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Mathematical Model of Stratified Deep Water Flow Under Modified Gravity Using Perturbation Method Analysis

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ABSTRACT

This paper presents mathematical model of stratified deep water under modified gravity using the perturbation method (series solution) which involves the use of a small parameter, often denoted as $g^{'}$, to represent the modification in gravity. Then the governing equations obtained from conserved momentum and continuity equation for the fluid motion are then expanded in terms of this small parameter, allowing for a simplified analysis of the effects of the modified gravity on stratified deep water. We employed perturbation method to allow for the solution of the governing equations in terms of a power series with the small Coriolis parameter *f*. In addition, we demonstrated that modified gravity is the major factor that necessitates stratification in deep water. One key aspect of this model is the assumption of small modifications to the gravity due to density disruption, which allows for the use of a perturbation expansion. Overall, the mathematical model of stratified deep water under modified gravity using the perturbation method provides a useful tool for studying the effects of modified gravity on the behavior of stratified deep water systems. The mathematical model of stratified deep water flow under modified gravity using the perturbation method analysis provides new insights into the dynamics of deep water systems. By considering the effects of modified gravity, the model captured the behavior of stratified flows in environments where the gravitational force is altered, such as in the presence of strong magnetic fields or in the context of planetary bodies with different gravitational properties, perturbation method allows for the analysis of small deviations from the base state, enabling the identification of instability mechanisms and the prediction of the emergence of new flow regimes. This reveals the conditions under which the stratified flow becomes unstable and transitions to turbulent or other complex states, which is crucial for understanding the dynamics of deep water systems. This model contributes to the advancement of our knowledge of stratified deep water flow under modified gravity, providing insights that can be applied in various fields, such as oceanography, geophysics, and planetary science.

1. INTRODUCTION

Stratification in deep water means the layering or division of water masses based on their temperature, density and salinity characteristics in deep ocean surroundings [1]. It plays a crucial role in oceanography with significant implications for the distribution of heat, nutrients, and dissolved gases in the ocean [2]. In deep water, stratification typically takes place in areas where there is limited vertical mixing between different water masses [3]. The major factors influencing stratification are temperature and salinity gradients. Cold and denser water tends to sink, but warm and less dense water rises [4]. Salinity also affects water density, with higher salinity usually leading to increased density. The primary mechanism responsible for deep water stratification is thermohaline circulation, which is equally known as the ocean conveyor belt [5]. This circulation is driven by differences in temperature and salinity, which in turn causes water masses to sink and rise, creating vertical stratification [6]. Deep water masses are formed in high-latitude regions, where the surface water cools and becomes denser due to low temperatures and high salinity [7]. This dense water sinks to the deep ocean, forming deep water masses. The stratification in deep water has important implications for the distribution of nutrients and dissolved gases [8, 9]. Deep water masses are mostly nutrient-rich owing to the processes like upwelling and mixing with organic matter sinking from the surface [10]. This nutrient-rich water supports diverse marine ecosystems in deep-sea environments [11]. Imperatively, deep water masses play a crucial role in the global carbon cycle by sequestering carbon dioxide from the atmosphere and transporting it to the deep ocean where it is in abundant [12, 13].

Additionally, deep water stratification is essential for studying climate patterns, ocean circulation, and the impact of climate change on the oceans [14, 15]. Scientists use various methods and instruments, including autonomous underwater vehicles (AUVs) and conductivity-temperature-depth (CTD) profilers, to measure temperature, salinity, and density profiles in the deep ocean and study stratification patterns [16]. Earth's atmosphere and oceans exhibit complex patterns of motion over a vast range of space and time. The deep water and its density gradients may strongly affect the hydrodynamics [17]. The Density stratification processes are therefore essential in geophysical flows, and equally they are key features in the biogeochemical mechanisms occurring in natural aquatic systems [18-21].

In the previous study [22], we worked on equations for deep water and developed model for waves moving both right and left but this model did not factor the effect of modified gravity. The Coriolis force always act at right angles to the direction of movement, which is to the right in the Northern Hemisphere and to the left in the Southern Hemisphere [23, 24].

To fully account for the description of the motion of mass conservation (continuity), the conservation of internal energy must also be added [15]. It can be expressed in terms of density or in terms of temperature and salinity. To make these motion more tractable and directly applicable to the flow circulation, various simplification and approximations are applied [16-18]. The simplifications are usually based on a scale analysis of the various terms in the equations. The vertical height or depth of the fluid layer is much larger than the horizontal scale of motion [25]. The Boussinesq approximations in which the density variations are assumed to be small compared to the mean value and are therefore neglected except in the buoyancy term of the equation [1, 19]. As result of this, the first approximation for the vertical component of the conservation of momentum can be reduced to a diagnostic equation for hydrostatic balance [9]. The second approximation, which is closely equivalent to assuming that sea water is incompressible, referring to, mass continuity can be reduced to a diagnostic equation for the conservation of volume [20-22]. The embedding of parameter in perturbation method and its application [26] is useful in the concept of series solution.

2. ASSUMPTIONS OF THE MODEL

We wish to formulate some necessary and basic assumptions to aid our model mathematically.

The following are the assumptions for the model:

i. The fluid is incompressible with continuous density stratification.

ii. In deep water flows the depth is infinite, so the vertical length scale (h) and the horizontal length scale (L) guarantee deep water regime when:

$\frac{L}{h} \ll 1(deep\ water\ assumption)$ but fails when $\frac{L}{h} \gg 1$

iii. The velocity components in the directions of increasing x, y and z will be denoted by u, v and w.

iv. We denoted depth-average velocity in the x direction as U=$u(x, y, t)$ and the depth-average velocity in the y- direction as $v = v\,(x; y; t\,)$. While the plane (z=0) can be chosen arbitrarily, it is usually positioned at mean water level.

v. Take the (x, y) horizontal plane as being parallel to the surface of the still water, and the depth of the water at a given point as h =(x, y, t)>0.

vi. Measuring down from this plane to the transition zone which is the thermocline, the point where the circular orbit of the deep water particles decrease at depth $z = -\zeta(x, y)$. The equation $z = -\zeta(x, y)$ is the equation for the bottom surface at which the diameter of the orbital path is zero at any instant. The interaction of deep water flow at this thermocline regime, which at this instant serves as the bottom condition is the layer of the ocean where the temperature changes most rapidly with varying depth.

vii. The Cartesian coordinates x, y and z will be used, in Cartesian coordinates for deep water waves, the z-axis typically measures the vertical direction, the x-axis is the horizontal direction, the y-axis represents another horizontal direction and the z-axis represents the vertical direction up-down.

viii. In deep water, the motion of water particle becomes circular as it moves from the perturbed surface through the strata.

ix. The variation in velocity along y- direction is constant because the flow is predominantly in horizontal direction so that partial derivative of velocity with respect to y is zero.

3. MATHEMATICAL FORMULATION OF THE PROBLEM

The formulation of stratified deep water can be done using various physical and mathematical principles. This is done by adopting conservation of momentum.

Equation to describe the principle of ocean dynamics and continuity equation to account for the movement of water masses with their impact on the vertical structure of the water column.

Figure 1 is the geometry of stratified deep water with circular pattern as it decays exponentially across the strata.

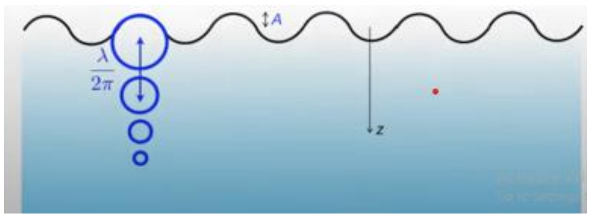

Figure 1. Exponential decay of deep water particles

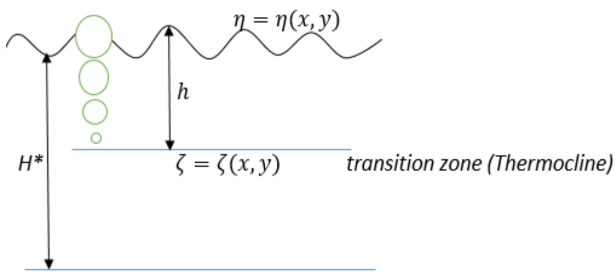

Figure 2. Symmetry of deep water decay at transition zone

Figure 2 is the geometry of deep water with infinite depth showing transition zone where the deep water is stratified under modified gravity.

Now we consider the dynamics of multiple layers of fluid stacked on top of each other in a deep water. This is a very old way for representation of continuous stratification, but it turns out to be a powerful model of many geophysical interesting phenomena. The pressure is continuous across the interface, but the density jumps discontinuously and this allows the horizontal velocity to have a corresponding discontinuity. In each layer pressure is given by the hydrostatic approximation, and so anywhere in the interior. We can find the pressure as many times as possible by integrating the hydrostatic approximation down from the top. This can be uniquely done by applying Boussinesq Equation. Thus, hydrostatic approximation is,

$$\frac{dp}{dz} = -gp$$

The multiple layers of stratified deep water stacked on top of each other,

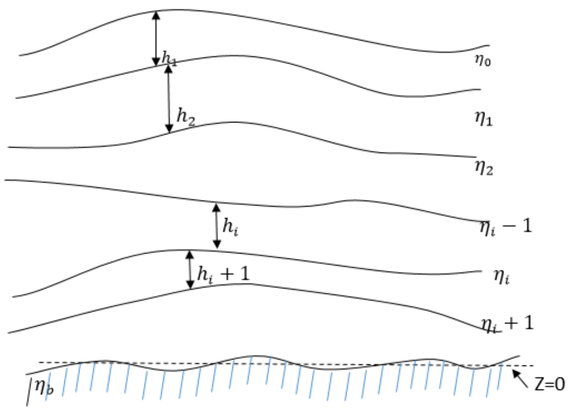

Figure 3. Multilayer system of stratified deep water

Figure 3 is the multilayer stratified deep water system depicting stacking of fluid in strata.

The layers are numbered from the top down. The coordinates of the interfaces are denoted by, and the layer thickness h_i.

$$\int dp = \int_{\eta_0}^{\eta_z} (-g\rho dz) \quad (1)$$

$$p_1 = -\rho_1 g(z - \eta_0)$$
$$p_1 = \rho_1 g(\eta_0 - z) \quad (2)$$

For the second layer, integrating from η_0 to η_1 and η_1 to z we have

$$p_2 = -\rho g \int_{\eta_0}^{\eta_1} dz - \rho g \int_{z}^{\eta_1} dz$$
$$p_2 = \rho_1 g(\eta_0 - \eta_1) + \rho_2 g(\eta_1 - z) = \rho_1 g\eta_0 - \rho_1 g'_1\eta_1 + \rho_2 g'_1\eta_1 - \rho_2 gz \quad (3)$$
$$p_2 = \rho_1 g\eta_0 + (\rho_2 - \rho_{1)} g'\eta_1 - \rho_2 gz$$

At $z = 0$, $p_2 gz = 0$, hence

$$p_2 = \rho_1 g\eta_0 + (\rho_2 - \rho_{1)} g'\eta_1$$

From Eq. (2), $p_1 = \rho_1 g\eta_0$ and

$$\rho_1 g'_1\eta_1 = g(\rho_2 - \rho_1)\eta_1 \quad (4)$$

From Eq. (4), we get

$$g'_1 = g\frac{(\rho_2 - \rho_1)}{\rho_1} = g\frac{\Delta\rho}{\bar{\rho}}$$

Similarly for nth layers, we have

$$g'_n = g\frac{(\rho_{n+1} - \rho_n)}{\rho_n} \quad (5)$$

Now for generality we can suppress n on g'_n to give g' and express

$$g' = g\frac{(\rho_{i+1} - \rho_i)}{\rho_i} \quad (6)$$

Eq. (6) is the general nth term for the expression of modified gravity.

And similarly for other levels. The term involving z is irrelevant for the dynamics, because only the horizontal motion is considered. Omitting this term, for the two layered model, the dynamical pressure is expressed as

$$p_1 = \rho_1 g\eta_0$$
$$p_2 = \rho_1 g\eta_0 + \rho_1 g'_1\eta_1 \quad (7)$$

η_1 can be assumed from the top down; hence
$\eta_0 = h_1 + h_2 + \eta_b$ and $\eta_1 = h_2 + \eta_b$

Therefore, the pressure in the two layers' system can be expressed as

$$p_1 = \rho_1 g\eta_0 = \rho_1 g(h_1 + h_2 + \eta_b)$$
$$p_2 = \rho_1 g\eta_0 + \rho_1 g'_1\eta_1 \quad (8)$$
$$= \rho_1 g(h_1 + h_2 + \eta_b) + \rho_1 g'_1(h_2 + \eta_b)$$

Finally, the mass conservation equation for each layer has the same form as the single-layer case, where η_b is the bottom elevation at thermocline regime, hence we have

$$\frac{Dh_n}{Dt} + h_n\nabla.u_n \quad (9)$$

Consider two layers of incompressible fluid under kinematic and dynamic boundary condition. We shall denote the upper layer 1 and the intermediate layer 2, with respective densities ρ_1 and ρ_2, the mean horizontal velocities u_1 and u_2 the thicknesses h_1 and h_2, with $h_1 + h_2 = H$, where H is the distance between the mean surface and the overall thermocline position. The pressure P is constant, the unknowns are therefore $h_1, h_2, u_1, u_2, \rho_1, \rho_2$, and all are functions of (x, t).

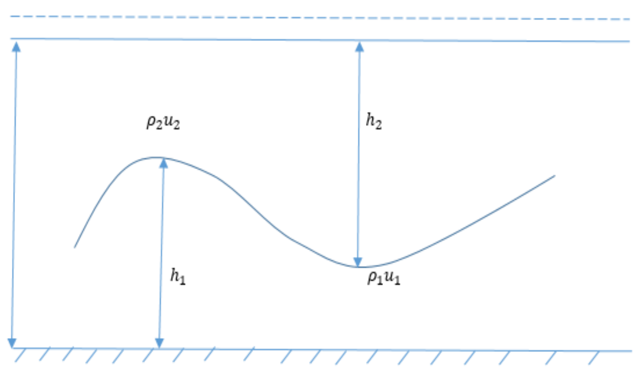

Figure 4. Two layers of stratified deep water

Figure 4 displays two layers of stratified deep water with continuous densities ρ_1, ρ_2 and horizontal velocities , u_1, u_2 with varying depths h_1 and h_2.

The corresponding equations describing the motion of the stratified deep water are:

$$\frac{\partial h_1}{\partial t}+\frac{\partial h_1 u_1}{\partial x}=0 \tag{10}$$

$$\frac{\partial h_2}{\partial t}+\frac{\partial h_2 u_2}{\partial x}=0 \tag{11}$$

$$(h_2u_2)_t+\left(h_2u_2^2+\frac{Ph_2}{\rho_2}+g\frac{P_2h_1h_2}{\rho_2}+g\frac{h_2^2}{2}\right)_x =-\frac{1}{\rho_2}(P+g\rho_2h_2)h_{2_x} \tag{12}$$

$$(h_1u_1)_t+\left(h_1u_1^2+\frac{Ph_1}{\rho_1}+g\rho_2h_2h_1+g\frac{h_1^2}{2}\right)_x =-\frac{1}{\rho_1}(P+g\rho_1h_1)h_{1_x} \tag{13}$$

Now if we incorporate the effect of gravity variation into the model at stratified condition then the model equations become:

$$\begin{aligned}
&\frac{\partial(h_1u_1)}{\partial t}+\frac{\partial\left(h_1u_1^2+\frac{g\frac{\rho_0-\rho_1}{\rho_1}h_1^2}{2}\right)}{\partial x} = -g\frac{\rho_1-\rho_0}{\rho_0}h_1\frac{\partial h_2}{\partial x}-g\frac{\rho_1-\rho_0}{\rho_0}h_1\frac{\partial(\xi)}{\partial x}\\
&+fu_1\frac{\partial(h_1v_1)}{\partial t}=-g\frac{\rho_1-\rho_0}{\rho_0}h_1\frac{\partial h_1}{\partial y}\\
&-g\frac{\rho_1-\rho_0}{\rho_0}h_1\frac{\partial(\xi)}{\partial x}+fu_1\frac{\partial h_1}{\partial t}+\frac{\partial(h_1u_1)}{\partial x}=0\\
&\frac{\partial(h_2u_2)}{\partial t}+\frac{\partial\left(h_2u_2^2+\frac{g\frac{\rho_2-\rho_1}{\rho_1}h_2^2}{2}+g\frac{\rho_2-\rho_1}{\rho_1}h_2h_1\right)}{\partial x} = -g\frac{\rho_1-\rho_2}{\rho_2}h_2\frac{\partial h_2}{\partial x}-g\frac{\rho_1-\rho_2}{\rho_2}h_2\frac{\partial(\xi)}{\partial x}\\
&+fu_2\frac{\partial(h_2v_2)}{\partial t}=-g\frac{\rho_2-\rho_1}{\rho_1}h_1\frac{\partial h_2}{\partial y}\\
&-g\frac{\rho_2-\rho_1}{\rho_1}h_2\frac{\partial(\xi)}{\partial y}+fu_2\frac{\partial h_2}{\partial t}+\frac{\partial(h_2u_2)}{\partial x}=0
\end{aligned} \tag{14}$$

4. ANALYSIS OF THE MODEL

Having obtained equations for the model which takes into account the necessary functions under modified gravity, the solution of the model equations can be provided using perturbation method. Now we consider the solutions of the deep water flow models using perturbation method (series solution).

The deep water flow model: With initial condition and boundary condition

$$\begin{aligned}
&u_1(x,0)=u_0, \quad u_x(0,y,t)=0\\
&u_x(l,t)=0, \quad u_2(x,0)=u_0\\
&u_2(x,t)=0, \quad u_x(x,t)=0\\
&u_{(2)x}(0,t)=0, \quad u_{2(x)}(l,t)=0
\end{aligned} \tag{15}$$

$h(x,0)=me^{-s(x^2)}-\xi(x)$, $h_x(u,t)$, where $\xi(x,y)=\beta sin(\alpha x)$ $0\le\alpha\le 90$, and the values of β and α depends on the size of the stratified layer at thermocline regime. Where α is the measure of strength t and β is the measure of stability of the system.

$$h(x,t=0)=me^{-s(x^2)}-\xi(x)$$

Consider the equation for total derivative of the system:

$$\begin{aligned}
&\frac{\partial h}{\partial t}+\frac{\partial(h_1u_1)}{\partial x}+\frac{\partial(h_2u_2)}{\partial x}+\frac{\partial(h_3u_3)}{\partial x}=0\\
&\frac{\partial h}{\partial t}+\frac{\partial(h_1u_1+h_2u_2+h_3u_3)}{\partial x}=0
\end{aligned} \tag{16}$$

That is,

$$\frac{dh_1}{dt}+h_1\frac{\partial(u_1)}{\partial x}=0 \tag{17}$$

$$\frac{dh_2}{dt}+h_2\frac{\partial(u_2)}{\partial x}=0 \tag{18}$$

$$\frac{dh_3}{dt}+h_3\frac{\partial(u_3)}{\partial x}=0 \tag{19}$$

Therefore,

$$\frac{dh_1}{dt}=-h_1\frac{\partial(u_1)}{\partial x} \tag{20}$$

$$\frac{dh_2}{dt}=-h_2\frac{\partial(u_2)}{\partial x} \tag{21}$$

$$\frac{dh_3}{dt}=-h_3\frac{\partial(u_3)}{\partial x} \tag{22}$$

Note that if $f=(x,y,t)\Rightarrow f=(x,t)$, since the variation in velocity in y- direction is very small and then neglected.

By product rule, we have:

$$\frac{df}{dt}=\frac{\partial f}{\partial t}.\frac{dt}{dt}+\frac{\partial f}{\partial x}.\frac{dx}{dt}+\frac{\partial f}{\partial y}.\frac{dy}{dt}$$

Now, we can rewrite the model equation as,

$$\begin{aligned}
&\frac{\partial(h_1u_1)}{\partial t}+\frac{\partial\left(h_1u_1^2+g(\frac{\rho_1-\rho_0}{\rho_0})h_1^2/2\right)}{\partial x} =-g\frac{\rho_1-\rho_0}{\rho_0}\frac{\partial\xi}{\partial x}+fu_1\\
&u_1\frac{\partial h_1}{\partial t}+h_1\frac{\partial u_1}{\partial t}+u_1^2\frac{\partial h_1}{\partial x}+2h_1u_1\frac{\partial u}{\partial x}+g\frac{\rho_1-\rho_0}{\rho_0}h_1\frac{\partial h_1}{\partial x}= =-g\frac{\rho_1-\rho_0}{\rho_0}\frac{\partial\xi}{\partial x}+fu_1
\end{aligned}$$

This is further simplified as:

$$u_1\left(\frac{\partial h_1}{\partial t}+u_1\frac{\partial h_1}{\partial x}\right)+h_1\left(\frac{\partial u_1}{\partial t}+2u_1\frac{\partial u_1}{\partial x}\right)+ g\frac{(\rho_1-\rho_0)}{\rho_0}h_1\frac{\partial h_1}{\partial x}=-g\frac{(\rho_0-\rho_1)}{\rho_1}\frac{\partial\xi}{\partial x}+fu_1 \tag{23}$$

From Eq. (23), recall the continuity equation for each of the stratification.

From the first stratification level,

$$\frac{\partial}{\partial t}(h_1) + \frac{\partial}{\partial x}(h_1 u_1) = 0$$
$$\Rightarrow \frac{\partial}{\partial t}(h_1) + h_1\frac{\partial}{\partial x}(u_1) + u_1\frac{\partial (h_1)}{\partial x} = 0 \quad (24)$$
$$\frac{\partial}{\partial t}(h_1) + u_1\frac{\partial (h_1)}{\partial x} = -h_1\frac{\partial}{\partial x}(u_1)$$

Substituting Eq. (24) into (23), we get

$$\begin{aligned}-u_1 h_1\frac{\partial u_1}{\partial t} + h_1\frac{\partial}{\partial x}(u_1) + 2u_1h_1\frac{\partial u_1}{\partial x}\\ + gh_1\frac{\rho_1-\rho_0}{\rho_0}\frac{\partial h_1}{\partial x}\\ = -gh_1\left(\frac{\rho_1-\rho_0}{\rho_0}\right)\frac{\partial \xi}{\partial x} + fu_1h_1\left(\frac{\partial}{\partial t}u_1 + u_1\frac{\partial}{\partial x}u_1\right)\\ + gh_1\left(\frac{\rho_1-\rho_0}{\rho_0}\right)\frac{\partial h_1}{\partial x}\\ = -g\left(\frac{\rho_1-\rho_0}{\rho_0}\right)\frac{\partial \xi}{\partial x} + fu_1\end{aligned} \quad (25)$$

From Eq. (25), $\left(\frac{\partial}{\partial t}u_1 + u_1\frac{\partial}{\partial x}u_1\right) = \frac{du_1}{dt}$
Eq. (25) becomes;

$$h_1\frac{du_1}{dt} + gh_1\left(\frac{\rho_1-\rho_0}{\rho_0}\right)\frac{\partial h_1}{\partial x} = -g\left(\frac{\rho_1-\rho_0}{\rho_0}\right)\frac{\partial \xi}{\partial x} + fu_1$$

Using Eq. (17) in Eq. (23), yields

$$\begin{aligned}h_1\frac{du_1}{dt} + g\frac{(\rho_1-\rho_0)}{\rho_0}h_1\frac{\partial h_1}{\partial x}\\ = -g\frac{(\rho_1-\rho_0)}{\rho_0}\frac{\partial \xi}{\partial x} + fu_1\end{aligned} \quad (26)$$

Similarly, Eq. (18) in Eq. (23) can be expressed as

$$\begin{aligned}u_2\left(\frac{\partial h_2}{\partial t} + u_2\frac{\partial h_2}{\partial x}\right) + h_2\left(\frac{\partial u_2}{\partial t} + 2u_2\frac{\partial u_2}{\partial x}\right)\\ +g\frac{(\rho_2-\rho_1)}{\rho_1}h_2\frac{\partial h_2}{\partial x} = -g\frac{(\rho_2-\rho_1)}{\rho_1}\frac{\partial \xi}{\partial x} + fu_2\end{aligned} \quad (27)$$

Using Eq. (18) in Eq. (27), we obtain

$$h_2\frac{du_2}{dt} + g\frac{(\rho_2-\rho_1)}{\rho_1}h_2\frac{\partial h_2}{\partial x} = -g\frac{(\rho_2-\rho_1)}{\rho_1}\frac{\partial \xi}{\partial x} + fu_2 \quad (28)$$

Again, Eq. (19) in Eq. (23) can be written as

$$\begin{aligned}u_3\left(\frac{\partial h_3}{\partial t} + u_3\frac{\partial h_3}{\partial x}\right) + h_3\left(\frac{\partial u_3}{\partial t} + 2u_3\frac{\partial u_3}{\partial x}\right)\\ +g\frac{(\rho_3-\rho_2)}{\rho_2}h_3\frac{\partial h_3}{\partial x} = -g\frac{(\rho_3-\rho_2)}{\rho_2}\frac{\partial \xi}{\partial x} + fu_3\\ h_3\frac{du_3}{dt} + g\frac{(\rho_2-\rho_1)}{\rho_1}h_3\frac{\partial h_3}{\partial x} = -g\frac{(\rho_2-\rho_1)}{\rho_1}\frac{\partial \xi}{\partial x} + fu_3\end{aligned} \quad (29)$$

Now let $0 < f \ll 1, and\ g = g\frac{(\rho_{i+1}-\rho_i)}{\rho_i} = af.$

The perturbation method (series solution) is a technique used to solve partial differential equations (PDEs) by assuming a solution in the form of an infinite series. By applying this method to solve the PDE of deep water waves, the Coriolis parameter is included on the left-hand side of the equation to account for the effect of rotation on the wave propagation. The inclusion of the Coriolis parameter is important because deep water waves can propagate in the presence of rotation, and this rotation can affect the wave speed and direction. Again, including the Coriolis parameter f on the left-hand side of the equation, the effect of rotation is taken into account, and the solution obtained will be more accurate. Since the Coriolis parameter is approximately 7.29×10^{-5}/s, Gaspard de Coriolis, (1835) which is $\ll 1$, we can equally write the series expansion of the velocities and height in terms of f.

Suppose the solution (u_1, u_2, h_1, h_2)

$$\left.\begin{aligned}u_1(x,t) = u_0(x,t) + fu_1(x,t) + \cdots\\ u_2(x,t) = u_0(x,t) + fu_2(x,t) + \cdots\\ u_3(x,t) = u_0(x,t) + fu_3(x,t) + \cdots\\ h_1(x,t) = h_0(x,t) + fh_1(x,t) + \cdots\\ h_2(x,t) = h_0(x,t) + fh_2(x,t) + \cdots\end{aligned}\right\} \quad (30)$$

Substituting Eq. (30) in Eqs. (17)-(19), we get

$$\left.\begin{aligned}\frac{d}{dt}(h_0 + fh_1 + \cdots) + (h_0 + fh_1 + \cdots)\\ \left(\frac{\partial}{\partial x}(u_0 + fu_1 + \cdots) + \frac{\partial}{\partial x}(u_0 + fu_1 + \cdots)\right) = 0\\ \frac{d}{dt}(h_0 + fh_2 + \cdots) + (h_0 + fh_2 + \cdots)\\ \left(\frac{\partial}{\partial x}(u_0 + fu_1 + \cdots) + \frac{\partial}{\partial x}(u_0 + fu_2 + \cdots)\right) = 0\end{aligned}\right\} \quad (31)$$

Eq. (31) gives

$$\left.\begin{aligned}\frac{d}{dt}h_0 + \frac{d}{dt}fh_1 + \cdots\\ +h_0\frac{\partial}{\partial x}u_0 + h_0\frac{\partial}{\partial x}fu_1 + fh_1\frac{\partial}{\partial x}u_0\\ +f^2h_1\frac{\partial}{\partial x}fu_1 + \cdots += 0\\ \frac{d}{dt}h_0 + \frac{d}{dt}fh_2 + \cdots\\ +h_0\frac{\partial}{\partial x}u_0 + h_0\frac{\partial}{\partial x}fu_2 + fh_2\frac{\partial}{\partial x}u_0\\ +f^2h_2\frac{\partial}{\partial x}fu_2 + \cdots += 0\end{aligned}\right\} \quad (32)$$

Substituting Eq. (30) in Eq. (26), we get,

$$\begin{aligned}(h_0 + fh_1 + \cdots)\frac{d}{dt}(u_0 + fu_1 + \cdots)\\ +af(h_0 + fh_1 + \cdots)\frac{\partial}{\partial x}(h_0 + fh_1 + \cdots\\ = -af\frac{\partial \xi}{\partial x} + f(u_0 + fu_1 + \cdots)\\ h_0\frac{d}{dt}u_0 + fh_0\frac{d}{dt}u_1 + fh_1\frac{d}{dt}u_0 + f^2h_1\frac{d}{dt}u_1 + \cdots\\ +afh_0\frac{\partial}{\partial x}h_0 + af^2h_0\frac{\partial}{\partial x}h_1 +\\ af^2h_1\frac{\partial}{\partial x}h_0 + af^3h_1\frac{\partial}{\partial x}h_1 + \cdots\\ += -af\frac{\partial \xi}{\partial x} + fu_0 + f^2u_1 + \cdots\end{aligned} \quad (33)$$

Substituting Eq. (30) in Eq. (28), we obtain

$$(h_0 + fh_1 + \cdots)\frac{d}{dt}(u_0 + fu_2 + \cdots) + af(h_0 + fh_2 + \cdots)\frac{\partial}{\partial x}\begin{pmatrix} h_0 \\ +fh_2 + \cdots \end{pmatrix} = -af\frac{\partial \xi}{\partial x} + f\begin{pmatrix} u_0 \\ +fu_1 + \cdots \end{pmatrix}$$

$$h_0\frac{d}{dt}u_0 + fh_0\frac{d}{dt}u_2 + fh_2\frac{d}{dt}u_0 + f^2h_2\frac{d}{dt}u_2 + \cdots + afh_0\frac{\partial}{\partial x}h_0 + af^2h_0\frac{\partial}{\partial x}h_2 + af^2h_2\frac{\partial}{\partial x}h_0 + af^3h_2\frac{\partial}{\partial x}h_2 + \cdots += -af\frac{\partial \xi}{\partial x} + fu_0 + f^2u_2 + \cdots \tag{34}$$

Substituting Eq. (30) in Eq. (29), we obtain:

$$h_0\frac{d}{dt}u_0 + fh_0\frac{d}{dt}u_3 + fh_3\frac{d}{dt}u_0 + f^2h_3\frac{d}{dt}u_3 + \cdots + afh_0\frac{\partial}{\partial x}h_0 + af^2h_0\frac{\partial}{\partial x}h_3 + af^2h_3\frac{\partial}{\partial x}h_0 + af^3h_3\frac{\partial}{\partial x}h_{3+\cdots+} = -af\frac{\partial \xi}{\partial x} + fu_0 + f^2u_2 + \tag{35}$$

Comparing the coefficient of powers of f from Eqs. (33)-(35), we have:

$$f^0: h_0\frac{du_0}{dt} = 0; \qquad u_1(x,0) = u_1 \tag{36}$$

$$h_0\frac{du_0}{dt} = 0; \qquad u_2(x,0) = u_2 \tag{37}$$

$$h_0\frac{du_0}{dt} = 0; \qquad u_3(x,0) = u_3 \tag{38}$$

$$\begin{aligned} \frac{dh_0}{dt} &= -h_0\left(\frac{\partial u_1}{\partial x}\right);\ h_0(x,0) = me^{-s(x^2)} - \xi(x,0) \\ \frac{dh_0}{dt} &= -h_0\left(\frac{\partial u_2}{\partial x}\right);\ h_0(x,0) = me^{-s(x^2)} - \xi(x,0) \\ \frac{dh_0}{dt} &= -h_0\left(\frac{\partial u_3}{\partial x}\right);\ h_0(x,0) = me^{-s(x^2)} - \xi(x,0) \end{aligned} \tag{39}$$

f^1:

$$\left.\begin{aligned} h_0\frac{du_1}{dt} + h_1\frac{du_0}{dt} + ah_0\frac{\partial h_0}{\partial x} &= -a\frac{\partial \xi}{\partial x} + u_0; \\ u_1(x,0) &= 0 \\ h_0\frac{du_2}{dt} + h_2\frac{du_0}{dt} + ah_0\frac{\partial h_0}{\partial x} &= -a\frac{\partial \xi}{\partial x} + u_0; \\ u_2(x,0) &= 0 \end{aligned}\right\} \tag{40}$$

$$\frac{dh_1}{dt} + h_0\frac{\partial u_1}{\partial x} + h_1\frac{\partial u_0}{\partial x} = 0 \tag{41}$$

$$\frac{dh_2}{dt} + h_0\frac{\partial u_2}{\partial x} + h_2\frac{\partial u_0}{\partial x} = 0 \tag{42}$$

f^2:

$$\left.\begin{aligned} h_1\frac{d}{dt}u_1 + ah_0\frac{\partial}{\partial x}h_1 + ah_1\frac{\partial}{\partial x}h_0 &= u_1; \\ u_1(x,0) &= 0 \\ h_2\frac{d}{dt}u_2 + ah_0\frac{\partial}{\partial x}h_2 + ah_2\frac{\partial}{\partial x}h_0 &= u_2; \\ u_2(x,0) &= 0 \end{aligned}\right\} \tag{43}$$

Now integrating Eq. (36) with respect to t, we have:

$$\begin{aligned} u_0(x,t) &= c_1 = const\ ant \\ u_0(x,0) &= c_1 = u_0 \\ u_0(x,t) &= u_0 \end{aligned} \tag{44}$$

Integrating Eq. (37) with respect to t, we have:

$$u_1(x,t) = c_2 = constant$$

$$u_1(x,0) = c_2 = u_1$$

Then

$$u_1(x,t) = u_1 \tag{45}$$

By using Eqs. (44) and (45), Eq. (37) reduces to

$$\frac{dh_0}{dt} = -h_0(0) = 0, \frac{dh_0}{dt} = 0,$$
$$h_0(x,0) = d - \xi(x,0) = me^{-s(x^2)} - \beta\sin(\alpha x) = me^{-sx^2} - \beta\sin(\alpha x)$$

Integrating with respect to, gives:

$$\begin{aligned} h_0(x,t) &= c_3 = constant \\ h_0(x,0) &= c_3 = me^{-s\,(x^2)} - \xi(x,0) \\ h_0(x,t) &= me^{-sx^2} - \beta\sin(\alpha x) \end{aligned} \tag{46}$$

Using Eqs. (45)-(47), Eq. (39) reduces to

$$\frac{du_1}{dt} = \frac{-h_1\frac{du_0}{dt} - ah_0\frac{\partial h_0}{\partial x} - a\frac{\partial \xi}{\partial x} + u_0}{h_0} = -\frac{h_1}{h_0}\frac{du_0}{dt} - a\frac{\partial h_0}{\partial x} - \frac{a}{h_0}\frac{\partial \xi}{\partial x} + \frac{u_0}{h_0}$$

$$\frac{du_1}{dt} = 2g\frac{\rho_1 - \rho_0}{\rho_0}msxe^{-s(x^2)} + g\frac{\rho_1 - \rho_0}{\rho_0}\alpha\beta cos\alpha x - \frac{g\frac{\rho_1 - \rho_0}{\rho_0}\alpha\beta cos\alpha x}{me^{-sx^2} - \beta\sin\alpha x} + \frac{u_0}{me^{-sx^2} - \beta\sin\alpha x}$$

Integrating with respect to t, we have

$$u_1 = \left(2g\frac{\rho_1 - \rho_0}{\rho_0}msxe^{-s(x^2)} + g\frac{\rho_1 - \rho_0}{\rho_0}\alpha\beta cos\alpha x - \frac{g\frac{\rho_1 - \rho_0}{\rho_0}\alpha\beta cos\alpha x}{me^{-sx^2} - \beta\sin\alpha x} + \frac{u_0}{me^{-sx^2} - \beta\sin\alpha x}\right)t + c_4$$

But $u_1(x,0)=0 \Longrightarrow c_4=0$

$$u_1(x,t) = +2gt\frac{\rho_1-\rho_0}{\rho_0}msxe^{-sx^2} +gt\frac{\rho_1-\rho_0}{\rho_0}\alpha\beta t cos\alpha x - t\frac{g\frac{\rho_1-\rho_0}{\rho_0}\alpha\beta cos\alpha x}{me^{-sx^2}-\beta\sin\alpha x} +t\frac{u_0}{me^{-sx^2}-\beta\sin\alpha x} \quad (47)$$

Eq. (47) is the speed of the stratified deep water at the first layer.

Using Eqs. (45)-(47), Eq. (39) reduces to

$$\frac{du_2}{dt} = \frac{-h_2\frac{du_0}{dt} - ah_0\frac{\partial h_0}{\partial x} - a\frac{\partial \xi}{\partial x} + u_0}{h_0} = -\frac{h_2}{h_0}\frac{du_0}{dt} - a\frac{\partial h_0}{\partial x} - \frac{a}{h_0}\frac{\partial \xi}{\partial x} + \frac{u_0}{h_0}$$

$$\frac{du_2}{dt} = 2g\frac{\rho_2-\rho_1}{\rho_1}msxe^{-s(x^2)} +g\frac{\rho_2-\rho_1}{\rho_1}\alpha\beta cos\alpha x - \frac{g\frac{\rho_2-\rho_1}{\rho_1}\alpha\beta cos\alpha x}{me^{-sx^2}-\beta\sin\alpha x} + \frac{u_0}{me^{-sx^2}-\beta\sin\alpha x}$$

Integrating with respect to t, we get

$$u_2 = \left(2g\frac{\rho_2-\rho_1}{\rho_1}msxe^{-s(x^2)} + g\frac{\rho_2-\rho_1}{\rho_1}\alpha\beta cos\alpha x - \frac{g\frac{\rho_2-\rho_1}{\rho_1}\alpha\beta cos\alpha x}{me^{-sx^2}-\beta\sin\alpha x} + \frac{u_0}{me^{-sx^2}-\beta\sin\alpha x}\right)t + c_5$$

But $u_2(x,0)=0 \Rightarrow c_5=0$

$$u_2(x,t) = +2gt\frac{\rho_2-\rho_1}{\rho_1}msxe^{-sx^2} +gt\frac{\rho_2-\rho_1}{\rho_1}\alpha\beta t cos\alpha x - t\frac{g\frac{\rho_2-\rho_1}{\rho_1}\alpha\beta cos\alpha x}{me^{-sx^2}-\beta\sin\alpha x} +t\frac{u_0}{me^{-sx^2}-\beta\sin\alpha x} \quad (48)$$

Now using Eqs. (45)-(48), Eq. (41) becomes:

$$\frac{dh_1}{dt} = -h_0\left(\frac{\partial u_1}{\partial x}\right) - h_1\left(\frac{\partial u_0}{\partial x}\right)$$

That is

$$\frac{dh_1}{dt} = -h_0\left(\frac{\partial u_1}{\partial x}\right)$$

This implies that,

$$\frac{\partial u_1}{\partial x} = \left(\frac{\begin{gathered}2amsxe^{-sx^2} - 4ams^2x^2e^{-sx^2} \\ +a\alpha^2\beta\sin(\alpha x) - \\ m\alpha\beta e^{-sx^2}(2cos\alpha x - sin\alpha x) + a\alpha^2\beta^2(\sin^2\alpha x + \cos^2\alpha x)\end{gathered}}{\left(me^{-sx^2}-\beta sin\alpha x\right)^2} + u_0\frac{(2mxe^{-sx^2}+\alpha\beta cos\alpha x)}{\left(me^{-sx^2}-\beta sin\alpha x\right)^2}\right)t$$

$$\frac{dh_1}{dt} = -\left(me^{-sx^2}-\beta sin\alpha x\right)\left(\frac{\begin{gathered}2amsxe^{-sx^2} - 4ams^2x^2e^{-sx^2} + \\ a\alpha^2\beta\sin(\alpha x) - \\ m\alpha\beta e^{-sx^2}(2cos\alpha x - sin\alpha x) \\ +a\alpha^2\beta^2(\sin^2\alpha x + \cos^2\alpha x)\end{gathered}}{\left(me^{-sx^2}-\beta sin\alpha x\right)^2} + u_0\frac{(2mxe^{-sx^2}+\alpha\beta cos\alpha x)}{\left(me^{-sx^2}-\beta sin\alpha x\right)^2}\right)t$$

Integrating with respect to t, we get

$$h_1 = -\left(me^{-sx^2}-\beta sin\alpha x\right)\left(\frac{\begin{gathered}2g\frac{\rho_1-\rho_0}{\rho_0}msxe^{-sx^2} - 4g\frac{\rho_1-\rho_0}{\rho_0}ms^2x^2e^{-sx^2} \\ +g\frac{\rho_1-\rho_0}{\rho_0}\alpha^2\beta\sin(\alpha x) - \\ mg\frac{\rho_1-\rho_0}{\rho_0}\beta e^{-sx^2}(2cos\alpha x - sin\alpha x) + a\alpha^2\beta^2\end{gathered}}{\left(me^{-sx^2}-\beta sin\alpha x\right)^2} + u_0\frac{(2mxe^{-sx^2}+\alpha\beta cos\alpha x)}{\left(me^{-sx^2}-\beta sin\alpha x\right)^2}\right)t^2/2 + c_7$$

Since $\sin^2\alpha x + \cos^2\alpha x = 1$

$$\begin{aligned}\therefore h_1(x,t) = &-t^2g\frac{\rho_1-\rho_0}{\rho_0}m^2sxe^{-2sx^2} \\ &+t^2\beta sin\alpha x g\frac{\rho_1-\rho_0}{\rho_0}msxe^{-sx^2} \\ &+2t^2g\frac{\rho_1-\rho_0}{\rho_0}m^2s^2x^2e^{-2sx^2} \\ &-2t^2\beta\sin(\alpha x)\,g\frac{\rho_1-\rho_0}{\rho_0}ms^2x^2e^{-sx^2} \\ &-\frac{1}{2}t^2me^{-sx^2}g\frac{\rho_1-\rho_0}{\rho_0}\alpha^2\beta\sin(\alpha x) \\ &+\frac{1}{2}t^2g\frac{\rho_1-\rho_0}{\rho_0}\alpha^2\beta^2\sin^2(\alpha x) \\ &+\frac{1}{2}t^2\frac{m^2\alpha\beta e^{-2sx^2}(2cos\alpha x - sin\alpha x) + me^{-sx^2}g\frac{\rho_1-\rho_0}{\rho_0}\alpha^2\beta^2}{\left(me^{-sx^2}-\beta sin\alpha x\right)^2} \\ &-\frac{1}{2}t^2\frac{m\alpha\beta^2\sin(\alpha x)\,e^{-sx^2}(2cos\alpha x - sin\alpha x) + g\frac{\rho_1-\rho_0}{\rho_0}\alpha^2\beta^3\sin(\alpha x)}{\left(me^{-sx^2}-\beta sin\alpha x\right)^2} \\ &-\frac{1}{2}t^2u_0me^{-sx^2}\frac{(2mxe^{-sx^2}+\alpha\beta cos\alpha x)}{\left(me^{-sx^2}-\beta sin\alpha x\right)^2} \\ &+\frac{1}{2}t^2u_{0\beta\sin(\alpha x)}\frac{(2mxe^{-sx^2}+\alpha\beta cos\alpha x)}{\left(me^{-sx^2}-\beta sin\alpha x\right)^2}\end{aligned} \quad (49)$$

Eq. (49) is the perturbed height of the first stratified deep water column in series form.

From Eq. (41):

$$\frac{dh_2}{dt} + h_0\frac{\partial u_2}{\partial x} + h_2\frac{\partial u_0}{\partial x} = 0; \frac{dh_2}{dt} = -h_0\left(\frac{\partial u_2}{\partial x}\right)$$

So

$$\frac{\partial u_2}{\partial x} = \left(\begin{array}{c} \dfrac{\begin{array}{c} 2amse^{-sx^2} - 4ams^2e^{-sx^2} \\ +a\alpha^2\beta\sin(\alpha x) - \\ m\alpha\beta e^{-sx^2}(2cos\alpha x - sin\alpha x) \\ +a\alpha^2\beta^2(\sin^2\alpha x + \cos^2\alpha x) \end{array}}{\left(me^{-sx^2} - \beta sin\alpha x\right)^2} \\ +u_0\dfrac{(2mse^{-sx^2} + \alpha\beta cos\alpha x)}{\left(me^{-sx^2} - \beta sin\alpha x\right)^2} \end{array}\right) t$$

$$\frac{dh_2}{dt} = -\left(me^{-sx^2} - \beta sin\alpha x\right)\left(\begin{array}{c} \dfrac{\begin{array}{c} 2amsxe^{-sx^2} - 4ams^2x^2e^{-sx^2} \\ +a\alpha^2\beta\sin(\alpha x) - \\ m\alpha\beta e^{-sx^2}(2cos\alpha x - sin\alpha x) \\ +a\alpha^2\beta^2(\sin^2\alpha x + \cos^2\alpha x) \end{array}}{\left(me^{-sx^2} - \beta sin\alpha x\right)^2} \\ +u_0\dfrac{(2mxe^{-sx^2} + \alpha\beta cos\alpha x)}{\left(me^{-sx^2} - \beta sin\alpha x\right)^2} \end{array}\right) t$$

Integrating with respect to *t*, we get

$$h_2 = -\left(me^{-sx^2} - \beta sin\alpha x\right) \left(\begin{array}{c} \dfrac{\begin{array}{c} 2amsxe^{-sx^2} - 4ams^2x^2e^{-sx^2} \\ +a\alpha^2\beta\sin(\alpha x) - \\ m\alpha\beta e^{-sx^2}(2cos\alpha x - sin\alpha x) + a\alpha^2\beta^2 \end{array}}{\left(me^{-sx^2} - \beta sin\alpha x\right)^2} \\ +u_0\dfrac{(2mxe^{-sx^2} + \alpha\beta cos\alpha x)}{\left(me^{-sx^2} - \beta sin\alpha x\right)^2} \end{array}\right) {t^2}/{2} + c_8$$

Since $\sin^2\alpha x + \cos^2\alpha x = 1$

$$\begin{aligned} \therefore h_2(t,x) = & -t^2 g\frac{\rho_2 - \rho_1}{\rho_1}m^2sxe^{-2sx^2} \\ & +t^2\beta sin\alpha x g\frac{\rho_2 - \rho_1}{\rho_1}msxe^{-sx^2} \\ & +2t^2 g\frac{\rho_2 - \rho_1}{\rho_1}m^2s^2x^2e^{-2sx^2} \\ & -2t^2\beta\sin(\alpha x)\, g\frac{\rho_2 - \rho_1}{\rho_1}ms^2x^2e^{-sx^2} \\ & -\frac{1}{2}t^2me^{-sx^2}g\frac{\rho_2 - \rho_1}{\rho_1}\alpha^2\beta\sin(\alpha x) \\ & +\frac{1}{2}t^2 g\frac{\rho_2 - \rho_1}{\rho_1}\alpha^2\beta^2\sin^2(\alpha x) \\ & +\frac{1}{2}t^2\frac{\begin{array}{c} m^2\alpha\beta e^{-2sx^2}(2cos\alpha x - sin\alpha x) \\ +me^{-sx^2}g\frac{\rho_2 - \rho_1}{\rho_1}\alpha^2\beta^2 \end{array}}{\left(me^{-sx^2} - \beta sin\alpha x\right)^2} \\ & -\frac{1}{2}t^2\frac{\begin{array}{c} m\alpha\beta^2\sin(\alpha x)\, e^{-sx^2}(2cos\alpha x - sin\alpha x) \\ +g\frac{\rho_2 - \rho_1}{\rho_1}\alpha^2\beta^3\sin(\alpha x) \end{array}}{\left(me^{-sx^2} - \beta sin\alpha x\right)^2} \\ & -\frac{1}{2}t^2u_0me^{-sx^2}\frac{(2mxe^{-sx^2} + \alpha\beta cos\alpha x)}{\left(me^{-sx^2} - \beta sin\alpha x\right)^2} \\ & +\frac{1}{2}t^2u_{0\beta\sin(\alpha x)}\frac{(2mxe^{-sx^2} + \alpha\beta cos\alpha x)}{\left(me^{-sx^2} - \beta sin\alpha x\right)^2} \end{aligned} \quad (50)$$

Eq. (50) is the series solution of the height for the second column**.**

6. RESULTS AND DISCUSSION

Figure 5 shows the effect of density on stratification. At density of 1000 kg/cm^3 the deep water is uniformly stratified. At 1037.13 kg/cm^3 the deep water is dense and stratified. At density of 1090 kg/cm^3 the deep water is denser and more stratified than at densities of 1000 kg/cm^3, 1037.13 kg/cm^3 and 1069 kg/cm^3. Higher density causes more stratification in deep water since stratification refers to the layering of water masses with different densities.

At $\beta = 0.05$, the deep water is less stable with speed of about (1.5-3.0 m/s). At $\beta = 0.1$ the stratified deep water is slightly stable than at $\beta = 0.05$ with speed of (2.0-3.0 m/s). At $\beta = 0.3$ the stability of the stratified deep water at second layer increases at speed (2.5-4.0 m/s) as shown in Figure 6.

At $\beta = 0.05$ for the first layer, stratification occurred at depth of 0.150m and time of 0.50 secs. At $\beta = 0.3$, the deep water is stratified at depth of 10m and time of 0.50 secs. At $\beta = 0.1$ for the second layer, the deep water is stratified at depth of 0.100 m and time 1.25 secs. At $\beta = 0.5$, the stratification in the second layer occurred at temperature that is below freezing. Figure 7 shows the variation of depth under modified gravity with different values of beta which were the measure of the stability of the system. In stratified deep water under modified gravity the larger beta value indicates a more stable system, which means the system is less likely to undergo significant changes when perturbed during stratification process. A smaller beta value indicates a less stable system which is likely susceptible to perturbations. The depth of the first stratified layer is influenced by the wave's interaction with the water surface as shown with the simulation.

At varied values of *α*=0.6283185307179586, 1.5707963267948966, 3.141592653589793 and 6.283185307179586 the strength of stratification increases and more layering occurred respectively.

Figure 8 shows how variation in alpha values which is the measure of strength of the system affects deep water stratification under modified gravity.

Figure 9 shows the oscillatory nature of the stratified deep water with unsteady amplitude which goes far to show that deep water is never stable. Figure 9(a) shows how velocity affect the amplitude of stratified deep water under modified gravity. Figure 9(b) shows the impact of depth on the amplitude of stratified deep water under modified gravity.

In Figure 10, at *α*=0.3141592653589793, the amplitude critically dampened. At *α*=1.5707963267948966 the amplitude rises with speed 0.06 m/s. When *α*=2.199114857512855 the speed rises above 0.06 m/s and system acquired maximum amplitude for the oscillation. Therefore, the higher value of α which is the strength of the system then the higher the amplitude of oscillation. Similarly, at $\beta = 0.05$ and $\beta = 0.1$ the speed of the oscillatory wave motion is far below 0.25 m/s. At $\beta = 0.3$, the speed rises above 0.25 m/s to 0.50 m/s. At $\beta = 0.5$, the speed rises from above 0.5 m/s to 1.00 m/s. This shows that increase in stability of the system increases the speed and amplitude of the wave in deep water.

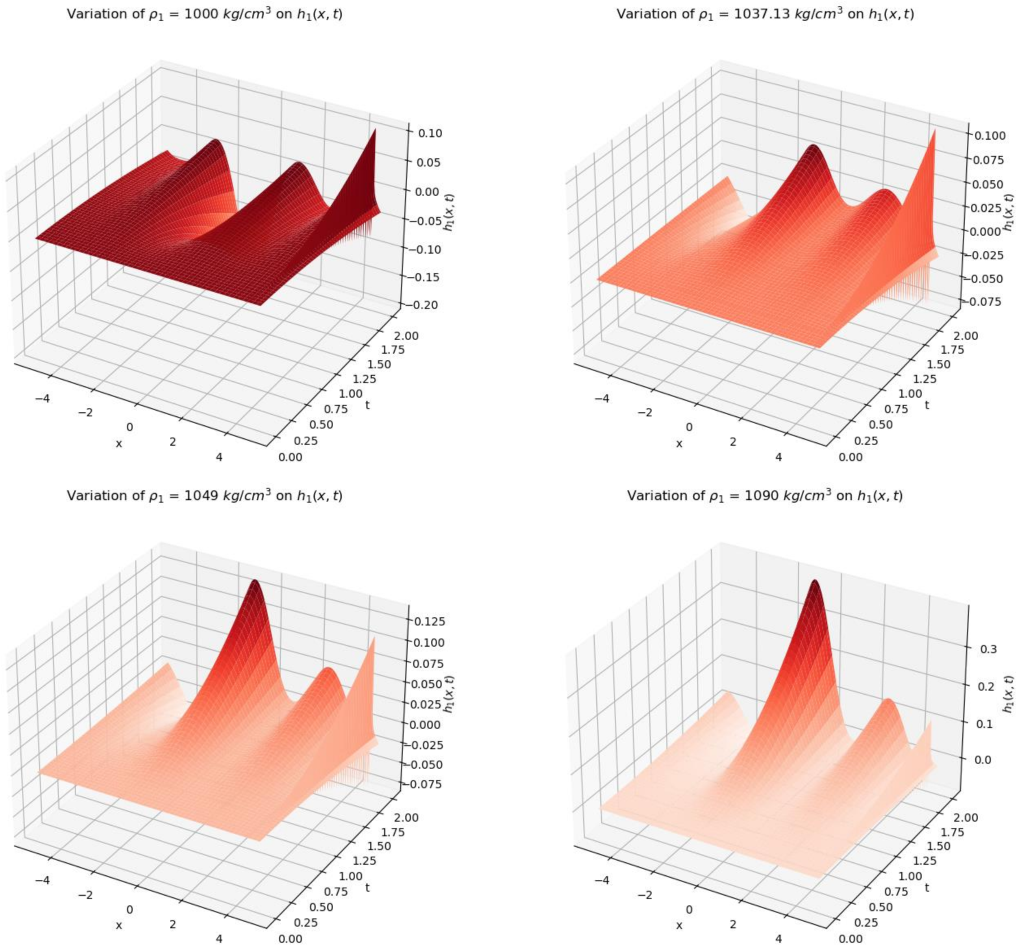

Figure 5. Effect of density on stratification/density variation with depth

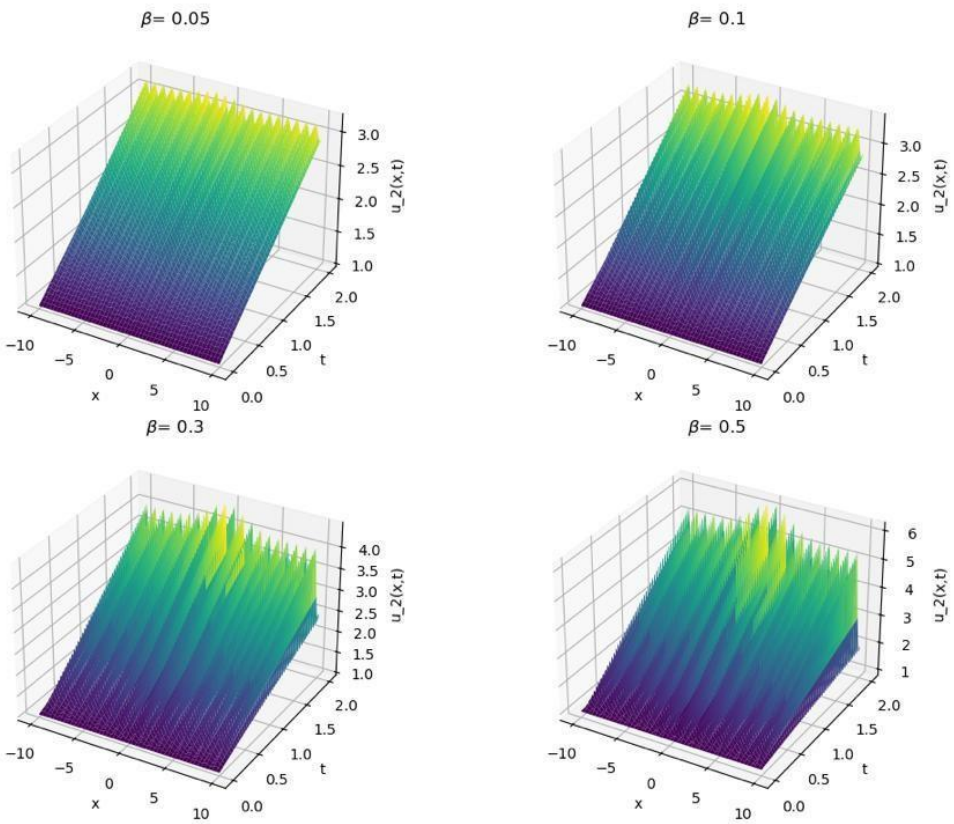

Figure 6. Effect of salinity and temperature on stratification

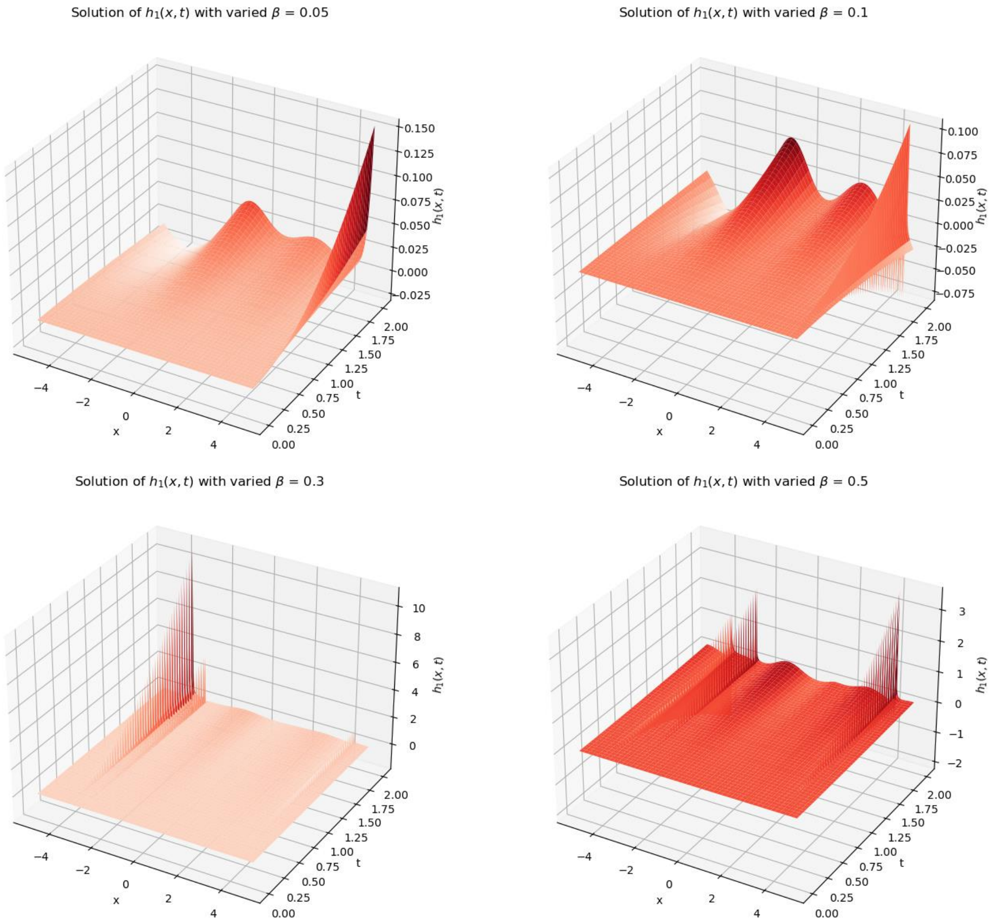

Figure 7. Variation of depth

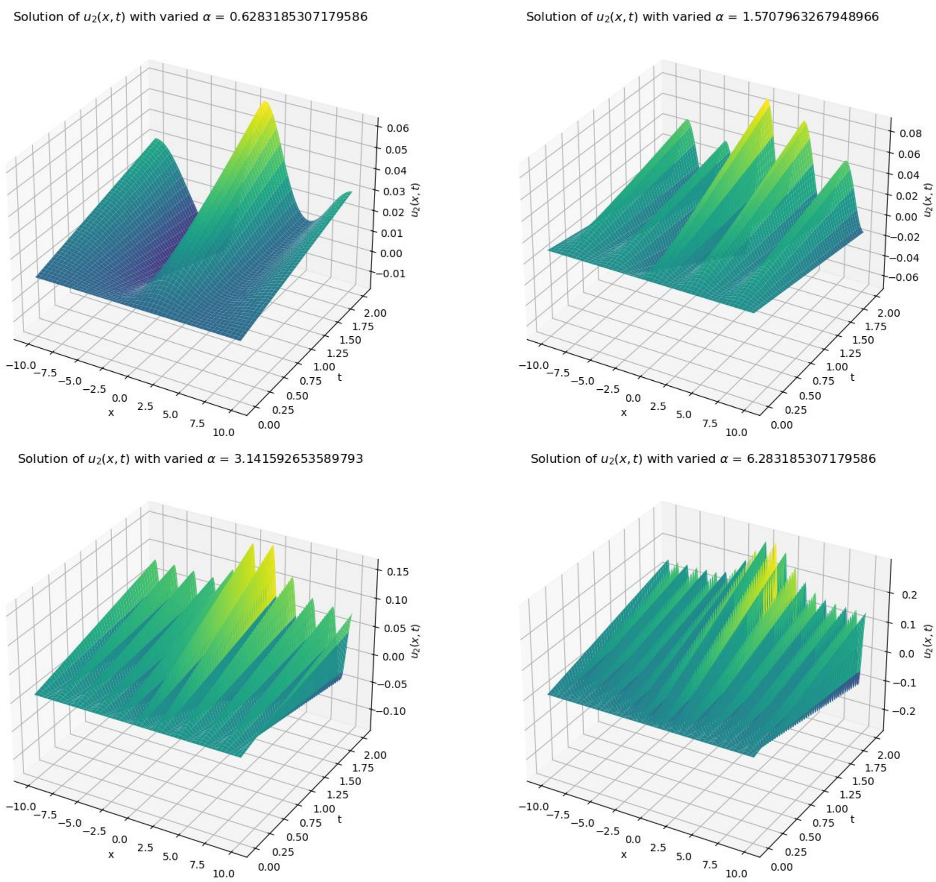

Figure 8. Measure of strength of the system

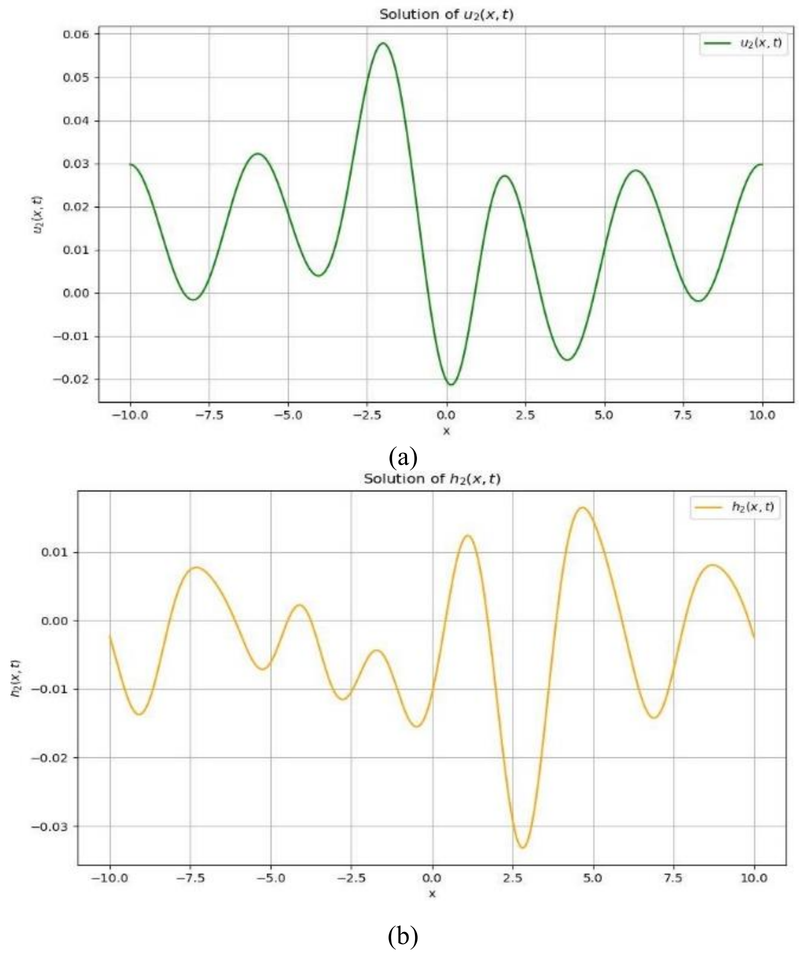

Figure 9. (a) The impact of velocity on amplitude under modified gravity; (b) The impact of depth on amplitude of deep water

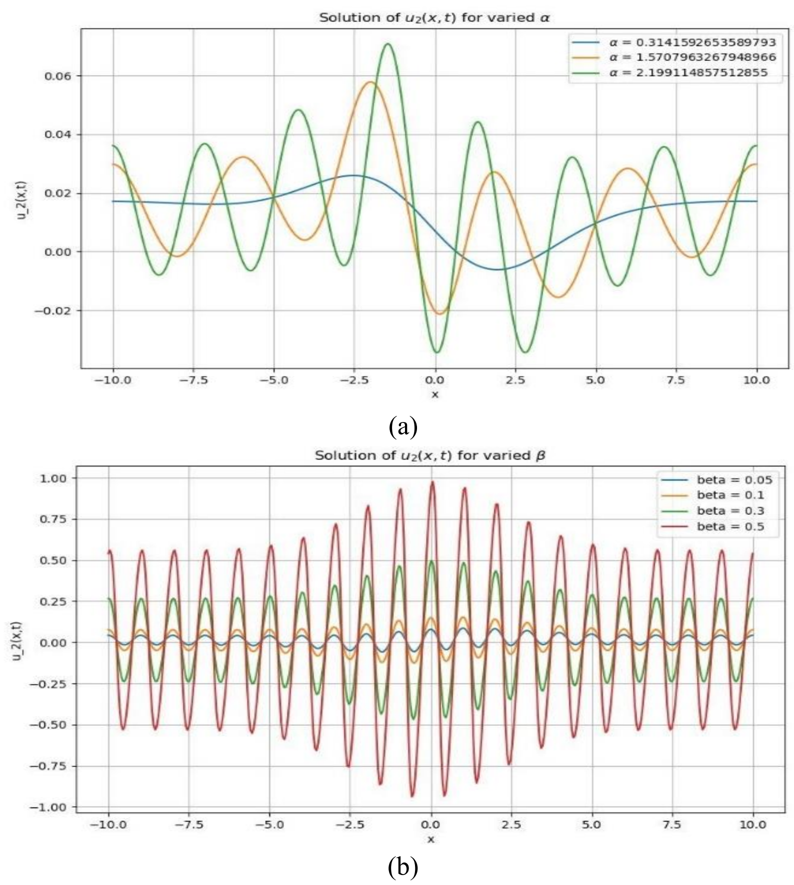

Figure 10. (a) Damping effect on the amplitude; (b) Deep water stability under modified gravity

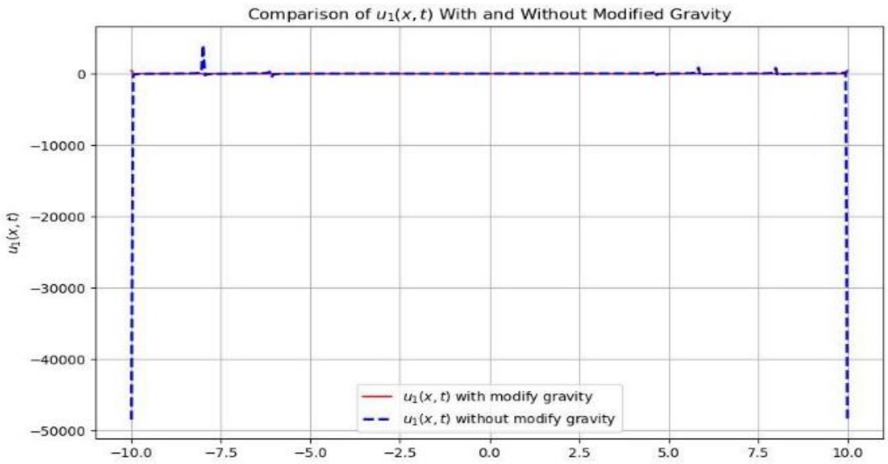

Figure 11. Effect of modified gravity on the first layer of the stratified deep water

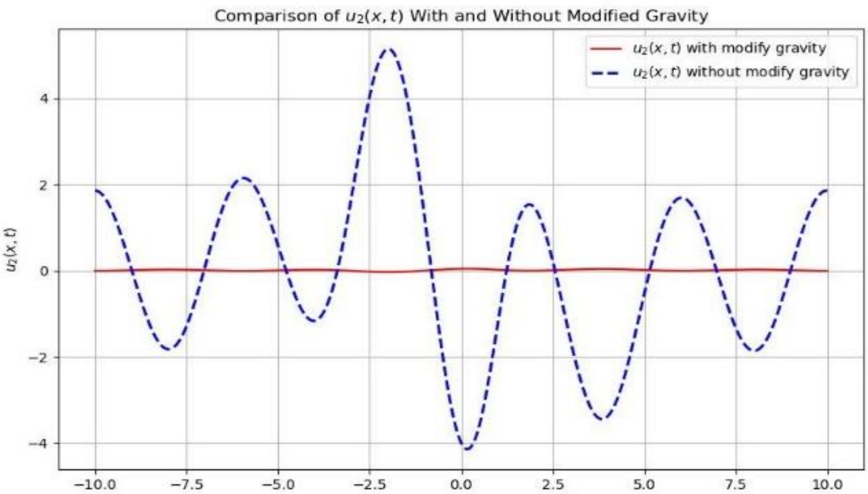

Figure 12. Effect of modified gravity on second layer of the stratified deep water

In Figure 10(a), this is damping effect on the amplitude of stratified deep water under modified gravity. As our simulation shows, a larger alpha value indicates a stronger perturbation, which can lead to more significant changes in the stratified deep water condition and a smaller alpha value indicates a weaker perturbation. In Figure 10(b), the stability of the stratified deep water under modified gravity. Our numerical simulation shows that larger beta value means stronger perturbation, which leads to more significant changes in the stratified deep water condition.

There is no vertical movement of water masses due to gravity modification. The spec observed when the gravity is not modified is at instant the deep water bubbles due to instability on the water body as shown in Figure 11.

The effect of speed in stratified deep water shows that without modified gravity, the system typically exhibits oscillatory motion which means that the system moves back and forth around an equilibrium position. However, when modified gravity is quickly introduced into the system, our simulated result shows that the behaviour changes to horizontal line along the x-axis. This signifies that the system is now moving with constant velocity in the horizontal direction, rather than oscillating as shown in Figure 12.

7. CONCLUSIONS

Our findings demonstrate that the mathematical model of stratified deep water flow under modified gravity, analyzed through the perturbation method, offers valuable insights into the behavior of deep water systems. By considering the effects of modified gravity, we understood the dynamics of stratified flows in environments where the gravitational force is altered. The perturbation method allowed us to identify instability mechanisms and predict the emergence of new flow regimes, which is crucial for understanding the complex behavior of deep water systems.

The key conclusions of our study are as follows:

1). The model captures the behavior of stratified flows in environments with altered gravitational properties, providing insights into the dynamics of deep water systems under modified gravity.

2). The perturbation analysis reveals the conditions under which the stratified flow becomes unstable and transitions to turbulent or other complex states, which is essential for understanding the dynamics of deep water systems.

3). Our findings have implications for various fields, such as oceanography, geophysics, and planetary science, where the understanding of stratified deep water flow under modified gravity is crucial.

Overall, our study contributes to the advancement of knowledge in the field of stratified deep water flow under modified gravity, offering new insights that can be applied in diverse scientific disciplines. The effect of modified gravity in stratified deep water indicate that the speed of the system is affected by both oscillatory and uniform motion. High density evidently causes more stratification in deep water. In deep water, denser water masses tend to sink and form distinct layers below less dense water masses and thereafter creates a stable stratification with different density and temperature profile for each layer.

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NOMENCLATURE

u=(u, v, w)	The three dimensional velocity vector
ρ	The density
p	The pressure
g	The gravity constant
$g^{'}$	Modified gravity
f	Coriolis parameter
Ω	The angular velocity
u	Velocity in the horizontal x direction
v	Velocity in the horizontal y direction
L	Length scale
R_0	Rossby number
(T)	Temperature
ρ_0, T_0, p_0	Are reference values of density, temperature and salinity respectively
h	Vertical length scale
H^*	Vertical height of deep water at thermocline
ς	Free surface elevation
x	Horizontal, x direction
y	Horizontal, y direction
z	Vertical z direction
t	Time
$\frac{D}{Dt}$	Material derivative
$h(x, y, t)$	The height of water surface from the same reference height
$\xi(x, y)$	The thermocline regime
H	Deep water dept
h	The water height above each stratified column
δx	Width in the $x-$ direction
δy	Width in the $y-$direction
u_1	Velocity in the first layer in the $x-$ direction
u_2	Velocity in the second layer in the $x-$direction
v_1	Velocity in the first layer in the $y-$ direction
v_2	Velocity in the second layer in the $y-$direction
α	Measure of strength of the system
β	Measure of stability of the system
F	Sum of all forces
m	Mass
a	Acceleration of the block of water
K	Wave number