International Information and Engineering Technology Association
Advancing the World of Information and Engineering

Mathematical Modelling of Engineering Problems
Vol. 11, No. 2, February, 2024, pp. 301-315
Journal homepage: http://iieta.org/journals/mmep

Complex Dynamics of a Novel Iterative Scheme Using Finite Difference Technique

Mani Sandeep Kumar Mylapalli[1*], Navya Kakarlapudi[1], Pravin Singh[2], Akkapeddi Chandra Sekhar[1], Pragathi Chaganti[1]

[1] Department of Mathematics, School of Science, GITAM (Deemed to be University), Visakhapatnam 530040, India
[2] Department of Mathematics, University of KwaZulu Natal, Durban 4001, South Africa

Corresponding Author Email: mmylapal@gitam.edu

https://doi.org/10.18280/mmep.110203

Received: 2 July 2023
Revised: 3 September 2023
Accepted: 10 September 2023
Available online: 27 February 2024

Keywords:
nonlinear equations, iterative method, functional evaluations, efficiency index, order of convergence, basins of attraction

ABSTRACT

This paper provides a novel seventh-order iterative method for identifying the zeros of nonlinear equations that use the weight function and finite difference techniques. It is free of the second derivative and consists of three functional evaluations and one of its first derivatives at each iteration. Theoretical study shows that the proposed approach has a seventh order of convergence and a corresponding error equation. In some examples, the performance and effectiveness of the novel iterative method were tested and compared with a few existing equivalent order methods. We analyze a wide range of real-world situations from several fields to test the applicability and effectiveness of the proposed methods, and numerical comparisons are made with a few recently developed techniques. Finally, by investigating its complex dynamical behavior using test functions, we demonstrate that the suggested method is superior to currently used alternative methods.

1. INTRODUCTION

Iterative approaches are widely used for locating the roots of nonlinear equations. Almost all fields of the sciences, engineering, applied mathematics, and computers involve these equations. Analytical methods cannot generally solve the zeros of nonlinear equations (either algebraic or transcendental). Hence, the solutions of the equations must be approached using iterative methods. Since most application problems in every field can build a suitable mathematical model, they can be reduced to a solvable nonlinear equation. Due to their importance, various iterative techniques have been created employing tools such as Taylor's series, interpolation formula, finite difference technique, Adomian polynomials, weight function technique, homotropy perturbation method, general quadrature formula, variational iteration method, decomposition method. Newton Raphson's Method [NR] [1] is one of the earliest techniques for locating the zeros of the nonlinear equation $h(x)=0$, and it is defined as:

$$x_{n+1} = x_n - \frac{h(x_n)}{h'(x_n)},\ h'(x_n) \neq 0 \tag{1}$$

with two functional evaluations, it exhibits second-order convergence, and the Efficiency Index is 1.414.

Numerous researchers have attempted to raise the order of convergence of Newton's technique to produce better outcomes. In this paper, we suggest a seventh-order iterative scheme based on the weight function and finite difference approximations for finding the solution of the nonlinear equation. Four function evaluations are performed during each iteration, and the efficiency index is 1.6265. In this study, we applied application problems in several branches to demonstrate the efficacy of our zero-finding method. We will compare our numerical calculations to well-known seventh-order methods. In addition, our suggested methods provide faster convergence and less residual error. Our approaches produce superior numerical results than those produced by the existing methods, which is how we proved the effectiveness and robustness of the suggested method. We also studied iterative processes' dynamic behavior in the complex plane.

The remainder of the study will be structured as follows: In section 1, we offered some fundamental guidelines for finding the roots of nonlinear equations. In section 2, we used a finite difference and weight function approach to create a novel seventh-order iterative algorithm. In Section 3, we theoretically demonstrate that the suggested scheme's order of convergence is seven. In section 4, we provide a number of numerical examples in various branches to highlight the benefits of the suggested technique and contrast it with other recent methods of similar and different order methods. Basins of attraction were looked at in section 5 to show the complex plane's dynamic behavior. The conclusion is provided in Section 6.

1.1 Definitions

1.1.1 Order of convergence [2]

Let x_1, x_2, x_3... be a series of real integers that converges to the root x_0 of a real function $h(x)$, which has a simple root of x_0. Then, $\lim_{n\to\infty} \frac{x_{n+1}-x_0}{(x_n-x_0)^p} \approx C$ is used to define the sequence's

$p \in R^+$ order of convergence. The asymptotic error constant is $C \neq 0$ in this case.

1.1.2 Error equation

The relationship $\varepsilon_{n+1} = C\varepsilon_n^p + O(\varepsilon_n^{p+1})$ is referred to as an error equation if $\varepsilon_n = x_n - x_0$ represents the error in the n[th] iteration.

1.1.3 Efficiency index [3]

Let *d* represent how often the function was utilized in the specified algorithm, and the order of convergence is indicated by *p*, then the efficiency index is defined as $E.I = p^{1/d}$.

1.1.4 Computational order of convergence (COC) [4]

Let x_{n-1}, x_n, x_{n+1} represent three sequential iterations that are close to the root, and let x be the root of the nonlinear equation $h(x)=0$. The COC can then be estimated to be below:

$$\rho = \frac{\log\left(\left|x_{n+1} - x\right| / \left|x_n - x\right|\right)}{\log\left(\left|x_n - x\right| / \left|x_{n-1} - x\right|\right)}$$

It is used to check the convergence order of a given iterative scheme.

1.1.5 Weight function [5]

The weight function is a real variable sufficiently differentiable function defined on a given interval. These are introduced in an iteration scheme via any arithmetical operation. The main reason for the intrusion of weight function to some specific entity in an iteration process is to enhance the behavior of the iterative method and assist in improving its order of convergence and computational efficiency.

2. SEVENTH ORDER CONVERGENT METHOD

Consider x^* be the exact root of $h(x)=0$ as well as x_n being the n[th] approximation root and ε_n being the error. Then:

$$x^* = x_n + \varepsilon_n \tag{2}$$

Now:

$$h\left(x^*\right) = 0 \tag{3}$$

By Taylor's series expansion:

$$h\left(x^*\right) = h(x_n) + \varepsilon_n h'(x_n) + \frac{\varepsilon_n^2}{2!} h''(x_n) + \ldots \tag{4}$$

Neglecting higher powers of ε_n, and from Eq. (3) and Eq. (4), we have $\varepsilon_n^2 h''(x_n) + 2\varepsilon_n h'(x_n) + 2h(x_n) = 0$, $\varepsilon_n = \left(-2h'(x_n) \pm \sqrt{4h'(x_n) - 8h(x_n)h''(x_n)}\right) \div 2h''(x_n)$.

On simplification:

$$\varepsilon_n = \frac{-2h(x_n)}{h'(x_n)} \cdot \frac{1}{1+\sqrt{1-2\rho_n}} \tag{5}$$

Putting x^* by x_{n+1} in $h(x)=0$ and from Eq. (5), we get:

$$x_{n+1} = y_n - H(\tau)\left[\frac{2h(y_n)}{h'(y_n)} \cdot \frac{1}{1+\sqrt{1-2\rho_n}}\right] \tag{6}$$

where, $y_n = x_n - \frac{h(x_n)}{h'(x_n)}$, $\rho_n = \frac{h'(x_n) - h'(y_n)}{h'(x_n)}$,

$h'(y_n) = 2h[y_n, x_n] - h'(x_n)$, $H(\tau) = 1 - \tau$ and $\tau = \frac{h(y_n)}{h(x_n)}$ is the weight function.

We may create the algorithm by employing Eq. (1) and Eq. (6) as the first and second steps, extending the above scheme by adding Newton's variation technique as the third step. Adding more sub-steps can help the method above perform better, and using multi-step iterative methods, the computational efficiency index can be raised. The motivation of this study is to determine whether it is possible to increase the computational efficiency index, basins of attraction, and rate of convergence of numerical algorithms without introducing new functions. To increase the higher order of convergence or better efficiency, we need to avoid calculating the high-order derivatives of the function. So, we compute $h'(z_n)$ using the following finite difference approximation: $h'(z_n) = h[z_n, y_n] + (z_n - y_n)h[z_n, y_n, x_n]$.

2.1 Algorithm

The computation for the iterative scheme is x_{n+1}:

$$\begin{aligned}
&1.\ y_n = x_n - \frac{h(x_n)}{h'(x_n)} \\
&2.\ z_n = y_n - H(\tau)\left[\frac{2h(y_n)}{h'(y_n)} \cdot \frac{1}{1+\sqrt{1-2\rho_n}}\right] \\
&\quad \text{where, } h'(y_n) = 2h[y_n, x_n] - h'(x_n) \\
&\quad \rho_n = \frac{h'(x_n) - h'(y_n)}{h'(x_n)},\ H(\tau) = 1 - \tau \text{ and } \tau = \frac{h(y_n)}{h(x_n)} \\
&3.\ x_{n+1} = z_n - \frac{h(z_n)}{h'(z_n)} \\
&\quad \text{where, } h'(z_n) = h[z_n, y_n] + (z_n - y_n)h[z_n, y_n, x_n]
\end{aligned} \tag{7}$$

The above scheme Eq. (7) is denoted as SR**.** It has one of its derivatives and three functional evaluations.

3. CONVERGENCE CRITERIA

In this section, we derive the convergence analysis of the developed method using Taylor's series method.

3.1 Theorem [6]

If x_0 is close to x^*, let $x_0 \in D$ be a single zero of a sufficiently differentiable function $h(x)$ for an open interval D. The procedure Eq. (7) then converges to the seventh order.

Proof: Let x^* represent the first zero of $h(x)=0$ and $x^* = x_n + \varepsilon_n$. Thus, $h(x^*)=0$

Using Taylor's method, expanding $h(x_n)$ and $h'(x_n)$ about x^*, we have:

$$h(x_n) = h'\left(x^*\right)\left(\varepsilon_n + c_2\varepsilon_n^2 + c_3\varepsilon_n^3 + c_4\varepsilon_n^4 + \ldots\right) \tag{8}$$

$$h'(x_n) = h'(x^*)\left(1 + 2c_2\varepsilon_n + 3c_3\varepsilon_n^2 + 4c_4\varepsilon_n^3 + \ldots\right) \quad (9)$$

Dividing Eq. (8) by Eq. (9), we get:

$$\frac{h(x_n)}{h'(x_n)} = \varepsilon_n - c_2\varepsilon_n^2 - \left(2c_3 - 2c_2^2\right)\varepsilon_n^3 - \left(3c_4 - 7c_2c_3 + 4c_2^3\right)\varepsilon_n^4 + \ldots \quad (10)$$

Substituting Eq. (10) in the first step of Eq. (7), we get:

$$y_n = x^* + Y$$

where (11)

$$Y = c_2\varepsilon_n^2 + \left(2c_3 - 2c_2^2\right)\varepsilon_n^3 + \left(3c_4 - 7c_2c_3 + 4c_2^3\right)\varepsilon_n^4 + \ldots$$

Expanding $h(y_n)$ about x^* by using Taylor's method, $h(y_n) = h'(x^*)(c_2\varepsilon_n^2 + (2c_3 - 2c_2^3)\varepsilon_n^3 + (3c_4 - 7c_2c_3 + 5c_2^3)\varepsilon_n^4 + \ldots)$; $h'(y_n) = h'(x^*)(1 + (2c_2^2 - c_3)\varepsilon_n^2 + (6c_2c_3 - 4c_2^3 - 2c_4)\varepsilon_n^3 + \ldots)$, Thus:

$$\frac{h(y_n)}{h'(y_n)} = c_2\varepsilon_n^2 + \left(2c_3 - 2c_2^3\right)\varepsilon_n^3 + \left(3c_2^3 - 6c_2c_3 + 3c_4\right)\varepsilon_n^4 + \ldots \quad (12)$$

$$\rho_n = 2c_2\varepsilon_n + \left(4c_3 - 6c_2^2\right)\varepsilon_n^2 + \left(6c_4 + 16c_2^3 - 20c_2c_3\right)\varepsilon_n^3 + \ldots \quad (13)$$

$$H(\tau) = 1 - c_2\varepsilon_n - \left(2c_3 - 3c_2^2\right)\varepsilon_n^2 - \left(3c_4 - 10c_2c_3 + 8c_2^3\right)\varepsilon_n^3 + \ldots \quad (14)$$

Substitute Eqs. (12)-(14), in second step of Eq. (7), we get:

$$z_n = x^* + Z \quad (15)$$

where,

$$Z = \left\{ -c_2c_3\varepsilon_n^4 + \left(c_2c_4 - c_3^2 + c_2^4\right)\varepsilon_n^5 + \begin{pmatrix} c_2c_5 + 6c_2^2c_4 + 4c_2c_3^2 \\ +5c_2^3c_3 - c_2^5 - c_3c_4 - 13c_2c_3c_4 \end{pmatrix}\varepsilon_n^6 + \ldots \right\}$$

Again, expanding $h(z_n)$ about x^* by using Taylor's expansion as follows:

$$h(z_n) = h'(x^*)\left(Z + c_2Z^2 + c_3Z^3 + \ldots\right) \quad (16)$$

$$h[z_n, y_n] = \frac{h(z_n) - h(y_n)}{z_n - y_n} = 1 + c_2(Y + Z) + c_3\left(Y^2 + YZ + Z^2\right) + \ldots \quad (17)$$

$$h[y_n, x_n] = \frac{h(y_n) - h(x_n)}{y_n - x_n} = 1 + c_2(Y + \varepsilon_n) + c_3\left(Y^2 + Y\varepsilon_n + \varepsilon_n^2\right) + \ldots \quad (18)$$

Therefore, from Eq. (17) and Eq. (18), we have:

$$h[z_n, y_n, x_n] = \frac{h[z_n, y_n] - h[y_n, x_n]}{z_n - x_n} = c_2 + c_3(Z + Y + \varepsilon_n) + c_4\left(Z^2 + Y^2 + \varepsilon_n^2 + Z\varepsilon_n + Y\varepsilon_n + ZY\right) + \ldots \quad (19)$$

From Eq. (17) and Eq. (19), we get:

$$h'(z_n) = h[z_n, y_n] + (z_n - y_n)h[z_n, y_n, x_n] = h'(x^*)\begin{Bmatrix} 1 + 2c_2Z + c_3ZY + c_3Z\varepsilon_n - c_3Y\varepsilon_n + \\ c_4Z\varepsilon_n^2 - c_4Y^3 - c_4Y\varepsilon_n^2 - c_4Y^2\varepsilon_n \end{Bmatrix} \quad (20)$$

Substituting Eq. (15), Eq. (16) and Eq. (20) in the third step of Eq. (7), we get $\varepsilon_{n+1} = (c_2^2c_3^2)\varepsilon_n^7 + O(\varepsilon_n^8)$.

As a result, we concluded that the order of convergence of SR is seven, and its efficiency index is $7^{1/4}$=1.6265.

4. NUMERICAL EXAMPLES

We will evaluate the effectiveness and convergence performance of our suggested algorithm in this part. We look at some real-time engineering uses for this. We will now compare our suggested approach (SR) to various seventh-order existing methods, and with other existing methods including SM, HM, NM, FM, PM, CM, IM, and EM regarding iterations, errors, and functional evaluations. All the numerical calculations are performed by mpmath library and an Intel(R) Core (TM) i5-10210U CPU clocked at 2.11 GHz with a 64-bit operating system, we employ PYTHON for all numerical operations. The stopping criterion $|f(x_n)|<\varepsilon$, with the required precision set to 690 decimal places and the tolerance set to $\varepsilon=10^{-199}$.

Consider the following existing seventh-order iterative methods for comparison:

In 2014, Al-Subaihi and Al-Qarni [7] suggested a seventh-order iterative method (SM) with four function evaluations given by:

$$\begin{aligned} y_n &= x_n - \frac{h(x_n)}{h'(x_n)} \\ z_n &= y_n + \frac{h(y_n)}{h'(x_n)} - 2\frac{h(x_n)h(y_n)}{h'(x_n)\left(h(x_n) - h(y_n)\right)} \\ x_{n+1} &= z_n - \frac{h(z_n)}{h[z_n, y_n] + h[z_n, x_n, x_n](z_n - y_n)} \end{aligned} \quad (21)$$

In 2021, Bawazir [8] proposed a seventh-order method (HM) with five functions given by:

$$\begin{aligned} y_n &= x_n - \frac{h(x_n)}{h'(x_n)} \\ z_n &= y_n - \frac{h(y_n)(1 + \mu/2)}{h'(y_n)} \\ x_{n+1} &= z_n + \frac{h(z_n)h(y_n)(1 + \mu/2)}{h'(y_n)\left(h(z_n) - h(y_n)\right)} \end{aligned} \quad (22)$$

where, $\mu = \frac{h(y_n)\left(h'(x_n)-h'(y_n)\right)}{h(x_n)h'(y_n)}$.

In 2016, Napassanan and Montri [9] developed a new scheme (NM) of order seven is given by:

$$\begin{aligned} y_n &= x_n - \frac{h(x_n)}{h'(x_n)} \\ z_n &= y_n - \frac{h(y_n)}{h'(y_n)} - \frac{h(y_n)^2 h''(y_n)}{2(h'(y_n))^3} \\ x_{n+1} &= z_n - \frac{(x_n - z_n)h(z_n)}{h(x_n)-2h(z_n)} \\ \text{where } h''(y_n) &= \frac{2}{y_n - x_n}\left(2h'(y_n)+h'(x_n)-3\frac{h(y_n)-h(x_n)}{y_n - x_n}\right) \end{aligned} \tag{23}$$

In 2019, Francisco et al. [10] presented a novel family of iterative methods (FM) of order seven to find the root of nonlinear equations given by:

$$\begin{aligned} y_n &= x_n - \frac{h(x_n)}{h'(x_n)} \\ z_n &= x_n - G(\eta)\frac{h(x_n)}{h'(x_n)}, \\ \text{where } G(\eta) &= 1+\frac{h(y_n)}{h(x_n)}+2\left(\frac{h(y_n)}{h(x_n)}\right)^2 \\ w_n &= z_n + \frac{h(z_n)}{h'(x_n)} \\ x_{n+1} &= z_n - \left(1-4\left(\frac{h(z_n)}{h(w_n)}\right)+8\left(\frac{h(z_n)}{h(w_n)}\right)^2\right)\frac{h(z_n)}{h'(x_n)} \end{aligned} \tag{24}$$

In 2019, a sixth-order iterative scheme (PM) proposed by Prem Chand et al. [11] is given by:

$$\begin{aligned} y_n &= x_n - \frac{h(x_n)}{h'(x_n)} \\ z_n &= x_n - \left(1+2\left(\frac{h(y_n)}{h(x_n)}\right)^2\right)\left(\frac{h(x_n)+h(y_n)}{h'(x_n)}\right) \\ x_{n+1} &= z_n - \left(1+2\left(\frac{h(y_n)}{h(x_n)}\right)\right)\frac{h(z_n)}{h'(x_n)} \end{aligned} \tag{25}$$

In 2020, Prem Chand et al. [12] presented an iterative method (CM) of order six is given by:

$$\begin{aligned} y_n &= x_n - \frac{2}{3}\frac{h(x_n)}{h'(x_n)} \\ z_n &= x_n - \left(-\frac{1}{4}+\frac{3}{4}\frac{h'(x_n)}{h'(y_n)}+\frac{1}{2}\frac{h'(y_n)}{h'(x_n)}\right)\cdot\frac{2h(x_n)}{h'(x_n)+h'(y_n)} \\ x_{n+1} &= z_n - \frac{h(z_n)}{h'(y_n)}\cdot\left(\frac{1}{2}+\frac{1}{2}\frac{h'(x_n)}{h'(y_n)}\right) \end{aligned} \tag{26}$$

In 2017, A sixth-order Iterative method (IM) is free from derivative for solving nonlinear equation is developed by Rahma et al. [13] and given by:

$$\begin{aligned} y_n &= x_n - \frac{h(x_n)}{h'(x_n)}, \quad x_{n+1} = y_n - \frac{h(y_n)}{h'(y_n)} \\ &- \frac{2h(y_n)^2 h'(y_n)h''(y_n)}{4(h'(y_n))^4 - 4h(y_n)(h'(y_n))^2 h''(y_n) + (h(y_n))^2(h''(y_n))^2} \end{aligned} \tag{27}$$

where, $h''(y_n) = \frac{2}{x_n - y_n}\left[3\frac{h(x_n)-h(y_n)}{x_n-y_n} - 2h'(y_n) - h'(x_n)\right]$.

In 2022. An efficient sixth-order Iterative method (EM) suggested by Sharma and Sunil [14] is given by:

$$\begin{aligned} y_n &= x_n - \frac{h(x_n)}{h'(x_n)} \\ x_{n+1} &= y_n - \frac{h(y_n)}{h'(y_n)} - \left(\frac{h(y_n)}{h'(y_n)}\left(\frac{-2M_1+M_2+M_3*M_4}{2M_1-2M_2-M_3*M_4}\right)\right)^2\frac{h''(y_n)}{2h'(y_n)} \end{aligned} \tag{28}$$

where,

$$M_1 = h(x_n)^3 h'(y_n)^2 h''(y_n), \; M_2 = h(x_n)^3 h(y_n)(h''(y_n))^2$$

$$M_3 = h'(x_n)h(y_n)h'(y_n), M_4 = h(x_n)^2 h''(y_n) - 2h(y_n)h'(x_n)^2$$

$$h''(y_n) = \frac{2}{x_n - y_n}\left[3\frac{h(x_n)-h(y_n)}{x_n - y_n} - 2h'(y_n) - h'(x_n)\right]$$

The analogy of an efficiency index of different iterative methods is shown in Table 1. Table 2 shows the roots of the test functions. For each test function and application, we offer the number of iterations, functional evaluations and first five absolute residual errors $|e_1|$, $|e_2|$, $|e_3|$, $|e_4|$, and $|e_5|$ in order to facilitate comparisons. Table 3 reports the comparison results for a few other techniques applied to application problems from various fields based on the number of function evaluations and inaccuracy.

Table 1. The analogy of the efficiency index

Method	P	N	E. I
SM	7	4	1.626
HM	7	5	1.475
NM	7	5	1.475
FM	7	5	1.475
PM	6	5	1.430
CM	6	4	1.565
IM	6	4	1.565
EM	6	4	1.565
SR	7	4	1.626

Note: P represents the order of the method, N represents the number of functional evaluations per iteration and E. I represent the efficiency index.

The following test functions (algebraic or transcendental) and their simple zeros are provided for our investigation [6]:

Table 2. Roots of the test functions

$h(z)$	Root
$h_1(x) = sin(2\cos x) - 1 - x^2 + e^{sin(x^3)}$	-0.784895987661
$h_2(x) = (x+2)e^x - 1$	-0.442854010023
$h_3(x) = x^2 + sin\left(\frac{x}{5}\right) - \frac{1}{4}$	0.409992017989
$h_4(x) = x^3 - 10$	2.154434690031
$h_{35}(x) = sin^2 x - x^2 + 1$	-1.404491648215

The efficacy and computational behavior of the provided methods are examined using the following application problems:

4.1 Azeotropic point of a binary solution [15]

To determine the azeotropic point of the nonlinear equation:

$$h_6(x)=\frac{PQ\left[Q(1-x)^2-Px^2\right]}{\left[x(P-Q)+Q\right]^2}+0.14845$$

where, the Van Laar equation's coefficients, which describe the phase equilibria of liquid solutions, are P and Q. We took, P=0.38969 and Q=0.55954 were used. The root of the above equation is 0.69147373574714144, is displayed in the table below.

4.2 Fractional conversion [11]

When using hydrogen feed that is transformed into ammonia at 500℃ and 250 atm, the fractional conversion of nitrogen can be calculated using the following equation:

$$h_7(x)=x^4-7.79075x^3+14.7445x^2+2.511x-1.674$$

The root lies in the range of 0 and 1 according to the definition of fractional conversion. Therefore, the real root is 0.2777595428417206.

4.3 Ideal and non-ideal gas laws [15]

The equation:

$$px=nRT$$

Stands for the ideal gas law, also known as the universal gas equation. Where p, x, n, R, T are pressure, volume, number of moles, a gas's universal constant, and temperature respectively. By resolving:

$$h_8(x)=\left(p+{}^{a}\!/\!_{x^2}\right)(x-b)-RT$$

can be used to calculate the molal volume.

For carbon dioxide, we use the values R=0.082054L atm/(mol K), where b=0.04267, T=300K and. Therefore, the nonlinear equation's root is 24.5125881284415006.

4.4 Parachutist's problem [15]

The total force for parachutists is calculated as:

$$F=mg-xv$$

where, m is the mass, g refers the acceleration due to gravity, x is the drag coefficient, and v is the parachutist's velocity, and from the above equation, we obtain the nonlinear equation:

$$h_9(x)=\frac{gm}{x}\left(1-e^{\frac{-x}{m}t}\right)-v$$

We suppose that the parameters will have values of g=9.8m/s^2, v=41m/s, m=68kg, and t=8s. Therefore, 12.533522848184467 is the root of the above nonlinear equation.

4.5 Study of multifactor effect problem [16]

The equation:

$$\begin{aligned} x(\mathrm{t}) &= x_0+\left(v_0+eE_0(\mathrm{m}\,w)^{-1}\sin(wt_0+\eta)\right)(t-t_0) \\ &\quad +eE_0\left(\mathrm{m}\,w^2\right)^{-1}\left(\cos(wt_0+\eta)+\sin(wt_0+\eta)\right) \end{aligned}$$

describes the moment of an electron in the space between two parallel plates. Where x_0 is the position of the electron, v_0 is the velocity, e refers the charge, RF electric field between plates at time t_0 is represented by $E_0\sin(wt_0+\eta)$, and the resting mass of an electron is m. Regarding the specific values, it can be reduced in polynomial form as:

$$h_{10}(x)=x-0.5\cos x-\frac{\pi}{4}$$

This function has a simple root at x=-0.3094661392082.

4.6 The vertical stress [15]

The vertical stress σ_x, which is generated at a place in the elastic continuum beneath the edge of a strip footing sustaining a constant pressure q and is specified by:

$$h_{11}(x)=\frac{x+\cos x\sin x}{\pi}-\frac{1}{4}$$

is one of the fundamental stresses experienced by finite subsurface constructions.

The nonlinear equation $h_{11}(x)$=0 has a root of 0.4160444988100767043.

4.7 Volume from Van der Waals equation [6]

An equation:

$$h(V)=pV^3-n(RT+Bp)V^2+n^2AV-n^3AB$$

represents the non-ideal gas in the Van der Waals equation.

Put V=x and for instance, the nonlinear polynomial function $h_{12}(x)$=40x^3-95.26535116x^2+35.28x-5.6998368.

It can be used to figure out how much 1.4 moles of benzene vapor will weigh at 500 C and 40 atm of pressure. This is due to the fact that benzene, A=18 and B=0.1154 are the Van der Waals constants. It has three roots, one of which, x≈1.9707842194070294, is real.

4.8 Blood rheology model [17]

The scientific field of blood rheology investigates the physical and flow characteristics of blood. Blood is considered a Caisson fluid since it is a non-Newtonian fluid. We assume the following function as a nonlinear equation to test the plug flow of Caisson fluids:

$$h_{13}(x)=\frac{1}{441}x^8-\frac{8}{63}x^5-0.0571428571x^4+\frac{16}{9}x^2-3.624489796x+0.3$$

The findings are displayed in the Table 3 as 0.0864335580522467, which is the root of $h_{13}(x)=0$.

4.9 A stirred tank reactor [18]

Think about the stirred tank's reactor. The reactor receives materials at rates of β and q-β, respectively. The equipment enhances mixed reaction as follows:

$$H_1+H_2\rightarrow H_3,\ H_3+H_2\rightarrow H_4$$
$$H_4+H_2\rightarrow H_5,\ H_4+H_2\rightarrow H_6$$

In their initial investigation of this intricate control system, Douglas [19] discovered the nonlinear polynomial equation shown below:

$$\frac{2.98\times(x+2.25)}{(x+1.45)\times(x+2.85)^2\times(x+4.35)}=\frac{1}{T_C}$$

where, T_c is the proportional controller's gain. By taking, we have T_c=0.

$$h_{14}(x)=x^4+11.50x^3+47.49x^2+83.06325x-51.23266875=0$$

The root of the above nonlinear equation is -1.45.

Table 3. The analogy of different methods

Method	x_0	n	$\lvert e_1\rvert$	$\lvert e_2\rvert$	$\lvert e_3\rvert$	$\lvert e_4\rvert$	$\lvert e_5\rvert$	$\lvert h(x_{n+1})\rvert$
$h_1(x)$	-0.9							
SM		4	0.115103	1.2E-07	3.00E-49	2.36E-340	---	6.67E-340
HM		4	0.115104	4.5E-08	1.76E-45	6.23E-270	---	1.75E-269
NM		4	0.115104	2.5E-08	1.62E-54	7.78E-378	---	2.19E-377
FM		4	0.115104	1.5E-07	2.60E-48	1.31E-333	---	3.71E-333
PM		5	0.1151	3.8E-06	1.80E-32	2.00E-190	5.47E-691	1.64E-690
CM		5	0.115102	1.8E-06	5.60E-35	6.01E-136	6.48E-439	1.83E-438
IM		5	0.115108	3.7E-06	9.41E-32	2.50E-185	4.78E-690	1.09E-690
EM		5	0.115042	6.2E-05	1.00E-17	6.81E-69	1.38E-273	3.89E-273
SR		4	0.115104	2.3E-08	4.30E-55	3.29E-382	---	9.27E-382
	-0.5							
SM			Divergent					
HM		5	0.277695	0.007201	3.13E-14	1.99E-82	1.32E-491	3.73E-491
NM		5	0.284602	0.000294	4.96E-26	1.90E-178	1.77E-690	1.09E-690
FM		5	0.23325	0.051650	2.70E-09	1.90E-60	1.45E-418	4.10E-418
PM		5	0.25856	0.026330	2.50E-09	1.30E-51	3.17E-305	8.94E-305
CM		5	0.29106	0.006160	9.60E-14	6.00E-69	7.22E-238	2.03E-237
IM		5	0.350526	0.065629	9.10E-07	2.07E-35	2.86E-207	8.10E-207
EM		6	0.29887	0.013980	2.40E-08	2.10E-31	1.00E-123	3.56E-492
SR		4	0.28495	4.9E-050	7.80E-32	2.00E-219	---	6.01E-219
$h_2(x)$	-0.2							
SM		4	0.15715	1.50E-06	4.80E-42	2.03E-290	---	3.34E-290
HM		4	0.157148	2.60E-06	4.99E-35	2.50E-207	---	4.10E-207
NM		4	0.157145	5.91E-07	4.73E-45	9.99E-312	---	1.64E-311
FM		4	0.15714	1.20E-06	4.50E-42	3.49E-290	---	5.74E-290
PM		5	0.1572	5.10E-05	3.90E-26	7.00E-153	3.14E-689	5.47E-691
CM		5	0.15715	7.50E-06	7.10E-32	1.00E-126	3.85E-411	6.32E-411
IM		5	0.157181	3.51E-05	3.12E-27	1.50E-159	1.09E-690	5.47E-691
EM		5	0.15723	8.80E-05	5.50E-18	9.00E-71	6.11E-282	1.00E-281
SR		4	0.15715	1.10E-07	9.70E-51	3.34E-352	---	5.48E-352
	-0.6							
SM		4	0.15715	1.50E-06	4.80E-42	2.03E-290	---	3.34E-290
HM		4	0.157148	2.60E-06	4.99E-35	2.51E-207	---	4.14E-207
NM		4	0.157145	5.91E-07	4.73E-45	9.99E-312	---	1.64E-311
FM		4	0.15714	1.20E-06	4.50E-42	3.49E-290	---	5.74E-290
PM		5	0.1572	5.10E-05	3.90E-26	7.00E-153	3.14E-689	5.47E-691
CM		5	0.15715	7.50E-06	7.10E-32	1.00E-126	3.85E-411	6.32E-411
IM		5	0.157181	3.51E-05	3.12E-27	1.50E-159	1.09E-690	5.47E-691
EM		5	0.15723	8.80E-05	5.50E-18	9.01E-71	6.11E-282	1.00E-281
SR		4	0.15715	1.10E-07	9.65E-51	3.34E-352	---	5.48E-352
$h_3(x)$	0.3							
SM		4	0.109992	6.96E-07	5.72E-46	1.46E-319	---	1.48E-319
HM		5	0.109995	3.12E-06	8.32E-34	3.03E-199	3.42E-691	5.47E-691
NM		4	0.109991	7.48E-07	2.32E-43	6.49E-299	---	6.62E-299
FM		4	0.109991	9.43E-07	5.86E-42	2.11E-288	---	2.15E-288
PM		5	0.110080	8.82E-05	7.62E-24	3.23E-138	2.73E-691	5.41E-691
CM		5	0.110008	1.63E-05	7.93E-29	2.90E-117	1.51E-382	1.54E-382
IM		5	0.110196	0.000204	1.97E-21	1.60E-123	1.37E-690	5.47E-691
EM		5	0.110110	0.000118	8.93E-17	2.93E-65	3.38E-259	3.45E-259
SR		4	0.109992	9.22E-10	9.26E-70	9.55E-490	---	9.74E-490

	0.5							
SM		4	0.090008	1.27E-08	3.87E-58	9.44E-405	---	9.63E-405
HM		4	0.090008	2.86E-07	4.92E-40	1.28E-236	---	1.31E-236
NM		4	0.090008	4.63E-08	8.08E-52	3.96E-358	---	4.04E-358
FM		4	0.090008	2.22E-07	2.38E-46	3.83E-319	---	3.91E-319
PM		5	0.090004	3.74E-06	4.47E-32	1.30E-187	8.89E-691	4.78E-691
CM		5	0.090007	1.20E-06	1.23E-35	1.11E-137	7.69E-444	7.84E-444
IM		5	0.090002	5.50E-06	7.53E-31	4.89E-180	1.36E-690	5.47E-691
EM		5	0.089988	1.95E-05	6.71E-20	9.31E-78	3.45E-309	3.52E-309
SR		4	0.090008	1.37E-11	1.51E-82	2.96E-579	---	3.02E-579
$h_4(x)$	3							
SM		4	0.154435	0.000106	2.76E-30	2.10E-209	---	2.99E-208
HM			Divergent					
NM		5	0.844733	0.000833	4.62E-24	7.50E-166	0	1.31E-689
FM		5	0.842430	0.003135	2.77E-19	1.20E-131	1.69E-689	1.31E-688
PM		5	0.835893	0.009672	2.58E-13	9.61E-77	2.56E-457	3.57E-456
CM		5	0.841309	0.004257	4.17E-16	4.50E-79	1.21E-268	1.69E-267
IM		7	0.301894	0.365484	0.178198	1.07E-05	1.17E-28	1.31E-689
EM		6	0.861193	0.015628	4.22E-10	2.46E-40	2.80E-161	6.77E-644
SR		4	0.845483	8.21E-05	2.79E-32	1.49E-224	---	2.10E-223
	1							
SM		6	0.154435	7.85E-08	3.26E-52	5.24E-35	5.24E-35	2.69E-241
HM			Divergent					
NM		4	0.154435	4.86E-08	1.07E-53	2.64E-373	---	3.68E-372
FM		4	0.154434	1.98E-07	1.10E-48	1.82E-337	---	2.54E-336
PM		5	0.154442	7.39E-06	5.31E-32	7.30E-189	2.29E-689	1.31E-689
CM		5	0.154436	1.33E-06	3.95E-37	8.20E-143	7.31E-460	1.47E-377
IM		6	0.658368	0.488293	0.007774	1.36E-11	4.91E-64	1.47E-377
EM		5	0.154434	2.79E-07	4.69E-29	3.70E-116	1.48E-464	2.07E-463
SR		4	0.154435	3.24E-09	4.19E-63	3.50E-441	---	3.52E-439
$h_5(x)$	-1							
SM		4	0.204531	3.89E-05	2.54E-32	1.27E-222	---	3.20E-222
HM		5	0.204454	3.77E-05	8.44E-28	1.10E-163	4.37E-690	8.48E-690
NM		4	0.204478	1.36E-05	3.20E-35	1.27E-242	---	3.17E-242
FM			Divergent					
PM			Divergent					
CM		5	0.426101	0.021610	1.08E-10	8.32E-57	2.20E-201	5.40E-201
IM		5	0.468371	0.063879	2.93E-07	3.71E-39	1.51E-230	3.75E-230
EM		6	0.451963	0.047468	3.57E-06	1.28E-22	2.10E-88	3.82E-351
SR		4	0.204491	3.00E-07	1.03E-48	5.65E-339	---	1.40E-338
	0.7							
SM		6	8.124874	7.410812	0.00957	1.19E-15	6.40E-106	1.64E-689
HM			Divergent					
NM		5	0.651895	0.052597	5.25E-10	4.04E-66	6.48E-459	5.45E-458
FM		13	191511.5	1548910	29617.30	5663.092	1082.98	1.64E-689
PM		18	1699089	1306983	3016178	6960553	1606314	2.49E-206
CM		6	1.384208	0.673621	0.006096	5.83E-14	7.09E-70	3.34E-240
IM		8	16.95328	13.33576	2.791647	0.121366	1.00E-05	8.48E-690
EM			Divergent					
SR		5	2.128140	0.023648	1.45E-14	6.40E-100	2.73E-691	6.45E-690
$h_6(x)$	0.9							
SM		4	0.208528	1.84E-06	9.90E-42	1.28E-288	---	1.41E-288
HM		4	0.208522	4.41E-06	4.70E-34	6.90E-202	---	7.57E-202
NM		4	0.208525	1.03E-06	8.62E-44	2.39E-303	---	2.63E-303
FM		4	0.208521	4.99E-06	2.31E-38	1.06E-264	---	1.17E-264
PM		5	0.208486	4.01E-05	2.86E-27	3.70E-160	3.01E-690	5.47E-691
CM		5	0.208518	8.18E-06	3.20E-32	6.70E-128	6.09E-415	6.71E-415
IM		5	0.208472	5.39E-05	1.91E-26	3.80E-155	1.50E-690	7.18E-691
EM		5	0.208517	9.54E-06	2.10E-22	4.91E-89	1.48E-355	1.62E-355
SR		4	0.208526	6.64E-07	1.64E-45	9.33E-316	---	1.03E-315
	0.2							
SM			Divergent					
HM		5	0.492151	0.000678	6.15E-21	3.40E-123	4.10E-691	7.18E-691
NM		5	0.490938	0.000535	8.55E-25	2.30E-170	6.84E-691	7.18E-691
FM		5	0.495749	0.004275	7.88E-18	5.70E-121	1.91E-690	1.57E-690
PM		5	0.498268	0.006794	6.68E-14	6.08E-80	3.46E-476	3.81E-476
CM		5	0.492947	0.001473	1.09E-18	2.69E-87	3.99E-293	4.39E-293
IM		5	0.522265	0.030791	6.38E-10	5.27E-56	1.66E-332	1.83E-332
EM		6	0.496602	0.005128	1.72E-11	2.20E-45	5.90E-181	7.18E-691
SR		4	0.491330	0.000138	2.79E-29	3.80E-202	---	4.23E-202
$h_7(x)$	0.1							

SM		5	0.178041	0.000282	6.97E-25	4.00E-169	1.98E-690	6.29E-690
HM		6	0.290677	0.112961	4.40E-05	6.24E-27	5.10E-158	2.74E-690
NM		5	0.177624	0.000135	2.03E-27	3.40E-187	4.10E-691	6.29E-690
FM		5	0.178224	0.000464	4.47E-23	3.39E-156	8.21E-691	6.29E-690
PM		5	0.183736	0.005976	9.21E-13	1.35E-71	1.35E-424	1.21E-423
CM		5	0.179821	0.002062	5.62E-16	1.73E-77	3.02E-263	2.71E-262
IM		6	0.179596	0.001837	4.96E-12	2.68E-46	2.30E-183	2.74E-690
EM			Divergent					
SR		4	0.177759	9.62E-08	3.98E-50	9.23E-348	---	7.34E-346
	0.5							
SM		4	0.222227	1.31E-05	3.36E-34	2.43E-234	---	2.18E-233
HM		5	0.226725	0.004485	6.36E-15	5.68E-86	2.88E-512	2.59E-511
NM		4	0.222235	5.37E-06	3.12E-37	7.06E-256	---	6.34E-255
FM		4	0.222225	1.57E-05	2.22E-33	2.60E-228	---	2.32E-227
PM		5	0.222075	0.000165	4.40E-22	1.60E-127	2.02E-689	2.73E-690
CM		5	0.222141	0.000099	7.30E-24	2.30E-102	6.69E-338	6.01E-337
IM		25	3.678586	0.192571	0.518845	0.4208610	0.209340	1.91E-690
EM		5	0.222204	0.000196	6.56E-16	8.22E-62	2.02E-245	1.81E-244
SR		4	0.222237	3.79E-06	5.79E-39	1.13E-268	---	1.02E-267
$h_8(x)$	4							
SM		4	20.47773	0.034860	7.57E-28	1.70E-207	---	1.71E-207
HM		5	20.56040	0.047813	1.02E-26	9.80E-175	4.38e-690	3.94E-689
NM		4	20.50939	0.003197	2.02E-35	8.19E-261	---	8.14E-261
FM		4	20.50714	0.005451	1.04E-35	9.55E-265	---	9.49E-265
PM		5	20.83466	0.322077	2.62E-17	8.20E-114	5.25E-689	6.13E-689
CM		5	20.64364	0.131050	2.36E-18	4.86E-93	4.21E-317	4.18E-317
IM		5	20.50714	0.005447	2.58E-27	2.90E-173	4.07E-688	3.94E-689
EM		5	19.14509	1.367500	1.31E-07	8.80E-36	1.80E-148	3.00E-599
SR		4	20.51028	0.002304	2.02E-36	8.33E-268	---	8.28E-268
	30							
SM		4	5.487412	3.93E-13	1.70E-104	3.11E-688	---	6.13E-689
HM		4	5.487412	1.46E-13	8.37E-96	2.97E-589	---	2.95E-589
NM		4	5.487412	2.15E-13	1.20E-106	0	---	3.94E-689
FM		4	5.487412	2.44E-15	3.80E-122	1.22E-688	---	6.13E-689
PM		4	5.487412	2.03E-10	1.79E-72	8.32E-445	---	8.27E-445
CM		4	5.487412	5.17E-09	9.75E-59	3.41E-214	---	3.40E-214
IM		4	5.487412	4.80E-10	1.20E-69	3.01E-427	---	2.98E-427
EM		5	5.487424	1.21E-05	6.32E-28	4.70E-117	1.51E-473	1.51E-473
SR		4	5.487412	2.14E-13	1.20E-106	1.75E-689	---	6.13E-689
$h_9(x)$	7							
SM		4	5.533826	0.000474	2.91E-32	9.51E-230	---	1.75E-229
HM		5	6.081900	0.548548	3.01E-09	3.36E-59	6.59E-359	1.21E-358
NM		4	5.533054	0.000298	5.54E-34	4.25E-242	---	7.82E-242
FM		4	5.531875	0.001477	1.99E-28	1.60E-202	---	2.99E-202
PM		5	5.525900	0.007453	8.94E-20	4.40E-106	6.25E-686	2.63E-689
CM		5	5.531462	0.001890	3.94E-24	1.20E-113	5.93E-352	1.09E-351
IM		5	5.518294	0.015058	2.04E-18	3.47E-25	6.44E-685	1.18E-684
EM		6	5.235217	0.298131	4.87E-06	2.70E-121	8.90E-102	7.15E-408
SR		4	5.533150	0.000202	1.35E-35	7.91E-254	---	1.46E-253
	15							
SM		4	2.466652	3.99E-06	8.74E-47	2.11E-331	---	3.88E-331
HM			Divergent					
NM		5	10.01989	0.013463	2.14E-22	5.40E-161	0	4.12E-352
FM		4	2.466643	5.14E-06	1.24E-45	5.70E-323	---	1.05E-322
PM		5	2.466799	0.000151	6.14E-30	2.80E-182	1.71E-688	2.63E-689
CM		5	2.466669	2.10E-05	7.40E-36	2.80E-182	1.72E-457	3.16E-457
IM		4	2.466679	0.000321	1.89E-34	8.10E-210	---	1.50E-209
EM		5	2.485734	0.019086	8.19E-11	2.78E-44	3.70E-178	2.63E-689
SR		4	2.466647	5.08E-07	8.48E-54	3.07E-381	---	5.65E-381
$h_{10}(x)$	-0.7							
SM		4	0.390538	3.65E-06	8.48E-42	3.06E-291	---	2.60E-291
HM		4	0.390541	7.45E-06	2.99E-34	1.30E-204	---	1.07E-204
NM		4	0.390533	4.96E-07	2.05E-48	4.17E-338	---	3.54E-338
FM		4	0.390531	3.34E-06	2.12E-41	8.59E-288	---	7.29E-288
PM		5	0.390669	0.000135	1.55E-25	3.60E-151	1.01E-689	2.74E-689
CM		5	0.390552	1.79E-05	1.66E-31	2.20E-126	5.46E-411	4.63E-411
IM			Divergent					
EM		5	0.390149	0.000385	3.81E-16	3.62E-64	3.00E-256	2.54E-256
SR		4	0.390534	2.36E-07	2.90E-51	1.22E-358	---	1.03E-358
	0.5							
SM		4	0.809435	3.07E-05	2.52E-35	6.22E-246	---	5.28E-246

HM		5	0.809378	8.84E-05	8.35E-28	5.90E-166	4.78E-691	5.47E-691
NM		4	0.809449	1.72E-05	1.23E-37	1.16E-262	---	9.85E-263
FM		5	0.809244	0.000222	1.22E-28	1.80E-198	4.65E-690	1.37E-690
PM		5	0.808076	0.001390	1.86E-19	1.10E-114	3.55E-686	3.01E-686
CM		5	0.808913	0.000553	1.44E-22	1.40E-99	1.48E-330	1.20E-330
IM		5	0.807733	0.001733	1.10E-18	7.10E-110	5.15E-657	4.37E-657
EM		6	0.815672	0.006206	2.57E-11	7.49E-45	5.40E-179	5.47E-691
SR		4	0.809466	4.49E-07	2.64E-49	6.36E-345	---	5.39E-345
$h_{11}(x)$	-0.6							
SM		4	0.183954	1.45E-06	1.18E-42	2.80E-295	---	1.49E-295
HM		4	0.183957	1.87E-06	7.43E-37	2.82E-219	---	1.51E-219
NM		4	0.183955	3.24E-07	4.68E-48	6.06E-334	---	3.23E-234
FM		4	0.183954	1.23E-06	4.04E-43	1.67E-298	---	8.91E-299
PM		5	1.071820	0.055776	2.52E-08	1.39E-46	3.99E-276	2.13E-276
CM		5	1.046144	0.030100	1.73E-10	1.64E-56	5.30E-201	2.80E-201
IM		6	1.152809	0.136785	2.02E-05	6.01E-29	4.20E-170	3.42E-691
EM		7	0.288316	0.726026	0.001703	1.53E-12	1.02E-48	5.47E-691
SR		4	0.183955	1.17E-07	4.23E-51	3.39E-355	---	1.81E-355
	0.9							
SM		5	0.497056	0.013100	5.51E-15	1.40E-101	1.43E-690	1.37E-691
HM		5	0.485554	0.001598	2.78E-19	7.80E-114	3.67E-681	1.95E-681
NM		5	0.482772	0.001183	4.06E-23	2.30E-159	0	1.37E-691
FM		8	1.380297	0.949824	3582.672	3583.486	0.101104	1.20E-387
PM		9	0.227427	991.2025	982.6279	8.940183	0.355279	5.57E-450
CM		5	0.509435	0.025479	4.71E-11	8.97E-59	8.70E-208	4.60E-208
IM		7	2.824133	10.48625	8.285516	0.139444	1.98E-06	1.00E-206
EM		6	0.527256	0.0433	5.42E-07	1.59E-26	1.20E-104	1.83E-417
SR		5	0.484290	0.000334	6.63E-27	7.91E-186	1.37E-691	3.42E-691
$h_{12}(x)$	2.5							
SM		8	0.528856	0.000360	2.17E-24	2.09E-62	2.01E-100	2.24E-212
HM		8	0.362811	0.142116	0.02434	5.14E-05	1.42E-26	1.18E-404
NM		8	0.526889	0.002327	6.25E-18	4.24E-54	4.08E-92	4.57E-204
FM		7	0.171201	0.000417	4.11E-23	3.10E-71	2.30E-119	5.07E-205
PM		10	0.513484	0.015732	3.70E-10	2.65E-38	8.13E-67	2.80E-207
CM		13	0.520835	0.008381	2.10E-12	2.06E-31	2.02E-50	2.20E-200
IM			Divergent					
EM		14	0.521721	0.007495	2.39E-09	2.25E-27	2.20E-46	2.30E-215
SR		5	0.529028	0.000187	1.06E-27	1.23E-92	1.63E-222	2.05E-220
	1.8							
SM		8	0.170925	0.000141	3.06E-27	2.90E-65	2.83E-103	3.16E-215
HM		7	0.216648	0.048712	0.002847	1.95E-12	1.30E-52	1.45E-291
NM		8	0.170745	3.97E-05	2.46E-23	2.37E-61	2.28E-99	2.56E-211
FM		7	0.171201	0.000417	4.11E-23	3.10E-71	2.30E-119	1.66E-213
PM		10	0.173303	0.002518	7.01E-15	2.15E-43	6.61E-72	2.30E-212
CM		13	0.171210	0.000426	1.05E-19	1.03E-38	1.01E-57	1.10E-207
IM		16	0.009717	0.010149	0.010655	0.011259	0.011994	1.90E-104
EM		13	0.172196	0.001412	3.09E-12	3.07E-31	0.011994	3.20E-200
SR		5	0.170782	0.000001	6.71E-40	4.86E-117	2.55E-271	3.20E-269
$h_{13}(x)$	-0.6							
SM		4	0.686277	0.000157	9.83E-30	3.80E-206	---	1.26E-205
HM		5	0.466648	6.93E-10	5.05E-63	7.55E-382	1.56E-333	5.17E-333
NM		5	0.686026	0.000407	7.61E-26	6.10E-178	5.13E-692	3.05E-691
FM		5	0.685246	0.001187	7.70E-22	3.80E-149	1.03E-691	4.79E-691
PM		5	0.681939	0.004494	6.25E-15	4.62E-86	7.52E-513	2.49E-512
CM		5	0.684256	0.002177	2.05E-17	1.88E-83	1.16E-281	3.87E-281
IM		6	0.422264	0.264083	8.61E-05	4.63E-25	1.10E-146	2.05E-691
EM		6	0.666446	0.019987	4.56E-08	1.28E-30	7.90E-121	3.74E-481
SR		4	0.686433	1.11E-07	4.52E-54	1.77E-379	---	2.86E-378
	-0.2							
SM		4	1.03E-691	9.25E-09	2.49E-59	2.51E-413	---	8.34E-413
HM		4	0.113566	7.86E-08	1.03E-44	5.30E-266	---	1.76E-265
NM		4	0.113566	1.45E-08	5.69E-57	7.91E-396	---	2.62E-395
FM		4	0.113566	6.88E-08	1.69E-51	9.29E-357	---	8.08E-356
PM		5	0.286325	0.000109	1.29E-24	3.00E-144	1.71E-692	2.05E-691
CM		5	0.286394	3.96E-05	7.52E-28	7.50E-115	7.46E-376	2.47E-375
IM		5	0.286309	0.000125	4.35E-24	7.70E-141	4.45E-691	1.36E-691
EM		6	0.285258	0.001175	5.63E-13	2.98E-50	2.34E-199	2.05E-691
SR		4	0.113566	3.68E-11	2.02E-78	6.36E-550	---	1.03E-548
$h_{14}(x)$	-1.3							
SM		4	0.150007	7.05E-06	4.63E-35	2.41E-239	---	1.37E-238
HM		5	0.152445	0.002445	3.79E-15	5.22E-86	3.53E-511	2.01E-510

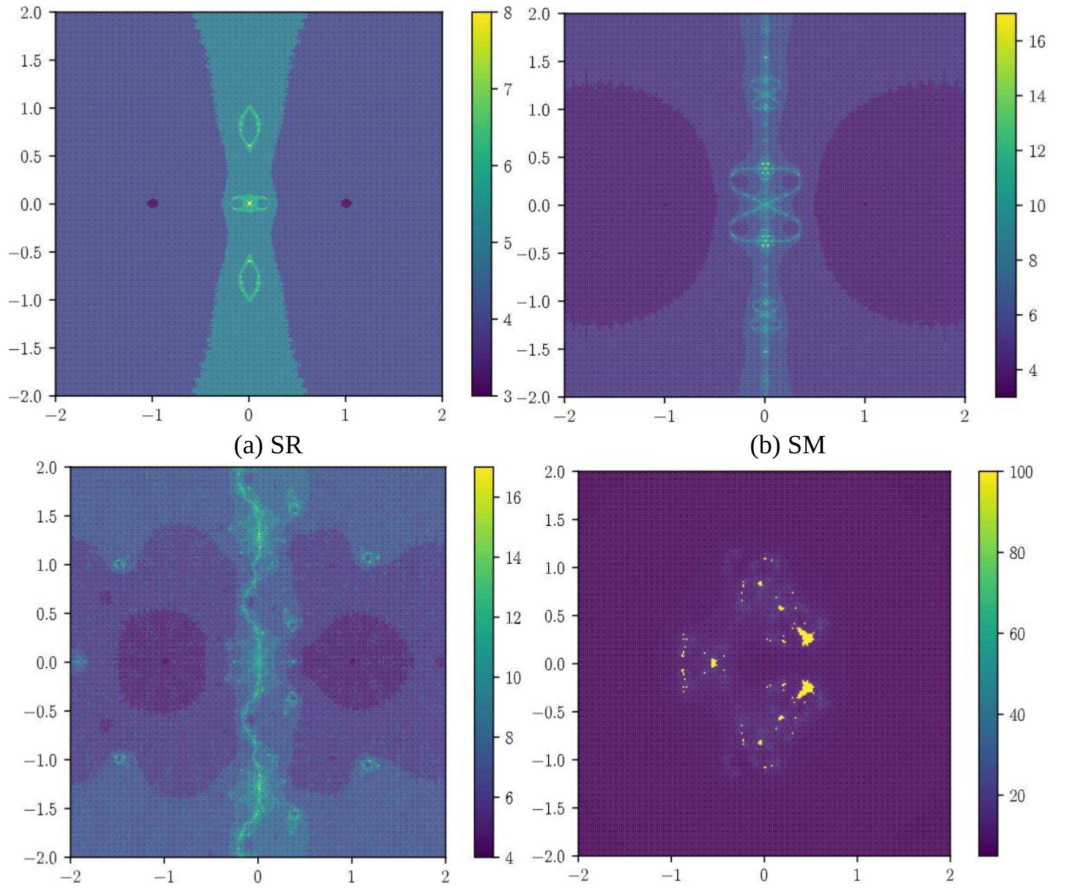
NM		4	0.149974	2.58E-05	4.10E-31	1.04E-211	---	5.90E-211
FM		5	0.149887	0.000113	6.95E-26	2.30E-174	1.37E-689	2.62E-689
PM		5	0.149496	0.000504	4.34E-18	1.80E-102	8.94E-609	5.08E-608
CM		5	0.149812	0.000188	2.60E-21	3.42E-94	7.73E-313	4.39E-313
IM		5	0.14476	0.00524	8.29E-12	1.47E-64	4.67E-381	2.65E-380
EM		6	0.149065	0.000935	3.05E-12	3.47E-46	5.80E-182	2.62E-689
SR		4	0.149998	2.00E-06	4.01E-40	5.25E-276	---	2.99E-275
	-1.5							
SM		4	0.05	1.08E-07	9.03E-48	2.61E-328	---	1.48E-327
HM		4	0.049999	6.35E-07	1.15E-36	4.04E-215	---	2.29E-214
NM		4	0.149974	0.000026	4.10E-31	1.04E-211	---	7.49E-339
FM		4	0.05	2.01E-07	3.83E-45	3.52E-309	---	2.00E-308
PM		5	0.050008	8.02E-06	7.11E-29	3.50E-167	1.25E-689	4.69E-409
CM		5	0.050001	1.40E-06	4.43E-34	1.70E-132	9.18E-428	6.12E-689
IM		5	0.050005	4.67E-06	4.71E-30	5.00E-174	1.69E-689	5.22E-427
EM		5	0.050033	3.32E-05	4.85E-18	2.22E-69	9.69E-275	2.62E-689
SR		4	0.05	3.71E-09	3.08E-59	5.36E-412	---	5.51E-274

Note: x_0 represents the starting approximation, n number of iterations, $|e_i|$ represents error and $|h(x_{n+1})|$ represents n[th] functional evaluation.

5. BASINS OF ATTRACTION

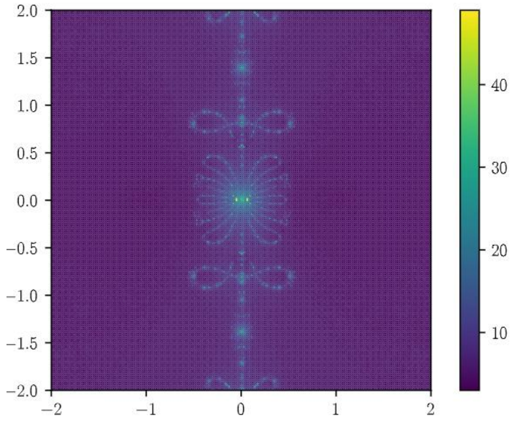
In this study, we will use basins of attraction to visually compare a few test polynomials in the complex domain. The starting points in the complex plane that travel to the root are known as the basins of attraction. For computations, let us consider a square region $[-2,2]\times[-2,2]\in C^2$ with 250×250 mesh points. We examine the iterative methods in all the mesh points z^0 in the square part. The stopping criterion is used, and 100 iterations are the most that can be made without reaching an attractive root. To illustrate the basins of attraction, we present three test polynomials to discuss the proposed method's efficiency in solving the roots of a complex function. The roots of $f_1(z)$=1-z^2, $f_2(z)$=1-z^3 and $f_3(z)$=1-z^4 are mapped with a dark violet color. The regions where the colors are purple are assigned to a more significant number of iterations to converge than with blue and yellow, representing that the method cannot find the root in that region. We compare our newly developed method (SR) with well-known seventh-order methods such as SM, HM, NM, FM, PM, CM, IM, and EM. All the computations(graphs) are made by PYTHON programming using the mpmath library. Figures 1-3 demonstrate that the suggested approach (SR) exhibits the greatest results and is rapidly convergent with the fewest iterations when compared to other methods.

$f_1(z)=1-z^2$

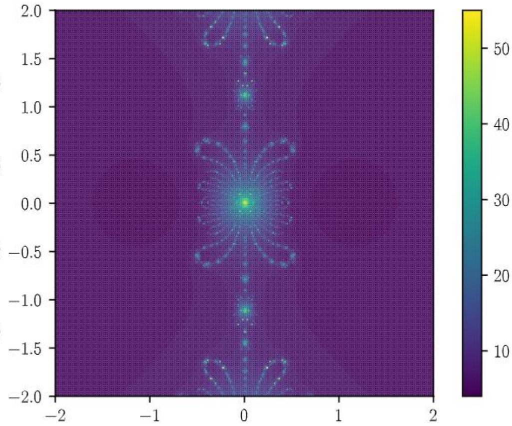


(a) SR (b) SM

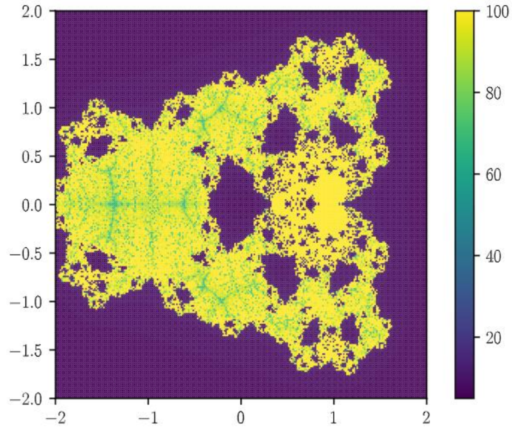
(c) HM (d) NM



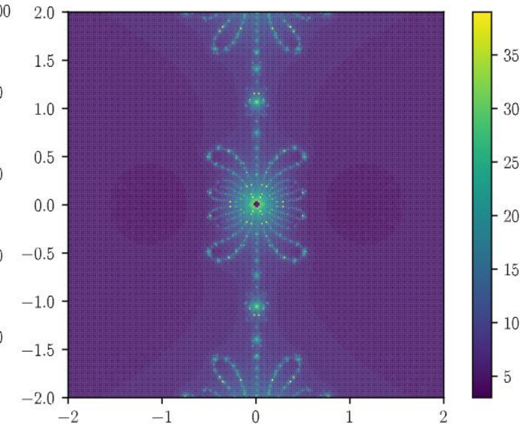
(e) FM



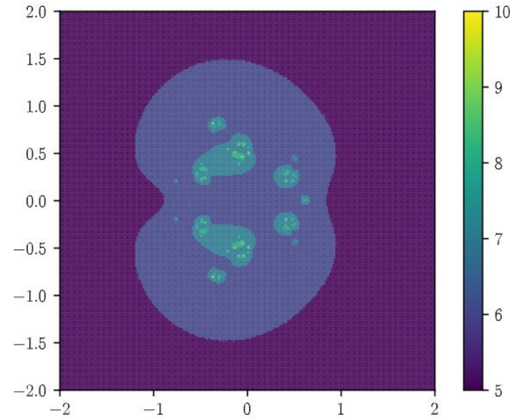
(f) PM



(g) CM



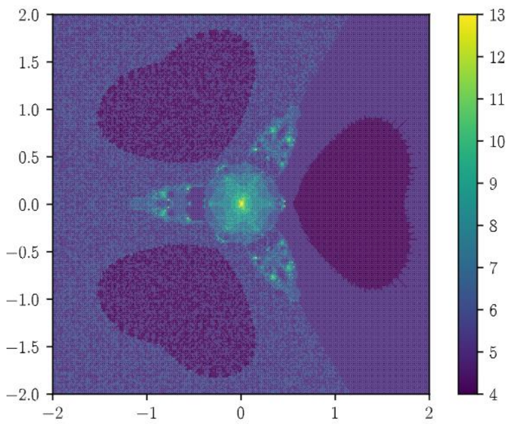
(h) IM



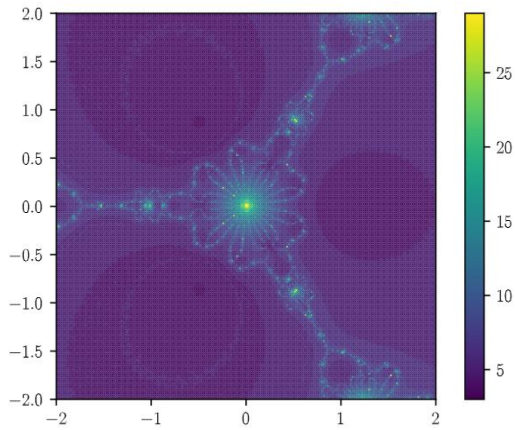
(i) EM

Figure 1. The polynomiographs obtained by the suggested method SR, SM, HM, NM, FM, PM, CM, IM, and EM for $f_1(z)$

$$f_2(z) = 1 - z^3$$



(a) SR



(b) SM

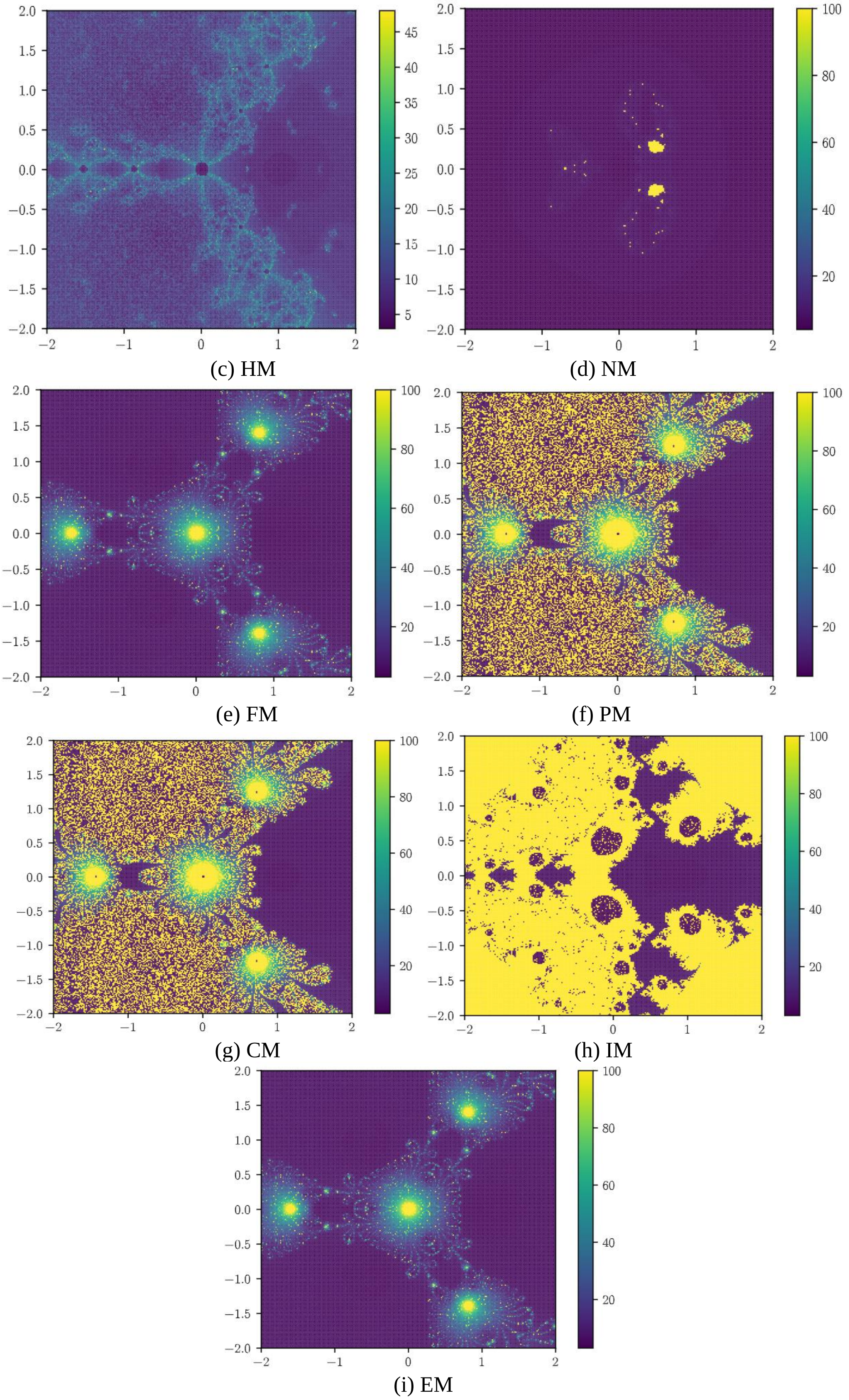


Figure 2. The polynomiographs obtained by the suggested method SR, SM, HM, NM, FM, PM, CM, IM, and EM for $f_2(z)$

5.1 Example 1

The first forecasts at or near the limits require the most iterations, as shown in Figures 1-3 (yellow lines in the figures). The smallest number of iterations is required when the first guess, z^0, is relatively close to the exact result.

The polynomiographs created using the polynomial approaches are displayed in Figure 1. We can see that the SR approach works well. Near the root, the methods SM, HM, IM, and EM exhibit some chaotic behaviour. In this scenario, the initial guess selection affects the performance of the procedures NM, FM, PM, and CM.

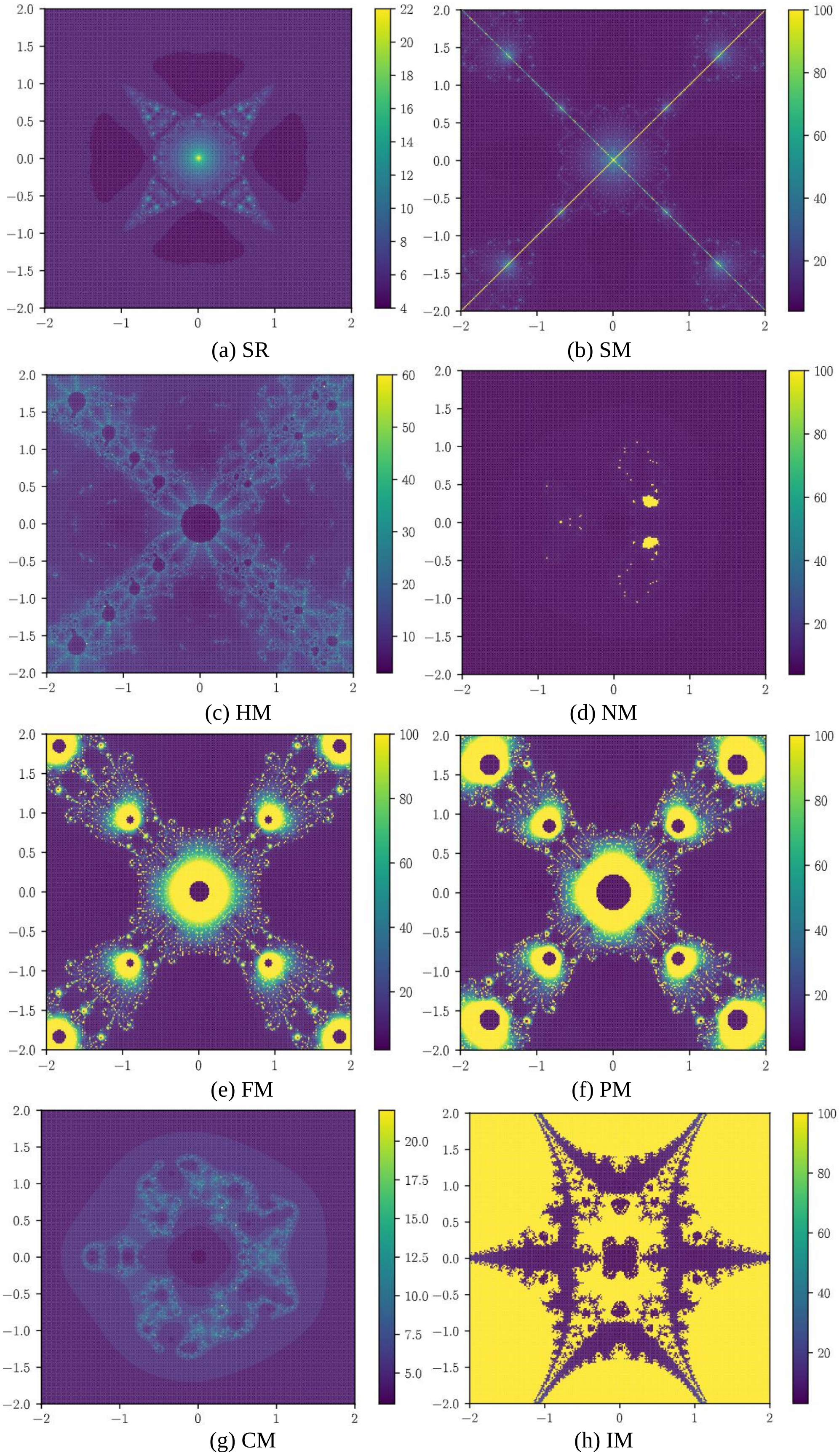
5.2 Example 2

The polynomiographs produced using the polynomial approaches are shown in Figure 2. We can see that the SR approach works well. Near the root, the methods SM, HM, NM, FM, PM, IM, and EM exhibit some chaotic behaviour.

5.3 Example 3

Figure 3 displays the polynomiographs created using the polynomial techniques. We can observe that the SR strategy is effective. The methods SM, HM, NM, FM, PM, CM, IM, and EM display some chaotic behaviour close to the root.

$f_3(z) = 1 - z^4$

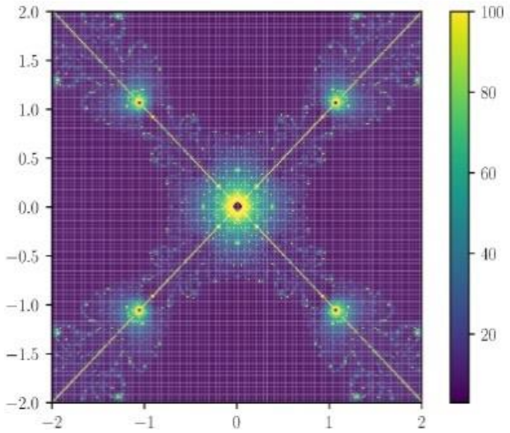


(a) SR (b) SM

(c) HM (d) NM

(e) FM (f) PM

(g) CM (h) IM



(i) EM

Figure 3. The polynomiographs obtained by the suggested methods SR, SM, HM, NM, FM, PM, CM, IM, and EM for $f_3(z)$

6. CONCLUSIONS

In this study, we have devised an effective finite difference approximation-based seventh-order iterative approach for solving nonlinear equations. The approaches call for four functions' evaluations to be computed, with a seven-fold order of convergence. This suggested method's order of convergence has undergone development. We test the proposed scheme and some other known schemes on a few examples, demonstrating the superiority of the proposed technique SR. Additionally, the suggested strategy and a few others already in use have been used to solve real-world issues. The outcomes show promise for the new approach SR and are intriguing. The results of the numerical tests indicate that the novel approach would be a valuable substitute for solving nonlinear equations. Finally, we also contrasted the complex plane basins of attraction of several seventh and sixth order approaches.

Future study will include:

• We are currently looking at the idea of using Newton's method to create optimal methods of any order.

• Additionally, we are looking into ways to solve systems of nonlinear equations without the need of derivatives.

ACKNOWLEDGMENT

The authors appreciate the assistance and materials provided by GITAM (Deemed to be University).

REFERENCES

[1] Traub, J.F. (1977). Iterative Methods for the Solution of Equations. Chelsea Publishing Company, New York.

[2] Wait, R. (1979). The Numerical Solution of Algebraic Equations. John Wiley and Sons, New York.

[3] Gautschi, W. (1997). Numerical Analysis: An Introduction. Birkhauser, Boston, M.A.

[4] Weerakoon, S., Fernando, T.G.I. (2000). A variant of Newton's method with accelerated third-order convergence. Applied Mathematical Letters, 13(8): 87-93. https://doi.org/10.1016/S0893-9659(00)00100-2

[5] Petković, M.S., Neta, B., Petković, L.D., Džunić, J. (2014). Multipoint methods for solving nonlinear equations: A survey. Applied Mathematics and Computation, 226: 635-660. https://doi.org/10.1016/j.amc.2013.10.072

[6] Kumar Mylapalli, M.S., Palli, R.K., Chaganti, P., Sri, R. (2022). An optimal fourth order iterative method for solving non-linear equations. IAENG International Journal of Applied Mathematics, 52(3): 732-741.

[7] Al-Subaihi, I.A., Al-Qarni, A.J. (2014). Higher-order iterative methods for solving nonlinear equations. Life Science Journal, 11(12): 85-91.

[8] Bawazir, H.M. (2021). Seventh and twelfth-order iterative methods for roots of nonlinear equations. Hadhramout University Journal of Natural & Applied Science, 18(1): 9-15.

[9] Napassanan, S., Montri, T. (2016). A note on three-step iterative method with seventh-order of convergence for solving nonlinear equations. The Thai Journal of Mathematics, 14(3): 565-573.

[10] Francisco, I.C., Cordero, A., Garrido, N., Torregrosa, J.R. (2019). Wide stability in a new family of optimal fourth-order iterative methods. Computational and Mathematical Methods, 2019(1): 1-14. https://doi.org/10.1002/cmm4.1023

[11] Prem Chand, B., Francisco, I.C., Garrido, N., Pankaj, J. (2019). Design and complex dynamics of Potra–Pták-type optimal methods for solving nonlinear equations and its applications. Mathematics, 7(942): 1-21. https://doi.org/10.3390/math7100942

[12] Prem Chand, B., Francisco, I.C., Pankaj, J. (2020). On the design and analysis of high-order Weerakoon-Fernando methods based on weight functions. Computational and Mathematical Methods, 2020(2): 1-17. https://doi.org/10.1002/cmm4.1114

[13] Rahma, Q., Imran, M., Syam, S. (2017). Another sixth-order iterative method free from derivative for solving multiple roots of a nonlinear equation. Applied Mathematical Sciences, 11(43): 2121-2129. https://doi.org/10.12988/ams.2017.76208

[14] Sharma, S., Sunil, P. (2022). Efficient sixth order iterative method free from higher derivatives for nonlinear equations. Journal of Mathematical and Computational Science, 12(46): 1-13. https://doi.org/10.28919/jmcs/695013

[15] Navya, K., Mani Sandeep Kumar, M., Pravin, S. (2023). Dynamical analysis of an optimal iterative scheme and its real-life applications. Engineering Letters, 14(3):

1160-1170.

[16] Navya, K., Mani Sandeep Kumar, M., Ramadevi Sri, Soujanya, M. (2023). Applications of an efficient iterative scheme for finding zeros of nonlinear equations and its basins of attraction. Communications in Mathematics and Applications, 14(1): 67-79. https://doi.org/10.26713/cma.v14i1.2113

[17] Amir, N., Rehman, M.A., Jihad, Y. (2021). Some real-life applications of a newly designed algorithm for nonlinear equations and its dynamics via computer tools. Complexity, 2021: 1-9. https://doi.org/10.1155/2021/9234932

[18] Mudassir, S., Rafiq, N., Nasreen, K. (2021). Inverse family of numerical methods for approximating all simple and roots with the multiplicity of nonlinear polynomial equations with engineering applications. Mathematical Problems in Engineering, 2021: 1-9. https://doi.org/10.1155/2021/3124615

[19] Douglas, J.M. (1972). Process Dynamics and Control. Prentice-Hall, Englewood Cliffs, NJ, USA.