

## DISTANCE BETWEEN UNCERTAIN RANDOM VARIABLES

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### ABSTRACT

In order to deal with indeterminacy data involving uncertainty and randomness, uncertain random variable is investigated by many scholars. As an extension of distance between uncertain variables, the definition of distance between uncertain random variables is proposed in this paper. Then, some formulas are provided to calculate distances between particular types of uncertain random variables

**Keywords:** uncertainty theory; uncertain random variable; chance measure; distance.

### 1. INTRODUCTION

In order to deal with indeterminacy phenomenon in daily life, several axiomatic systems have been founded. In 1933, Kolmogorov [5] founded an axiomatic system of probability theory to describe random phenomena. If there are enough historical data, we can employ probability theory to estimate probability distribution. Sometimes, it is difficult to collect observed data when some unexpected events occur. In this case, people have to invite experts to estimate the belief degree of each event's occurrence. However, some counterintuitive consequences may occur if we employ probability theory or fuzzy set theory to model the belief degree [11]. For dealing with belief degree legitimately, an axiomatic system named uncertainty theory was proposed by Liu [7] in 2007. In addition, the product uncertain measure was defined by Liu [9] in 2009.

The concept of uncertain variable and uncertainty distribution were proposed by Liu [7]. Then, a sufficient and necessary condition of uncertainty distribution was proved by Peng and Imamura [18] in 2010. To describe the relationship between uncertain measure and uncertain distribution, a measure inversion theorem was presented by Liu [10] from which the uncertain measures of some events would be calculated via the uncertainty distribution. After proposing the concept of independence [9], Liu [10] presented the operational law of uncertain variables. The concepts of expected value, variance, moments and distance of uncertain variable were proposed by Liu [10]. Besides, a useful formula was presented by Liu and Ha [13] to calculate the expected values of monotone functions of uncertain variables. In order to characterize the uncertainty of uncertain variables, Liu [9] proposed the concept of entropy in 2009. After that, Dai and Chen citeDai12 proved the positive linearity of entropy and gave some formulas to calculate the entropy of monotone function of uncertain variables. Chen and Dai [1] discussed

the method to select the uncertainty distribution using the maximum entropy principle. In order to make an extension of entropy, Chen, Kar and Ralescu [2] proposed a concept of cross-entropy for comparing an uncertainty distribution against a reference uncertainty distribution.

In 2013, Liu [14] proposed chance theory by defining uncertain random variable and chance measure in order to describe a system that involved both uncertainty and randomness. Some related concepts of uncertain random variables such as chance distribution, expected value, and variance were also presented by Liu [14]. As an important contribution to chance theory, Liu [15] proposed a basic operational method of uncertain random variables. After that, uncertain random variables were discussed widely. A law of large numbers was presented by Yao and Gao [23]. Besides, Yao and Gao [22] proposed an uncertain random process. As applications of chance theory, Liu [15] proposed uncertain random programming. Uncertain random risk analysis was presented by Liu [16]. Besides, chance theory was applied into many fields and many achievements were obtained, such as uncertain random reliability analysis (Wen and Kang [20]), uncertain random logic (Liu eLiuY13d), uncertain random graph (Liu [12]), and uncertain random network (Liu [12]).

In this paper, the distance between uncertain random variables is studied. In the first section, the uncertainty theory and uncertain random variables are introduced. In the following section, the definition of distance between uncertain random variables is presented and some formulas are proposed to calculate the distance between specific type's uncertain random variables. In addition, the method is illustrated by examples.

### 2. PRELIMINARY

As a branch of axiomatic mathematics, uncertainty theory aims to deal with human uncertainty. We will first present some basic concepts of uncertain theory and chance theory.

## 2.1 Uncertain Variables

Suppose that  $\Gamma$  is a nonempty set and  $L$  is a  $\sigma$ -algebra on  $\Gamma$ . Each element in  $L$  is called an event. A set function  $M$  from  $L$  to  $[0,1]$  that satisfies normality axiom, duality axiom, subadditivity axiom (Liu [7]) and product axiom (Liu [9]) is called an uncertain measure. In general,  $(\Gamma, L, M)$  is called an uncertainty space. An uncertain variable  $\xi$  is defined as a measurable function from  $(\Gamma, L, M)$  to  $R$ . That is to say, the set  $\xi^{-1}(B) = \{\gamma \in \Gamma \mid \xi(\gamma) \in B\}$  is an event, for any Borel set  $B$  of real numbers.

Liu [7] presented the definition of uncertainty distribution  $\Phi$  to represent uncertain variables, in which  $\Phi$  is defined as

$$\Phi(x) = M\{\xi \leq x\}$$

for any real number  $x$ .

**Definition1.** (Liu [7]) If the uncertainty distribution of an uncertain variable  $\xi$  is

$$\Phi(x) = \begin{cases} 0, & \text{if } x < a \\ (x-a)/(b-a), & \text{if } a \leq x < b \\ 1, & \text{if } x \geq b, \end{cases}$$

Where  $a$  and  $b$  are real numbers with  $a < b$ , then  $\xi$  is called linear uncertain variable and its uncertainty distribution is denoted by  $L(a, b)$ .

**Definition2.** (Liu [9]) Let  $\xi_1, \xi_2, \dots, \xi_n$  be uncertain variables. If

$$M\left\{\bigcap_{i=1}^n (\xi_i \in B_i)\right\} = \bigwedge_{i=1}^n M\{\xi_i \in B_i\}$$

For any Borel sets  $B_1, B_2, \dots, B_n$  of real numbers, then

$$E[\xi] = \int_0^1 f(\Phi_1^{-1}(\alpha), \dots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), \dots, \Phi_n^{-1}(1-\alpha)) d\alpha. \quad (3)$$

**Definition3.** (Liu [7]) Let  $\xi$  and  $\eta$  be two uncertain variables. The distance between  $\xi$  and  $\eta$  is defined as

$$d(\xi, \eta) = E[|\xi - \eta|] = \int_0^{+\infty} M\{|\xi - \eta| \geq r\} dr. \quad (4)$$

Besides, the distance  $d(\cdot, \cdot)$  satisfies no negativity, identification, symmetry and triangular inequality.

## 2.2 Uncertain Random Variables

In 2013, Liu [14] first proposed chance theory, which is a mathematical methodology for modeling complex systems with both uncertainty and randomness, including chance measure, uncertain random variable, chance distribution, operational law, expected value and so on. The chance space

we say  $\xi_1, \xi_2, \dots, \xi_n$  are independent uncertain variables.

**Theorem1.** (Liu [7]) Let  $f(x_1, x_2, \dots, x_n)$  be strictly monotone increasing with respect to  $x_1, x_2, \dots, x_m$  and strictly monotone decreasing with respect to  $x_{m+1}, x_{m+2}, \dots, x_n$ . Let  $\xi_1, \xi_2, \dots, \xi_n$  be independent regular uncertain variables and  $\xi = f(\xi_1, \xi_2, \dots, \xi_n)$ . Then, the inverse uncertainty distribution of  $\xi$   $\Psi^{-1}(\alpha)$  can be calculated by

$$\Psi^{-1}(\alpha) = f(\Phi_1^{-1}(\alpha), \dots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), \dots, \Phi_n^{-1}(1-\alpha)),$$

In which  $\Psi_1^{-1}(\alpha), \Psi_2^{-1}(\alpha), \dots, \Psi_n^{-1}(\alpha)$  are inverse uncertainty distributions of  $\xi_1, \xi_2, \dots, \xi_n$ , respectively.

In order to describe the mean value of an uncertain variable  $\xi$  by uncertain measure, Liu [7] defined the expected value of  $\xi$  as

$$E[\xi] = \int_0^{+\infty} M\{\xi \geq r\} dr - \int_{-\infty}^0 M\{\xi \leq r\} dr \quad (1)$$

Provided that at least one of the two integrals is finite. In addition, the expected value can be calculated by

$$E[\xi] = \int_0^{+\infty} (1 - \Phi(x)) dx - \int_{-\infty}^0 \Phi(x) dx. \quad (2)$$

Furthermore, the expected value of a function of  $n$  uncertain variables can be calculated by inverse uncertainty distributions.

**Theorem2.** (Liu and Ha [13]) Let  $\xi_1, \xi_2, \dots, \xi_n$  be  $n$  independent regular uncertain variables. If the conditions of Theorem 1 hold, then the expected value of uncertain variable  $\xi = f(\xi_1, \xi_2, \dots, \xi_n)$  is

is refer to the product  $(\Gamma, L, M) \times (\Omega, A, Pr)$ , in which  $(\Gamma, L, M)$  is an uncertain space and  $(\Omega, A, Pr)$  is a probability space.

**Definition4.** (Liu [14]) Let  $(\Gamma, L, M) \times (\Omega, A, Pr)$  be a chance space, and let  $\Theta \in L \times A$  be an event. Then the chance measure of  $\Theta$  is defined as

$$Ch\{\Theta\} = \int_0^1 Pr\{\omega \in \Omega \mid M\{\gamma \in \Gamma \mid (\gamma, \omega) \in \Theta\} \geq r\} dr$$

Liu ([14]) proved that a chance measure satisfies normality, duality, and monotonicity properties, that is

$$(a) Ch\{\Gamma \times \Omega\} = 1, Ch\{\emptyset\} = 0;$$

- (b)  $Ch\{\Theta\} + Ch\{\Theta^c\} = 1$  For any event  $\Theta$  ;  
(c)  $Ch\{\Theta_1\} \leq Ch\{\Theta_2\}$  For any event  $\Theta_1 \subset \Theta_2$  .

**Lemma1.** (Hou ([4])) The chance measure is sub additive.  
That is, for any countable sequence of events  $\Theta_1, \Theta_2, \dots$ , we have

$$Ch\left\{\bigcup_{i=1}^{\infty} \Theta_i\right\} \leq \sum_{i=1}^{\infty} Ch\{\Theta_i\}$$

Roughly speaking, an uncertain random variable is a measurable function of uncertain variables and random variables.

**Definition5.** (Liu [14]) An uncertain random variable is a function  $\zeta$  from a chance space  $(\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{A}, Pr)$  to the set of real numbers *i.e.*,  $\{(\gamma, \omega) \in \Gamma \times \Omega \mid \zeta(\gamma, \omega) \in B\}$  is an event for any Boral set  $B$  .

Note that an uncertain random variable  $\zeta(\gamma, \omega)$  is a bivariate function on  $\Gamma \times \Omega$  . Specifically, both random variables and uncertain variables are degenerated uncertain random variables.

The uncertain random arithmetic is defined as follows.

**Definition6.** (Liu [14]) Let  $f : R^n \rightarrow R$  be a measurable function, and  $\zeta_1, \zeta_2, \dots, \zeta_n$  be uncertain random variables on the chance space  $(\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{A}, Pr)$  . Then,  $\zeta = f(\zeta_1, \zeta_2, \dots, \zeta_n)$  is an uncertain random variable defined as

$$\zeta(\gamma, \omega) = f(\zeta_1(\gamma, \omega), \zeta_2(\gamma, \omega), \dots, \zeta_n(\gamma, \omega))$$

for all  $(\gamma, \omega) \in (\Gamma \times \Omega)$  .

**Definition7.** (Liu [14]) Let  $\zeta$  be an uncertain random variable. The chance distribution of  $\zeta$  is defined by  $\Phi(x) = Ch\{\zeta \leq x\}$  for any  $x \in R$  .

**Theorem3.** (Liu [15]) Let  $\eta_1, \eta_2, \dots, \eta_m$  be independent random variables with probability distributions  $\Psi_1, \Psi_2, \dots, \Psi_m$  , and let  $\tau_1, \tau_2, \dots, \tau_n$  be independent uncertain variables with uncertainty

$$E[\zeta] = \int_{R^m} \int_0^1 f(x, y_1, \dots, y_m, \Upsilon_1^{-1}(\alpha), \dots, \Upsilon_k^{-1}(\alpha), \Upsilon_{k+1}^{-1}(1-\alpha), \dots, \Upsilon_n^{-1}(1-\alpha)) d\alpha d\Psi_1(y_1) \dots d\Psi_m(y_m)$$

Provided that  $f(\eta_1, \eta_2, \dots, \eta_m, \tau_1, \tau_2, \dots, \tau_n)$  is a strictly increasing function with respect to  $\tau_1, \tau_2, \dots, \tau_k$  and strictly decreasing function with respect to  $\tau_{k+1}, \tau_{k+2}, \dots, \tau_n$  .

### 3. DISTANCE BETWEEN UNCERTAIN RANDOM VARIABLES

**Definition1.** The distance between uncertain random

$$d(\zeta_1, \zeta_2) = E[|\zeta_1 - \zeta_2|] = \int_0^{+\infty} Ch\{|\zeta_1 - \zeta_2| \geq r\} dr = \int_0^{+\infty} M\{|\zeta_1 - \zeta_2| \geq r\} dr.$$

It means that the definition of distance between uncertain

distributions  $\Upsilon_1, \text{Upsilonpsilon}_2, \dots, \Upsilon_n$  respectively. Then the uncertain random variable

$$\zeta = f(\eta_1, \eta_2, \dots, \eta_m, \tau_1, \tau_2, \dots, \tau_n)$$

Has a chance distribution

$$\Phi(x) = \int_{R^m} F(x; y_1, y_2, \dots, y_m) d\Psi_1(y_1) d\Psi_2(y_2) \dots d\Psi_m(y_m)$$

Where  $F(x; y_1, y_2, \dots, y_m)$  is the uncertainty distribution of the uncertain variable

$$f(y_1, y_2, \dots, y_m, \tau_1, \tau_2, \dots, \tau_n)$$

And is determined by its inverse function

$$F^{-1}(\alpha; y_1, y_2, \dots, y_m) = f(y_1, y_2, \dots, y_m, \Upsilon_1^{-1}(\alpha), \dots, \Upsilon_k^{-1}(\alpha), \Upsilon_{k+1}^{-1}(1-\alpha), \dots, \Upsilon_n^{-1}(1-\alpha))$$

Provided that  $f(\eta_1, \eta_2, \dots, \eta_m, \tau_1, \tau_2, \dots, \tau_n)$  is a strictly increasing function with respect to  $\tau_1, \tau_2, \dots, \tau_k$  and strictly decreasing function with respect to  $\tau_{k+1}, \tau_{k+2}, \dots, \tau_n$  .

**Definition8.** (Liu [14]) Let  $\zeta$  be an uncertain random variable. Then its expected value is defined by

$$E[\zeta] = \int_0^{+\infty} Ch\{\zeta \geq r\} dr - \int_{-\infty}^0 Ch\{\zeta \leq r\} dr$$

Provided that at least one of the two integrals is finite.

**Definition9.** (Liu [14]) Let  $\zeta$  be an uncertain random variable with chance distribution  $\Phi$  . If the expected value of  $\zeta$  exists, then

$$E[\zeta] = \int_0^{+\infty} (1 - \Phi(x)) dx - \int_{-\infty}^0 \Phi(x) dx.$$

**Theorem4.** (Liu [14]) Let  $\eta_1, \eta_2, \dots, \eta_m$  be independent random variables with probability distributions  $\Psi_1, \Psi_2, \dots, \Psi_m$  , and let  $\tau_1, \tau_2, \dots, \tau_n$  be independent uncertain variables with uncertainty distributions  $\Upsilon_1, \text{Upsilonpsilon}_2, \dots, \Upsilon_n$  respectively. Then the uncertain random variable

$$\zeta = f(\eta_1, \eta_2, \dots, \eta_m, \tau_1, \tau_2, \dots, \tau_n)$$

Has an expected value

variables  $\zeta_1$  and  $\zeta_2$  is defined as

$$d(\zeta_1, \zeta_2) = E[|\zeta_1 - \zeta_2|]. \tag{5}$$

That is,

$$d(\zeta_1, \zeta_2) = \int_0^{+\infty} Ch\{|\zeta_1 - \zeta_2| \geq r\} dr.$$

**Remark1.** If the uncertain random variables  $\zeta_1$  and  $\zeta_2$  degenerate to uncertain variables, then

random variables is consistent with uncertain variables.

**Remark2.** If the uncertain random variables  $\zeta_1$  and  $\zeta_2$  degenerate to random variables, then

$$d(\zeta_1, \zeta_2) = E[|\zeta_1 - \zeta_2|] = \int_0^{+\infty} Ch\{|\zeta_1 - \zeta_2| \geq r\} dr = \int_0^{+\infty} Pr\{|\zeta_1 - \zeta_2| \geq r\} dr.$$

It means that the definition of distance between uncertain random variables is consistent with random variables.

**Theorem1.** Let  $\zeta_1, \zeta_2$  and  $\zeta_3$  be uncertain random variables, and let  $d(\cdot, \cdot)$  be the distance. Then we have

- (a)(No negativity)  $d(\zeta_1, \zeta_2) \geq 0$ ;
- (b)(Identification)  $d(\zeta_1, \zeta_2) = 0$  if and only if  $\zeta_1 = \zeta_2$ ;
- (c)(Symmetry)  $d(\zeta_1, \zeta_2) = d(\zeta_2, \zeta_1)$ ;
- (d)(Triangle Inequality)  
 $d(\zeta_1, \zeta_2) \leq 2d(\zeta_1, \zeta_3) + 2d(\zeta_2, \zeta_3)$ .

**Proof.** The proofs of parts (a), (b) and (c) are trivial. Now we prove the part (d). By using the definition of distance and Lemma 2.1, we get

$$\begin{aligned} d(\zeta_1, \zeta_2) &= E[|\zeta_1 - \zeta_2|] = \int_0^{+\infty} Ch\{|\zeta_1 - \zeta_2| \geq r\} dr \\ &\leq \int_0^{+\infty} Ch\{|\zeta_1 - \zeta_3| + |\zeta_2 - \zeta_3| \geq r\} dr \\ &\leq \int_0^{+\infty} Ch\{(|\zeta_1 - \zeta_3| \geq r/2) \cup (|\zeta_2 - \zeta_3| \geq r/2)\} dr \\ &\leq \int_0^{+\infty} 2Ch\{(|\zeta_1 - \zeta_3| \geq r)\} + 2Ch\{(|\zeta_2 - \zeta_3| \geq r)\} dr \\ &= 2E[|\zeta_1 - \zeta_3|] + 2E[|\zeta_2 - \zeta_3|] = 2d(\zeta_1, \zeta_3) + 2d(\zeta_2, \zeta_3). \end{aligned}$$

**Example1.** Let  $\Gamma = \{\gamma_1, \gamma_2, \gamma_3\}$ . Define  $M\{\emptyset\} = 0, M\{\Gamma\} = 1$  and  $M\{\Lambda\} = 1/2$  for any subset  $\Lambda$  ( $\Lambda \neq \emptyset, \Gamma$ ). Let  $\Omega = \{\omega_1, \omega_2\}$ . Define  $Pr\{\omega_1\} = 1/3, Pr\{\omega_2\} = 2/3$ . We set uncertain random variables  $\zeta_1$  and  $\zeta_2$  as follows,

$$\zeta_1(\omega) = \begin{cases} \xi, & \text{if } \omega = \omega_1 \\ \eta, & \text{if } \omega = \omega_2, \end{cases} \quad \zeta_2(\omega) = \begin{cases} \eta, & \text{if } \omega = \omega_1 \\ \xi, & \text{if } \omega = \omega_2, \end{cases}$$

In which

$$\xi(\gamma) = \begin{cases} 1, & \text{if } \gamma = \gamma_1 \\ 1, & \text{if } \gamma = \gamma_2 \\ 0, & \text{if } \gamma = \gamma_3, \end{cases} \quad \eta(\gamma) = \begin{cases} 0, & \text{if } \gamma = \gamma_1 \\ -1, & \text{if } \gamma = \gamma_2 \\ -0, & \text{if } \gamma = \gamma_3. \end{cases}$$

It is easy to verify that  $d(\zeta_1, \zeta_2) = 3/2$ ,  $d(\zeta_1, \zeta_3) = 1/2$  and  $d(\zeta_2, \zeta_3) = 1/2$ . Thus

$$d(\zeta_1, \zeta_2) = \frac{3}{2}(d(\zeta_1, \zeta_3) + d(\zeta_2, \zeta_3)).$$

In the following, we will discuss the method to obtain distance from chance distributions. Let  $\zeta_1$  and  $\zeta_2$  be uncertain random variables with chance distributions  $Y_1(x)$  and  $Y_2(x)$ , respectively. If  $\zeta_1 - \zeta_2$  has a chance distribution  $Y(x)$ , then the distance is

$$\begin{aligned} d(\zeta_1, \zeta_2) &= \int_0^{+\infty} Ch\{|\zeta_1 - \zeta_2| \geq x\} dx \\ &= \int_0^{+\infty} Ch\{(\zeta_1 - \zeta_2 \geq x) \cup (\zeta_1 - \zeta_2 \leq -x)\} dx \\ &\leq \int_0^{+\infty} (Ch\{\zeta_1 - \zeta_2 \geq x\} + Ch\{\zeta_1 - \zeta_2 \leq -x\}) dx \\ &\leq \int_0^{+\infty} (1 - Y(x) + Y(-x)) dx. \end{aligned}$$

We stipulate that the distance between  $\zeta_1$  and  $\zeta_2$  is

$$d(\zeta_1, \zeta_2) = \int_0^{+\infty} (1 - Y(x) + Y(-x)) dx. \quad (6)$$

**Remark3.** Mention that (5) is not a precise formula but a stipulation. The calculation formula of distance between uncertain random variables in the rest of this paper is refer to (5).

**Remark4.** Let  $\eta_1$  and  $\eta_2$  be random variables with probability distributions  $\Psi_1(x)$  and  $\Psi_2(x)$ , respectively. If  $\eta_1 - \eta_2$  has a probability distribution  $Y(x)$ , then the distance between  $\eta_1$  and  $\eta_2$  is

$$\begin{aligned} d(\eta_1, \eta_2) &= \int_0^{+\infty} Ch\{|\eta_1 - \eta_2| \geq x\} dx \\ &= \int_0^{+\infty} Pr\{|\eta_1 - \eta_2| \geq x\} dx \\ &= \int_0^{+\infty} Pr\{(\zeta_1 - \zeta_2 > x) \cup (\zeta_1 - \zeta_2 \leq -x) \cup (\zeta_1 - \zeta_2 = x)\} dx \\ &= \int_0^{+\infty} (Pr\{\zeta_1 - \zeta_2 > x\} + Pr\{\zeta_1 - \zeta_2 \leq -x\} + Pr\{\zeta_1 - \zeta_2 = x\}) dx \\ &= \int_0^{+\infty} (1 - Y(x) + Y(-x)) dx \end{aligned}$$

That means (5) is a precise formula when the uncertain random variables degenerate to random variables.

**Theorem2.** Let  $\eta_1, \eta_2, \dots, \eta_m, \tau_1, \tau_2, \dots, \tau_p$  be independent random variables with probability distributions  $\Psi_1(x), \Psi_2(x), \dots, \Psi_{m+p}$ , and let  $\tau_1, \tau_2, \dots, \tau_n, \xi_1, \xi_2, \dots, \xi_q$  be independent uncertain variables with uncertainty distributions  $Y_1(x), Y_2(x), \dots, Y_{n+q}(x)$  respectively. Let  $\zeta_1$  and  $\zeta_2$  be two uncertain random variables, in which

$$\zeta_1 = f_1(\eta_1, \eta_2, \dots, \eta_m, \tau_1, \tau_2, \dots, \tau_n),$$

$$\zeta_2 = f_2(\tau_1, \tau_2, \dots, \tau_p, \xi_1, \xi_2, \dots, \xi_q).$$

Then, the distance between  $\zeta_1$  and  $\zeta_2$  is

$$f_1(y_1, y_2, \dots, y_m, \tau_1, \tau_2, \dots, \tau_n) - f_2(y_{m+1}, y_{m+2}, \dots, y_{m+p}, \xi_1, \xi_2, \dots, \xi_q)$$

And is determined by its inverse function

$$\begin{aligned} F^{-1}(\alpha; y_1, y_2, \dots, y_{m+p}) &= f_1(y_1, y_2, \dots, y_m, Y_1^{-1}(\alpha), \dots, Y_k^{-1}(\alpha), Y_{k+1}^{-1}(1-\alpha), \dots, Y_n^{-1}(1-\alpha)) \\ &\quad - f_2(y_{m+1}, y_{m+2}, \dots, y_{m+p}, Y_{n+1}^{-1}(1-\alpha), \dots, Y_{n+t}^{-1}(1-\alpha), Y_{n+t+1}^{-1}(\alpha), \dots, Y_{n+q}^{-1}(\alpha)) \end{aligned}$$

provided that  $f_1(\eta_1, \eta_2, \dots, \eta_m, \tau_1, \tau_2, \dots, \tau_n)$  is a strictly increasing function with respect to  $\tau_1, \tau_2, \dots, \tau_s$  and strictly decreasing function with respect to  $\tau_{s+1}, \dots, \tau_n$  and  $f_2(\tau_1, \tau_2, \dots, \tau_p, \xi_1, \xi_2, \dots, \xi_q)$  is a strictly increasing function with respect to  $\xi_1, \xi_2, \dots, \xi_t$  and strictly decreasing function with respect to  $\xi_{t+1}, \dots, \xi_n$ .

**Proof.** It follows from Theorem 3 immediately.

**Corollary1.** Let  $\eta$  be a random variable with probability distribution  $\Phi(x)$  and  $\tau$  be an uncertain variable with uncertainty distribution  $\Psi(x)$ . Let  $Y(x)$  be the chance distribution of  $\tau - \eta$ , and then we have

$$\begin{aligned} d(\eta, \tau) &= \int_0^{+\infty} (1 - Y(x) + Y(-x)) dx \\ &= \int_0^{+\infty} \left( 1 - \int_{-\infty}^{+\infty} \Psi(x+y) d\Phi(y) + \int_{-\infty}^{+\infty} \Psi(-x+y) d\Phi(y) \right) dx \end{aligned}$$

$$d(\zeta_1, \zeta_2) = \int_0^{+\infty} (1 - Y(x) + Y(-x)) dx$$

$$\begin{aligned} &= \int_0^{+\infty} [1 - \int_{R^{m+p}} F(x; y_1, \dots, y_{m+p}) d\Psi_1(y_1) \cdots d\Psi_{m+p}(y_{m+p}) \\ &\quad + \int_{R^{m+p}} F(-x; y_1, \dots, y_{m+p}) d\Psi_1(y_1) \cdots d\Psi_{m+p}(y_{m+p})] dx, \end{aligned} \tag{7}$$

Where  $F(x; y_1, y_2, \dots, y_{m+p})$  is the uncertainty distribution of the uncertain variable

**Proof.** Note that  $Y(x) = \int_{-\infty}^{+\infty} \Psi(x+y) d\Phi(y)$ . It follows from Theorem 2 immediately.

**Example2.** Let  $\eta$  be a random variable with probability distribution  $\Phi(x)$  and  $c$  be a real number with uncertainty distribution  $\Psi(x)$ . Suppose that  $Y(x)$  represents the chance distribution of  $c - \eta$ . According to Corollary 1, we have

$$Y(x) = \int_{-\infty}^{+\infty} \Psi(x+y) d\Phi(y) = \int_{c-x}^{+\infty} d\Phi(y) = 1 - \Phi(c-x),$$

In which

$$\Psi(x) = \begin{cases} 1, & \text{if } x \geq c \\ 0, & \text{if } x < c. \end{cases}$$

Then we have

$$d(\eta, c) = \int_0^{+\infty} (1 - \Upsilon(x) + \Upsilon(-x)) dx = \int_0^{+\infty} (1 - \Phi(c+x) + \Phi(c-x)) dx$$

**Example3.** Let  $c$  be a real number with probability distribution  $\Phi(x)$  and  $\tau$  be an uncertain variable with uncertainty distribution  $\Psi(x)$ . Suppose that  $\Upsilon(x)$  represents the chance distribution of  $\tau - c$ . According to Corollary 1, we have

$$\Upsilon(x) = \int_{-\infty}^{+\infty} \Psi(x+y) d\Phi(y) = \Psi(x+c),$$

in which

$$d(c, \tau) = \int_0^{+\infty} (1 - \Upsilon(x) + \Upsilon(-x)) dx = \int_0^{+\infty} (1 - \Psi(c+x) + \Psi(c-x)) dx$$

**Example4.** Let  $b$  and  $c$  be two real numbers with distribution functions  $\Psi(x)$  and  $\Phi(x)$ . Suppose that  $\Upsilon(x)$  represents the chance distribution of  $b - c$ . According to Corollary 1, we have

$$\Upsilon(x) = \int_{-\infty}^{+\infty} \Psi(x+y) d\Phi(y) = \Psi(x+c) = \begin{cases} 1, & \text{if } x \geq b-c \\ 0, & \text{if } x < b-c, \end{cases}$$

In which

$$\Phi(x) = \begin{cases} 1, & \text{if } x \geq c \\ 0, & \text{if } x < c, \end{cases} \quad \Psi(x) = \begin{cases} 1, & \text{if } x \geq b \\ 0, & \text{if } x < b. \end{cases}$$

Then we have

$$d(c, b) = \int_0^{+\infty} (1 - \Upsilon(x) + \Upsilon(-x)) dx = |b - c|$$

It means that the definition of distance between uncertain

$$= \int_0^{+\infty} \left( 1 - \int_{R^2} F(x; y_1, y_2) d\Phi_1(y_1) d\Phi_2(y_2) + \int_{R^2} F(-x; y_1, y_2) d\Phi_1(y_1) d\Phi_2(y_2) \right) dx$$

Where

$F^{-1}(\alpha; y_1, y_2) = y_1 - y_2 + \Psi_1^{-1}(\alpha) - \Psi_2^{-1}(1-\alpha)$  is the inverse distribution of the uncertain variable  $y_1 - y_2 + \tau_1 - \tau_2$ .

**Proof.** It follows from Theorem 2 immediately.

**Example5.** Let  $\eta$  be a random variable with probability distribution  $\Phi(x)$  and  $\tau$  be an uncertain variable with uncertainty distribution  $\Psi(x)$ , in which

$$\Phi(x) = \begin{cases} 1 - \exp(-x), & \text{if } x \geq 0 \\ 0, & \text{if } x < 0, \end{cases}$$

$$\int_0^{+\infty} \Psi(x+y) d\Phi(y) = \begin{cases} x+1 - \exp(-1+x), & \text{if } 0 \leq x < 1 \\ 1, & \text{if } x \geq 1, \end{cases}$$

And

$$\int_0^{+\infty} \Psi(-x+y) d\Phi(y) = \exp(-x) - \exp(-1-x), x \geq 0.$$

According to Corollary 1, we have

$$\Phi(x) = \begin{cases} 1, & \text{if } x \geq c \\ 0, & \text{if } x < c. \end{cases}$$

Then we have

random variables is consistent with real numbers.

**Corollary2.** Let  $\eta_1$  and  $\eta_2$  be two independent random variables with probability distributions  $\Phi_1(x)$  and  $\Phi_2(x)$ , respectively. Let  $\tau_1$  and  $\tau_2$  be two independent uncertain variables with uncertainty distributions  $\Psi_1(x)$  and  $\Psi_2(x)$ . Suppose that  $\Upsilon(x)$  represents the chance distribution of  $(\eta_1 + \tau_1) - (\eta_2 + \tau_2)$ . Then we have

$$d(\eta_1 + \tau_1, \eta_2 + \tau_2) = \int_0^{+\infty} (1 - \Upsilon(x) + \Upsilon(-x)) dx$$

(8)

$$\Psi(x) = \begin{cases} 1, & \text{if } x \geq 1 \\ x, & \text{if } 0 \leq x < 1 \\ 0, & \text{if } x < 0. \end{cases}$$

We have



$$\begin{aligned}
 d(\eta, \tau) &= \int_0^{+\infty} \left( 1 - \int_{-\infty}^{+\infty} \Psi(x+y) d\Phi(y) + \int_{-\infty}^{+\infty} \Psi(-x+y) d\Phi(y) \right) dx \\
 &= \int_0^1 (-2x + \exp(-1+x) + 1 - \exp(-x-1)) dx + \int_1^{+\infty} (\exp(-x) - \exp(-1-x)) dx \\
 &= 1/2 + \exp(-1) - \exp(-1)^2 = 0.7325.
 \end{aligned}$$

#### 4. CONCLUSIONS

In this paper, the concept of distance between uncertain variables was expanded to uncertain random variables. Based on the subadditivity of chance measure and expected value of uncertain random variables, several properties of the distance were proved. Then, the effectiveness of this method was illustrated by an example.

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#### REFERENCES

- [1] Chen X. and Dai W., Maximum entropy principle for uncertain variables. *International Journal of Fuzzy Systems*, vol.13, No.3, pp.232-236, 2011.
- [2] Chen X., Kar S., and Ralescu D., Cross-entropy measure of uncertain variables. *Information Sciences*, Vol.201, pp.53-60, 2012.
- [3] Dai W. and Chen X., Entropy of function of uncertain variables. *Mathematical and Computer Modelling*, vol.55, Nos.3-4, pp.754-760, 2012.
- [4] Hou Y., Subadditivity of chance measure. *Journal of Uncertainty Analysis and Applications*, vol.2:14, 2014.
- [5] Kolmogorov AN, *Grundbegriffe der wahrscheinlichkeitsrechnung*, Julius Springer, Berlin, 1933.
- [6] Li X. and Liu B., On distance between fuzzy variables. *Journal of Intelligent and Fuzzy Systems*, vol.19, No.3, pp.197-204, 2008.
- [7] Liu B., *Uncertainty Theory*, 2nd Edition, Springer-Verlag, Berlin, 2007.
- [8] Liu B., *Theory and Practice of Uncertain Programming*, 2nd Edition, Springer-Verlag, Berlin, 2009.
- [9] Liu B., Some research problems in uncertainty theory. *Journal of Uncertain Systems*, vol.3(1), pp.3-10, 2009.
- [10] Liu B., *Uncertainty Theory: A Branch of Mathematics for Modeling Human Uncertainty*, Springer-Verlag, Berlin, 2010.
- [11] Liu B., Why is there a need for uncertainty theory. *Journal of Uncertain Systems*, vol.6, No.1, pp. 3-10, 2012.
- [12] Liu B., *Uncertain random graphs and uncertain random networks*, Technical Report, 2013.
- [13] Liu Y. and Ha M., Expected value of function of uncertain variables. *Journal of Uncertain Systems*, vol.4, No.3, pp.181-186, 2010.
- [14] Liu Y., Uncertain random variables: A mixture of uncertainty and randomness. *Soft Computing*, vol.17, No.4, pp.625-634, 2013.
- [15] Liu Y., Uncertain random programming with applications. *Fuzzy Optimization and Decision Making*, vol.12, No.2, pp.153-169, 2013.
- [16] Liu Y., Risk index in uncertain random risk analysis, <http://orsc.edu.cn/online/130403.pdf>.
- [17] Liu Y., *Uncertain random logic and uncertain random entailment*, Technical Report, 2013.
- [18] Peng Z. and Iwamura K., A sufficient and necessary condition of uncertainty distribution. *Journal of Interdisciplinary Mathematics*, vol.13, No.3, pp.277-285, 2010.
- [19] Sheng Y., Stability in the p-th moment for uncertain differential equation, *Journal of Intelligent and Fuzzy Systems*, to be published, <http://orsc.edu.cn/online/1211-20.pdf>.
- [20] Wen M. and Kang R., Reliability analysis in uncertain random system, <http://orsc.edu.cn/online/120419.pdf>.
- [21] Yang L. and Liu B., On continuity theorem for characteristic function of fuzzy variable. *Journal of Intelligent and Fuzzy Systems*, vol.17, No.3, pp. 325-332, 2006.
- [22] Yao K. and Gao J, Law of large numbers for uncertain random variables, <http://orsc.edu.cn/on-line/120401.pdf>.
- [23] Yao K. and Gao J, Some concepts and theorems of uncertain random process, <http://orsc.edu.cn/on-line/120402.pdf>.
- [24] Yao K. and Chen X., A numerical method for solving uncertain differential equations, *Journal of Intelligent and Fuzzy Systems*, to be published, <http://orsc.edu.cn/online/110913.pdf>.
- [25] Zhu Y. and Ji X., Expected values of functions of fuzzy variables. *Journal of Intelligent and Fuzzy Systems*, vol.17, No.5, pp.471-478, 2006.

