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Enhanced Characterization of Rough Semigroup Ideals: Extension and Analysis

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ABSTRACT

Rough set theory (RST) is a formal theory derived from logical properties of information systems. Rough set theory extends traditional set theory by defining a subset of a universe through the use of a pair of sets referred to as the lower and upper approximations. It is a mathematical approach for dealing with ambiguities and imprecisions in a variety of situation. Since its introduction by Zdislaw Pawlak in the late eighties of the previous century, it has evolved into pure and applied directions from mathematical, logical, and computational perspectives. The area of rough set theory in computational mathematics is rapidly developing. As far as vagueness and imprecision are concerned, rough set theory is basically a mathematical approach. An equivalence relation is a key concept in rough set models. Approximations at the lower and upper levels are constructed based on equivalence classes. There is wide application of algebraic systems in sequential machines, formal languages, arithmetic codes, and error-correction algorithms. The study of any set will be effective if an algebraic structure is developed for it. In the context of semigroups research, rough set theory can be used to analyse and understand the properties and relationships within semigroups. Semigroups and related algebraic structures and their properties can be explored more deeply when rough set theory is applied. The aim of this paper is to extend the concept of rough semigroup ideals. It has already been shown that some properties of rough (left, right) ideals in semigroups can be obtained by extending the notion of a left (right) ideal in a semigroup. As a result of considering h-ideals in semigroups, rough upper h-ideals (left & right) have been introduced here along with their properties. Also, the results related to rough semi-lattices and rough quotient semigroups are given. These concepts are explained with suitable examples.

1. INTRODUCTION

The principal and central concept of rough sets was taken up by Pawlak in the year 1982 [1]. It's a suitable tool for studying uncertain and imprecise knowledge in information systems in a mathematical way. The theory and applications of rough set theory have been extensively researched over the past few years. In rough set algebra, two additional approximation operators are added to the set algebra. It can therefore be said that rough set theory extends set theory. Rough sets play a fundamental role in artificial intelligence and cognitive sciences, especially in machine learning, data mining and knowledge discovery, and pattern recognition. Data that is imprecise and noisy can be analysed using rough set approaches.

Mathematically, a system consists of a set that establishes relations between its elements or subsets in some way. The system categorizes data based on partial order as a key categorizing factor. It uses any two elements of a set as well as any two subsets of a set. Each pair of elements in the set does not need to have it defined. This type of relation is referred to as a totally ordered.

It is possible to approximate any set with lower or upper approximation by dividing the set in the universe by using relations, either partial order relations or totally ordered relations. As we consider our universe as our system, it may be impossible to fit everyone into a single cell in our universe if we divide it up by characters which is why an approximate set of characteristics may exist for any group or individual, where the characters that are entirely appropriate for them are called lower approximation. The characters that are appropriate for them are called upper approximation. A rough set is one with non-empty differences between upper and lower approximations, otherwise it is crisp. So far, many results have been produced by various experts while evaluating the algebraic aspects of the rough set.

Bonikowsaki [2], Iwinski [3], Pomykala and Pomykala [4] and Biswas and Nanda [5] gave an algebraic approach to rough sets. Kuroki and Wang [6] evaluated rough group with respect to approximation of lower and upper bounds of any subset of group with respect to its normal subgroup.

Based on arbitrary binary relations, Liu and Zhu [7] presented the lower and upper approximations. Also, Pawlak and Skowron [8] provided some extensions to rough sets.

Using fuzzy ideals with thresholds in ordered semigroups, Hussain et al. [9] proposed generalized roughness for fuzzy filters. Hussain et al. [10] assessed the roughness of fuzzy ideals in ordered semigroups by using isotone and monotone mappings also gave the concept of rough Pythagorean fuzzy ideals in semigroups

In particular, in semigroups the idea of semigroup rough ideals was initiated by Kuroki [11]. Based on fuzzy ideals, Wang and Zhan [12] introduced the concept of rough semigroups. Bagırmaz and Ozcan [13] gave the concept of rough semigroups and its homomorphism, also introduced ideals and bi-ideals in rough set. Soft sets have been used to introduce roughness in semigroups by Arabi and Talebi [14]. An approach to generalized rough approximation spaces based on maximal neighbourhoods and ideals was given by Hosny et al. [15]. Guler et al. [16] provided rough approximations via ideals for various topologies. Sangeetha and Sathish [17] defined rough groups using upper and lower approximations to rough sets within a finite universe.

Our focus will be on roughness in semigroups ideal, semi lattices, and quotient semigroups, motivated by the study of roughness in algebraic systems and partially ordered sets. A rough semigroup ideal is extended in this paper along with results related to rough semilattices and rough quotient semigroups. With the help of appropriate examples, these concepts are explained.

In this paper, the organization is as follows: Section 2 covers rough sets and their basic concepts, in section 3, concepts of rough semigroups and rough ideals were discussed, in section 4, the concept of rough h-ideals was introduced, in section 5, rough semi-lattices and in section 6, rough quotient semigroups were discussed. Some results from these concepts have been proved and verified with relevant examples.

2. BASIC TERMINOLOGIES OF ROUGH SETS

Definition 2.1 [1]

An approximation space is composed of a set with finite elements Ω, called universe along with an equivalence relation (a relation which satisfies reflexive, symmetric and transitive properties) $\sim$ on Ω and it is represented by $K=(\Omega, \sim)$.

Definition 2.2 [1]

A family of subsets F={C_1, C_2, C_3, ……, C_n} of Ω are said to be a classification of Ω if:

$\cdot C_1 \cup C_2 \cup \ldots . \cup C_n = \Omega$

$\cdot C_i \cap C_j = \phi$, for i≠j

Also, in approximation space and for an element k in the universe, $\sim$ induces a class of equivalence $[k]_\sim$.

Definition 2.3 [1]

Consider an approximation space where A is any subset of universe, then:

$\cdot \Omega^A = \{a_i | [a_i]_\sim \cap A \neq \phi\}$

$\cdot \Omega_A = \{a_i | [a_i]_\sim \subseteq A\}$

$\cdot BN_A = \Omega^A - \Omega_A$

are called approximations of upper, lower & boundary region of A with respect to $\sim$ respectively and A is said to be rough if BN_A is non empty otherwise it is crisp.

3. ROUGH SEMIGROUPS IDEAL

Definition 3.1 [5]

When a non-empty set is closed and associative under a binary operation, it is a semigroup and represented by (S, *).

Definition 3.2 Subsemigroup [5]

If $AA \subset A$, for any subset A of S, then A is called sub semigroup.

Definition 3.3 [5]

Relation ρ on the semigroup is a congruence relation if $a_1, b_1 \in S$ then $(a_1, b_1) \in \rho$ implies $(a_1x, b_1x) \in \rho$ & $(xa_1, xb_1) \in \rho$ for all $x \in S$. Furthermore, ρ is said to be complete if $[a_1]_\rho[b_1]_\rho=[a_1 b_1]_\rho$.

Definition 3.4 [8]

A_1, a non-empty subset of S. Then:

$\cdot \rho_{A_1} = \{x_1 \in S | [x_1]_\rho \subseteq A_1\}$

$\cdot \rho^{A_1} = \{x_1 \in S | [x_1]_\rho \cap A_1 \neq \phi\}$

called ρ-approximations (lower & upper) of A_1 respectively. If $\rho^{A_1} - \rho_{A_1} \neq \phi$, then A is rough with respect to ρ.

Definition 3.5 [8]

Let ρ_1, ρ_2 be complete congruence relations on S and A_1, B_1 are non-empty subsets of S. The following results are due to Kuroki [11]:

$\cdot \rho_{1_A}\rho_{1_B} \subseteq \rho_{1_{AB}}$

$\cdot (\rho_1 \cap \rho_2)^A \subseteq \rho_1^A \cap \rho_2^A$

$\cdot (\rho_1 \cup \rho_2)^A = \rho_1^A \cup \rho_2^A$

$\cdot (\rho_1 \cap \rho_2)_A = \rho_1^A \cap \rho_2^A$

$\cdot (\rho_1 \cup \rho_2)_A \supseteq \rho_1^A \cup \rho_2^A$

$\cdot \rho_1 \subseteq \rho_2 \Rightarrow \rho_{1_A} \supseteq \rho_{2_B}$

$\cdot \rho_1 \subseteq \rho_2 \Rightarrow \rho_1^A \subseteq \rho_2^B$

Definition 3.6 [5]

If $BS \subseteq B$, then B is an ideal (left), if $SB \subseteq B$, then B is an ideal (right) of the semigroup and if both are true then B is two-sided ideal.

Definition 3.7 Rough subsemigroup [8]

Let S be a semigroup. A non-empty subset A of S is a rough upper (lower) sub semigroup if the corresponding ρ^A (ρ_A) is sub semigroup of S respectively. If both ρ^A and ρ_A is subsemigroup of S then A is rough sub semigroup of S.

Definition 3.8 Rough ideals [8]

If ρ^{A_1} is a right [left, two-sided] ideal of S, then A_1 is rough upper right [left, two-sided] ideal of S. If ρ_{A_1} is a right [left, two-sided] ideals of S, then A_1 is a rough lower right [left, two sided] ideal of S.

Definition 3.9

If $A_1$is a subsemigroup of S then it is rough upper subsemigroup of S. Also, if A_1 is right [left, two-sided ideal] of S, then A_1 is rough upper right [left, two-sided ideal] of S.

4. ROUGH h-IDEALS

Definition 4.1 Rough upper ideal

Let A be any non-empty subset of a semigroup S and ρ be a congruence relation on S also, ρ^A is a subsemigroup of S. A is said to be rough upper ideal (left) if $\rho^A S \subseteq \rho^A$ and rough upper ideal (right) if $S\rho^A \subseteq \rho^A$ and it is rough upper ideal if it satisfies the both conditions.

Definition 4.2 Rough upper h-ideal

A is said to be rough upper h-ideal (left) if $\rho^A S=\rho^A$ and rough upper h-ideal (right) if $S\rho^A=\rho^A$ and it is rough upper h-ideal if it satisfies the both conditions.

Theorem 4.1

If X_1 is a sub semigroup, h-ideal(left) of the semigroup, then X_1 is rough upper h-ideal(left).

Proof:

Given X_1, h-ideal(left) of the semigroup, so $X_1S=X_1$.

$X_1S \subseteq X_1$ & $X_1 \subseteq X_1S$.

$\Rightarrow X_1$ is ideal (left) of S. Also, we have $\rho^{X_1}\rho^S \subseteq \rho^{X_1S}$.

Since, $\rho^S=S$ and by above, $\rho^{X_1}S \subseteq \rho^{X_1}$

$\Rightarrow X_1$ is rough upper ideal (left) of the semigroup S.

Conversely, let $x \in \rho^{X_1}$, also $x \in \rho^S = S$.

Since $X_1S \subseteq X_1$, $x \in \rho^{X_1}S$ completes the converse.

Theorem 4.2

If X_1 is a sub semigroup, h-ideal(right) of the semigroup, then X_1 is rough upper h-ideal(right).

Proof:

Given X_1, h- ideal(right) of the semigroup, then $SX_1=X_1$

$SX_1 \subseteq X_1$ & $X_1 \subseteq SX_1$

$\Rightarrow X_1$, ideal(right) of S. Also, we have $\rho^S\rho^{X_1} \subseteq \rho^{SX_1}$.

Since, $\rho^S=S$ and by above, $S\rho^{X_1} \subseteq \rho^{X_1}$.

$\Rightarrow X_1$ is rough upper ideal (right) of S.

Conversely, $x \in \rho^{X_1}$, also $x \in \rho^S = S$.

Since $SX_1 \subseteq X_1$ which implies $x \in S\rho^{X_1}$ proves the converse.

Example 4.1

$S=\{2, 3, 1\}$, semigroup with binary operation $*$ given by:

*	2	3	1
2	2	2	2
3	2	3	2
1	2	2	1

(2), (23), (21) & S are h-ideals of S and are represented by hi_1, hi_2, hi_3 & hi_4 respectively.

ρ- congruence classes are given by $\{\{2\}, \{3\}, \{1\}\}$ then $\rho^{hi_1} = hi_1, \rho^{hi_2} = hi_2, \rho^{hi_3} = hi_3, \rho^{hi_4} = hi_4$.

Hence hi_1, hi_2, hi_3 *and* hi_4 are rough upper h-ideal of S.

Also, $\rho_{hi_1} = hi_1, \rho_{hi_2}, \rho_{hi_3}$& ρ_{hi_4} are empty, which implies hi_1 is rough lower h-ideal.

Example 4.2

$S=\{2, 3, 1\}$, a semigroup with binary operation $*$ given by:

*	2	3	1
2	2	2	2
3	2	3	3
1	2	3	1

(2), (23), & S are h-ideals of S and are represented by hi_1, hi_2 & hi_3 respectively.

ρ-congruence classes are $\{\{2\}, \{3\}, \{1\}\}$ then $\rho^{hi_1} = hi_1, \rho^{hi_2} = hi_2, \rho^{hi_3} = hi_3$.

Hence hi_1, hi_2 & hi_3 are rough upper $h$ideal of S.

Also, $\rho_{hi_1} = hi_1$ *and* ρ_{hi_2}, ρ_{hi_3} are empty, which implies hi_1 is rough lower h-ideal.

Example 4.3

$S=\{1, 2, 3\}$, a semigroup with binary operation $*$ given by:

*	1	2	3
1	1	1	1
2	1	1	1
3	1	1	3

The h-ideals of S are (1), (13), & S represented as Z_1, Z_2 & Z_3 respectively.

ρ congruence classes are given by $\{\{1,2\}, \{3\}\}$ then $\rho^{Z_1} = \{1,2\}, \rho^{Z_2} = Z_3, \rho^{Z_3} = Z_3$.

Hence Z_2 & Z_3 are rough upper h-ideal.

Now, ρ_{Z_1}& ρ_{Z_2} *are empty*, $\rho_{Z_3} = Z_3$, which implies Z_3 is rough lower h-ideal of semigroup.

Example 4.4

$S=\{1, 2, 3\}$ with binary operation $*$ given by:

*	1	2	3
1	1	2	2
2	2	1	1
3	2	1	1

The h-ideals of S are (12) & S represented as X_1 & X_2 respectively.

ρ congruence classes are given by $\{\{1\}, \{2,3\}\}$ then $\rho^{X_1} = X_1$, $\rho^{X_2} = X_2$.

Hence rough upper h-ideal of S are X_1 & X_2.

ρ_{X_1} is empty & $\rho_{X_2} = X_2$, which implies A_2 is rough lower h-ideal of S.

Corollary 4.1

ρ_1, a congruence relation on a semigroup and B_1 is h-ideal (right, left) of S. Hence B_1 is rough upper h-ideal (right, left) but there is no need to hold the converse.

Example 4.5

$S=\{0, 1, 2, 3\}$ along with binary operation $*$ given by:

*	0	1	2	3
0	0	0	0	3
1	0	3	3	0
2	0	3	0	0
3	0	0	0	0

ρ congruence classes are given by $\{\{0,3\}, \{1\}, \{2\}\}$ then $A_1 = \{3\} \subseteq S$, $\rho^{A_1} = \{0,3\}$, $\rho^{A_1}S = \rho^{A_1}$.

which implies A_1 is rough upper h-ideal (left) of the semigroup. But $A_1S \neq A_1$, it is not h-ideal (left) of S.

Theorem 4.3

If A_1, B_1 are h- ideals [left] of S, then $\rho^{A_1}\rho^{B_1}$ is rough upper h-ideal [left].

Proof:

We have $A_1S=A_1$, $B_1S= B_1$.

Also, A_1 and B_1 are rough upper ideal (left) of S. (Theorem 4.1)

$\rho^{A_1}S = \rho^{A_1}$& $\rho^{B_1}S = \rho^{B_1}$.

Now, $\rho^{A_1}\rho^{B_1}S = \rho^{A_1}(\rho^{B_1}S) = \rho^{A_1}\rho^{B_1}$.

(subsets of S holds the associative property)

Hence $\rho^{A_1}\rho^{B_1}$ is rough upper h-ideal [left] of S.

Theorem 4.4

If A, B are h-ideals [right] of the semigroup S then $\rho^A\rho^B$ is rough upper h-ideal [right] of S.

Proof:

We have $A=SA$ & $B=SB$.

Also, both are rough upper ideal (right) of S. (Theorem 4.2)

$S\rho^A=\rho^A$, $S\rho^B=\rho^B$.

Now, $S\rho^A\rho^B=(S\rho^A)\rho^B=\rho^A\rho^B$.

$\Rightarrow\rho^A\rho^B$ is rough upper h-ideal [right] of the given semigroup.

Theorem 4.5

The collection of all rough upper ideals(left) of the semigroup forms rough upper ideal (right) of the set of all congruence subsets of the semigroup.

Proof:

A, an ideal(left) of the given semigroup.

$\Rightarrow\rho^A$ is rough upper ideal (left).

Denote $\boldsymbol{\Theta}$ as a collection of possible rough upper ideal (left) and η, a congruence class of all possible subsets of semigroup.

Now $\rho^A \in \Theta$, $B \in \eta$.

Now, $(B\rho^A)S = B(\rho^A S) \subseteq B\rho^A, hence\ \eta\Theta \subseteq \Theta$.

Theorem 4.6

The collection of all rough upper h-ideals (left) of the semigroup forms rough upper ideal (right) of the set of all congruence subsets of the semigroup S.

Proof:

Denote Π as the collection of all rough upper h-ideal (left) of the semigroup and ζ, congruence class of all subsets of the semigroup.

Let A be any h-ideal which implies $\rho^A \in \Pi$, $B \in \zeta$.

Now, $(B\rho^A)S = B(\rho^A S) = B\rho^A, hence\ \zeta\Pi \subseteq \Pi$.

Theorem 4.7

For any rough upper ideal (left) X of semigroup, if Y is any h-ideal (left) of A, then ρ^Y is rough upper ideal (left).

Proof:

Let X be any ideal (left) of the semigroup, we have $XS \subseteq X$

Since Y is a h-ideal (left) of X which implies ρ^Y is rough upper h-ideal (left) of X.

$\rho^Y A=\rho^Y$, now $\rho^Y S = \rho^Y XS \subseteq \rho^Y A = \rho^Y$.

Theorem 4.8

For any rough upper ideal (right) X of semigroup, if Y is any h-ideal (right) of A, then ρ^Y is rough upper ideal (right).

Proof:

Let X be any ideal (right) of S then $SX \subseteq X$. Also, Y, a h-ideal (right) of X.

implies ρ^Y is rough upper h-ideal(right) of X, $X\rho^Y=\rho^Y$, $S\rho^Y = SX\rho^Y \subseteq X\rho^Y = \rho^Y$.

Hence ρ^Y is rough upper ideal (right).

Theorem 4.9

Let A_1, A_2, A_3, ..., A_m be any finite set of two-sided ideals of S. ρ, a congruence relation on S and $\rho^{A_1}, \rho^{A_2}, \ldots.\rho^{A_m}$ are rough upper ideals (two sided) of S then $\bigcap_1^m \rho^{A_i} \neq \phi$.

Proof:

Using the method of induction

Given $\rho^{A_1}, \rho^{A_2}, \ldots.\rho^{A_m}$ are rough upper ideals (two sided) of S then

$\rho^{A_1}\rho^{A_2} \subseteq \rho^{A_1}S \subseteq \rho^{A_1}$, also $\rho^{A_1}\rho^{A_2} \subseteq S\rho^{A_2} \subseteq \rho^{A_2}$

$\therefore\ \rho^{A_1}\rho^{A_2} \subseteq \rho^{A_1} \cap \rho^{A_2}$.

Assume the result is true for m-1 terms.

$\rho^{A_1}\rho^{A_2} \ldots\ldots \rho^{A_{m-1}} \subseteq \bigcap_1^{m-1} \rho^{A_i}$

Now, $\rho^{A_1}\rho^{A_2} \ldots\ldots \rho^{A_{m-1}}\rho^{A_m} \subseteq \bigcap_1^{m-1} \rho^{A_i} \cap A_m \subseteq \bigcap_1^m \rho^{A_i}$

Since $\rho^{A_1}, \rho^{A_2}, \ldots.\rho^{A_m}$ are non-empty $\therefore \bigcap_1^m \rho^{A_i} \neq \phi$

Theorem 4.10

Let A_1, A_2, ..., A_n be any finite set of ideals (left) and $\rho^{A_1}, \rho^{A_2} \ldots.\rho^{A_n}$ are rough upper ideals (left) of S then $\bigcup_1^m \rho^{A_i}$ is ideal (left) of the semigroup.

Proof:

$\{\rho^{A_i}\}$, collection of rough upper ideal (left).

Now, $(\bigcup\rho^{A_i})S = \bigcup(\rho^{A_i}S)$

Using method of induction

Since $\rho^{A_1} \cup \rho^{A_2} = \rho^{A_1 \cup A_2}$

Also, $\rho^{A_1}S \subseteq \rho^{A_1}, \rho^{A_2}S \subseteq \rho^{A_2} \Rightarrow \rho^{A_1}S \cup \rho^{A_2}S \subseteq \rho^{A_1} \cup \rho^{A_2}$.

Assume the result is true for any m.

$\bigcup_{i=1}^m \rho^{A_i}S \subseteq \bigcup_1^m \rho^{A_i}$.

Consider $\bigcup_{i=1}^{m+1} \rho^{A_i}S = \bigcup_{i=1}^m \rho^{A_i}S \cup \rho^{A_{m+1}}S \subseteq \bigcup_1^{m+1} \rho^{A_i}$

Hence for m+1 the result is true. Hence by induction, $\bigcup_1^m \rho^{A_i}$ is an ideal (left) of S.

Theorem 4.11

Let A_1, A_2, ..., A_n be any finite set of ideals (left) of a semigroup S and $\rho^{A_1}, \rho^{A_2} \ldots.\rho^{A_n}$ are rough upper ideals (left) of S then either $\bigcap_1^n \rho^{A_i} = \phi$ or an ideal (left) of S.

Proof:

$\{\rho^{A_i}\}$ is a collection of rough upper ideal (left) of S

Suppose $\cap\, \rho^{A_i}$ is non empty.

$\cap\, \rho^{A_i} \supseteq \rho^{\cap A_i}$

$\rho^{\cap A_i} \subseteq \cap\, \rho^{A_i}$

$\rho^{\cap A_i}S \subseteq \cap\, \rho^{A_i}S \subseteq \cap\, \rho^{A_i}$

Theorem 4.12

Let ρ be a congruence relation on S. If A_1 is a h-ideal (left) of S and B_1 is an ideal (right) of S, then $\rho^{A_1} \subseteq \rho^{B_1}$.

Proof:

Now, $A_1S = A_1 \Rightarrow A_1 \supseteq A_1S\ \&\ SA_1$

$A_1B_1 \subseteq A_1S \subseteq A_1\ \&\ A_1 \subseteq A_1S \subseteq A_1B_1$

Then, $A_1B_1 \subseteq A_1\ \&\ A_1 \subseteq A_1B_1 \Rightarrow A_1 = A_1B_1$

Also, $A_1B_1 \subseteq SB_1 \subseteq B_1 \Rightarrow A_1B_1 \subseteq B_1$

$\rho^{A_1B_1} \subseteq \rho^{B_1} \Rightarrow \rho^{A_1} \subseteq \rho^{B_1}$

5. ROUGH SEMILATTICE

Definition 5.1 Semilattices

Semilattices are defined as posets P satisfying the following characteristics.

·A least upper bound (lub) is given to each subset of two or more elements.

·A greatest lower bound (glb) is given to each subset of two or more elements.

Lower semilattices are those that satisfy only the second condition, while upper semilattices are those that satisfy only the first condition.

The powerset of any set is a semilattice with respect to partial order inclusion. Here, we consider a collection of rough upper (lower) ideals (left/right) and it forms semilattice and hence called rough upper (lower) semi – lattice.

Definition 5.2 Rough semi – lattices

$\{A_1, A_2, \ldots, A_n\}$ are collection of finite set of ideals (left) of S, now the collection of rough upper ideals (left) of S, $\{\rho^{A_i}\}$ where i=1, 2, …., n forms rough upper semi-lattice with respect to partial order inclusion. In a similar manner rough lower semi-lattice defined.

With suitable examples, the following theorems 5.1 & 5.2 explain when rough upper ideals become rough semilattices.

Theorem 5.1

The set of rough upper ideal (left) of S forms a rough upper semilattice with respect to inclusion.

Proof:

Consider the finite collection of sets of rough upper ideal (left) of S.

If $A_1, A_2, \ldots, A_n$ are ideals (left) of S & ρ, an equivalence relation on S.

Then $\rho^{A_1}, \rho^{A_2}, \ldots . \rho^{A_n}$ are rough upper ideal (left) of S.

Since the union of rough upper ideal (left) of S is a rough upper ideal (left) and smallest set containing all of them is its union.

Hence union of ρ^{A_i} $i = 1,2, \ldots n$ serves as least upper bound.

Hence it forms rough upper semilattice with respect to inclusion.

Theorem 5.2

The set of rough lower ideal (left) of S forms a rough lower semilattice with respect to inclusion.

Proof:

If $A_1, A_2, \ldots, A_n$ are ideals (left) of S & ρ, an equivalence relation on S.

Then $\rho_{A_1}, \rho_{A_2} \ldots . \rho_{A_n}$ are rough lower ideal (left) of S. (if it is non-empty).

Since ϕ is contained in any set. Infimum exists and ϕ serves as greatest lower bound and it forms a rough lower semilattice with respect to inclusion.

Example 5.1

Let S={1, 2, 3, 4} be a semigroup with respect to $*$ given by

*	1	2	3	4
1	1	1	1	1
2	2	2	2	2
3	1	1	1	2
4	4	4	4	4

The ideals (left) of S are (1), (2), (4), (12), (14), (24), (123), (124), S and are represented by $A_1, A_2, A_3 \ldots . A_9$ respectively. These forms a semilattice with respect to inclusion.

Let ρ congruence classes are given by {{1,2,4}, {3}}.

$\rho^{A_i} = A_8, for\ i = 1,2,3,4,5,6,8$

ρ^{A_7} & $\rho^{A_9} = A_9$

A_8 and A_9 are rough upper ideal (left).

So, there is an upper semilattice whose lub is A_9 and glb is A_8. (Figure 1).

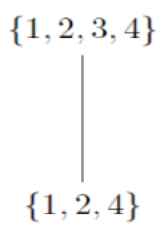

Figure 1. Hasse diagram

Similarly, lower approximation of above is given by

$\rho_{A_i} = \phi, for\ i = 1,2,3,4,5,6,8$

$\rho_{A_7} = \{3\}$& $\rho_{A_9} = A_9$

So, there is a lower semilattice whose lub is A_9 and glb is ϕ. (Figure 2).

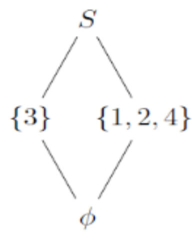

Figure 2. Hasse diagram

Example 5.2

S={1, 2, 3, 4, 5, 6}, along with binary operation $*$ given by

*	1	2	3	4	5	6
1	1	1	1	1	1	1
2	1	1	1	1	1	1
3	1	1	1	1	1	1
4	1	1	2	1	2	2
5	1	1	1	1	1	1
6	1	1	2	1	2	2

The ideals (left) of S are (1), (12), (13), (15), (124), (123), (126) and S represented by $A_1, A_2, A_3, A_4, A_5, A_6, A_7, A_8$ respectively and it forms a semilattice with respect to inclusion.

Let ρ congruence classes are {{1,2,3}, {3}, {5,6}}

$\rho^{A_i} = A_5\ for\ i = 1,2,5$, $\rho^{A_i} = A_8\ for\ i = 4,7,8$

ρ^{A_3}& $\rho^{A_6} = \{1,2,3,4\}$

Hence, there is a rough upper semilattice whose lub is A_8 and glb is A_5 (Figure 3).

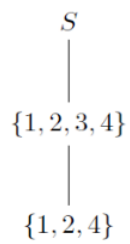

Figure 3. Hasse diagram

Similarly lower approximation is given by

$\rho_{A_i} = \phi, for\ i = 1,2,4,7$, $\rho_{A_i} = \{3\}\ for\ i = 3,6$, $\rho_{A_5} = \{12,4\}$, $\rho_{A_8} = A_8$ (Figure 4).

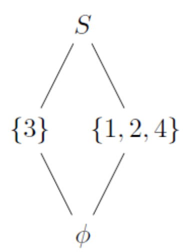

Figure 4. Hasse diagram

Definition 5.3. Minimal rough upper ideal

Let A be an ideal (right, left) of a semigroup. ρ, a relation(congruence) on S and which divides S into classes of equivalence. Now, ρ^A called minimal rough upper ideal (right, left) of S if there exists a $B \subseteq \rho^A$ then ρ^B is not a rough upper (left, right) ideal otherwise ρ^B=ρ^A.

Theorem 5.3

Any minimal rough upper ideal (left) is a rough upper h-ideal (left) of a semigroup.

Proof:

If ρ^A, a minimal rough upper ideal (left), then $\rho^A S \subseteq \rho^A$.

If $B \subseteq \rho^A$ then ρ^B never be a rough upper ideal (left) of S otherwise $\rho^B=\rho^A$.

Consider $\rho^A SS = (\rho^A S)S \subseteq \rho^A S$

$\Rightarrow\rho^A S$ is rough upper ideal (left) of S.

Since ρ^A is minimal rough upper ideal (left), $\rho^A S=\rho^A$

$\Rightarrow\rho^A$ is rough upper h-ideal (left) of S.

Theorem 5.4

If ρ^A is any rough upper ideal (right) of S & ρ^B is minimal rough upper ideal (left) of ρ^A, then ρ^B is rough upper h-ideal (left) of S.

Proof:

Since ρ^A is any rough upper ideal (right) of S then $S\rho^A \subseteq \rho^A$.

Also, ρ^B is minimal rough upper ideal (left) of ρ^A, then $\rho^B\rho^A \subseteq \rho^B$.

Now, $\rho^B S\, \rho^A = \rho^B(S\rho^A) \subseteq \rho^B\rho^A \subseteq \rho^B$.

Since ρ^B is minimal rough upper ideal (left) of ρ^A. Hence $\rho^B S=\rho^B$.

Theorem 5.5

Every minimal rough upper ideal (right) is a rough upper h-ideal (right).

Proof:

If ρ^A is minimal rough upper ideal (right), then $S\rho^A \subseteq \rho^A$.

If $\boldsymbol{B \subseteq \rho^A}$, then ρ^B is never be a rough upper ideal (right) otherwise $\rho^B=\rho^A$.

Now, $\boldsymbol{SS\rho^A = S(S\rho^A) \subseteq S\rho^A}$

$\Rightarrow S\rho^A$ rough upper ideal (right) of S because of minimality of ρ^A.

$S\rho^A=\rho^A$. Hence ρ^A is rough upper h-ideal (right) of S.

6. ROUGH QUOTIENT SEMIGROUP

Definition 6.1 [18]

ρ, a relation(equivalence) on a semigroup. The statements below are therefore implied by each other.

·ρ is congruence

·If $a\rho b$, then $ac\rho bc$ & $ca\rho cb, \forall c \in S$

·The collection of all equivalence classes forms a quotient semigroup S/ρ with operation $[a]_\rho.[b]_\rho=[ab]_\rho$

Definition 6.2 [8]

The product of two binary relations ρ_1 & ρ_2 on S is represented by ρ_1. P_2 and is given by $\rho_1.\rho_2 = \{(\alpha,\beta) \in S\,X\,S : (\alpha,\gamma) \in \rho_1 \& (\gamma,\beta) \in \rho_2\ for\ some\ \gamma \in S\}$

If ρ_1 & ρ_2 are relations (congruence) on S, then $\rho_1.\rho_2$ is also congruence if $\rho_1.\rho_2=\rho_2.\rho_1$.

Theorem 6.1 [8]

If ρ_1 & ρ_2 are relations (congruence) on S and if $\rho_1.\rho_2=\rho_2.\rho_1$ and A be any subsemigroup of S then $\rho_1^A\rho_2^A \subseteq (\rho_1.\rho_2)^A$.

Theorem 6.2

If τ_1 & τ_2 are congruence relation on S and if $\tau_1.\tau_2=\tau_2.\tau_1$ and H is any h-ideal (left) of S then $\tau_{1_H}\tau_{2_H} \subseteq (\tau_1.\tau_2)_H$

Proof:

Since H is h-ideal (left) of S then $HS=H$

Let $\boldsymbol{xy \in \tau_{1_H}\tau_{2_H}}$

$\boldsymbol{x \in \tau_{1_H} y \in \tau_{2_H}}$

$\boldsymbol{[z_1]_{\tau_1} \subseteq H, [z_2]_{\tau_2} \subseteq H}$ where $\boldsymbol{x \in [z_1]_{\tau_1}, y \in [z_2]_{\tau_2}}$

$\boldsymbol{x \in H, y \in S \Rightarrow xy \in H}$

$\boldsymbol{y \in H, x \in S \Rightarrow yx \in H}$

Also $\boldsymbol{x \in \tau_{1_H} y \in \tau_{2_H}}$

$\boldsymbol{\Rightarrow (x, z_1) \in \tau_1, (y, z_2) \in \tau_2}$

$\boldsymbol{\Rightarrow (xy, z_1y) \in \tau_1, (z_1y, z_1z_2) \in \tau_2}$

$\boldsymbol{\Rightarrow (xy, z_1z_2) \in \tau_1.\tau_2}$

$\boldsymbol{\Rightarrow [z_1z_2]_{\tau_1.\tau_2} \subseteq H}$

$\boldsymbol{\Rightarrow xy \in (\tau_1.\tau_2)_H}$

Hence $\boldsymbol{\tau_{1_H}\tau_{2_H} \subseteq (\tau_1.\tau_2)_H}$

7. RESULTS AND DISCUSSIONS

The theory of rough sets can be applied to the algebraic system - semigroups. Rough upper h-ideals (left & right) in semigroups are an extension of h-ideals in semigroups, with some properties proved for them. We have also extended the results proposed by Kuroki [11], proved some new properties of rough upper ideal (left & right) of a semigroup. In rough quotient semigroups, Kuroki [11] proved that $\rho_1^A\rho_2^A \subseteq (\rho_1.\rho_2)^A$ if A is a subsemigroup. We have extended the same for lower approximation of congruence classes if A is h- ideal (left) of S. The minimal rough upper ideal (left, right, h-ideal) has been discussed. As far as rough set algebraic structure is concerned, rough upper ideals (left, right, bi-ideal) have been employed so far, but here we have introduced rough h-ideals by combining rough set concepts with semigroup ideal concepts. A distinction is made between the relatively new ideas of rough set theory and those that are already well-established. The rough set theory and semigroup theory were connected via novel ideas.

8. CONCLUSIONS

The rough sets theory is crucial to both pure and applied mathematics. In this paper, we focused our study on rough sets algebraic properties as they relate semigroups. Semigroup S is considered as a universe set together with a congruence relation on S forms an approximation space. A connection was made between rough sets and semigroup ideals in this paper. Kuroki [11] introduced the notion of rough upper and lower ideals (left, right, bi ideal). In fact, we considered a h-ideal over a semigroup, and used it to form rough upper and lower h-ideals. Following that, we examined rough h-ideals' properties and results by using suitable examples. As a further extension, we apply rough concepts to semilattices and quotient semigroups The extensive properties of rough semigroup ideals have been discussed. The concept of rough upper and lower semilattices have been illustrated with suitable examples. Further rough quotient semigroup has been discussed. The relative newness of rough set theory is distinguished from the more established ideas. This work may be extended in a similar way to other algebraic structures as well.

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