
128	1.2555809075678	1.2556300021847	1.2556300818076	1.2556300825199	1.2556300825610	1.2556300825493	
256	1.2556126873498	1.2556300682897	1.2556300824849	1.2556300825506	1.2556300825520	1.2556300825518	1.2556300825518

4. THE DISCUSSION

It is clear according to tables of results of this research that when we calculate the approximate value for the integration when integrands with singular derivatives and Singular Integrands with by the composite Boole rule when we are using Romberg accelerating Using the calculated errors formula with rule we can get a best results with respect to convergence to value of integration with a few number from subintervals Thus, we can depend on method in a calculation the integrals

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