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Trade Credit and Preservation Technologies: An Inventory Replenishment Model for a Sustainable Supply Chain

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ABSTRACT

Achieving sustainability within today's competitive environment is highly challenging. Therefore, we proposed a supply chain inventory replenishment model incorporating a finite planning horizon. So, to enhance their profit and lessen the total cost and carbon, this research study examines the investment in green (carbon offset) and preservation technologies. Additionally, we analyzed the trade credit duration granted by suppliers to the retailers. Carbon offsets/green technology represent a prevalent and significant measure to reduce carbon emissions. Time becomes a critical factor influencing demand rates in this context, while the degradation of materials affects a vast number of business sectors. Therefore, the cost of investing in preservation or green technology to control the deterioration of the materials, and reduce environmental emissions, the cost for ordering, the holding cost, and the replenishment cycle duration are all calculated. Consequently, a numerical iterative algorithm is prepared to identify the optimized solution for the supply chain approach for inventory control and management challenges. The optimality and uniqueness of the parameters of the proposed research study are furnished with a theoretical, mathematical, tabular, and pictorial analysis. Also, proposed research studies are provided with managerial implications that provide practical insights for industry practitioners. In conclusion, this research not only contributes valuable theoretical insights but also offers a tangible framework applicable to real-world scenarios.

1. INTRODUCTION

In today's competitive environment, awareness of sustainable supply chain management is rapidly growing. Sustainable development is crucial for the enterprise's progress. Global climate is getting worse due to carbon emissions. The emphasis of the world today is on environmental issues because of the large increase in CO_2 emissions led by global industrialization. Most developed nations are actively working to reduce carbon emissions through implementing innovative technology and regulatory measures such as carbon taxes and cap policies [1]. To reduce global warming and avert its disastrous impacts on mankind, the Kyoto Protocol, which first looked into effect on February 16, 2005, established emission caps for developed countries. This protocol proposed three adaptable emission reduction approaches. Enterprises are granted the right to emit or discharge a certain amount of the designated pollutant in the form of emissions permits (emission caps) by a central authority or governmental agency. Once the limit or cap on emissions has been set, they will be taxed for any additional emissions [2, 3].

In a supply chain, collaboration enables individuals to generate and benefit from increased profits, lower costs, and reduced carbon emissions. This collaboration may be influenced by the trade credit period, a vital component of the industry. In today's more competitive environment, both retailers and suppliers can profit through the trade credit period. Suppliers should offer a certain credit period (permissible delay) to retailers, and in return, they can enhance their market sales and generate additional profit. Furthermore, retailers will never face the problem of being out of stock; they can even earn interest during the delay in payment. Collaboration in supply chains has emerged as a key strategy for enhancing profitability, reducing costs, and mitigating carbon emissions.

Nowadays, organizations and governments emphasize the importance of building and operating supply chains with a specific focus on reducing carbon emissions. Therefore, it is necessary to adopt emission-reduction technologies, such as carbon offset, while considering supply chain emission trading regulations in the study [4].

As we are aware, everything gradually undergoes deterioration over time. Pospíšil et al. [5] emphasizes the commercial and economic significance of various materials, including metals, polymer blends, and organic biomaterials. This underscores the importance of investigating their deteriorating behavior concerning emission concerns. Therefore, we cannot ignore deterioration, which is a significant aspect of daily life. Natural processes, including the deterioration of goods, lead to a decrease in a product's usefulness.

The majority of deteriorating goods, such as vegetables, flowers, and metals, gradually decline with time. Deterioration of materials can be controlled through preservation techniques. Additionally, investments in green technology can be made to lower carbon emissions. Furthermore, by investing in high-quality and fast-moving preservation approaches (PRA) and green approaches (GRA), inventory degradation and greenhouse gas emissions can be reduced [6].

Define the research gap and research problem

While existing research has considered models with certain features, such as demand dependence on time, pricing, and trade credit duration, this study addresses a significant gap. Several researchers have developed models incorporating features such as demand dependence on time, price, and trade credit duration, along with pricing with default risk, backorders, green approaches (carbon offset), preservation approaches, and different payment options in the study [7].

Most researchers have focused on developing inventory models with various parameters under infinite planning. However, the investigation of demand dependence on time with carbon emission policies, preservation technology under a finite planning horizon, and different replenishment cycle times has not been explored in previous studies. The purpose of the proposed research is to develop a model that considers preservation and carbon emission reduction policies for planning scenarios with finite planning horizons and unequal replenishment lengths—a unique approach not explored in previous studies.

The subsequent sections of this research article are organized as follows: Section 2 reviews the literature, identifying a research gap; Section 3 describes the assumptions and abbreviations/notations. Section 4 discusses the derivation of the mathematical model, while Section 5 presents a numerical example, examines the findings, and analyzes the sensitivity analysis. Finally, Section 6 explains the study's conclusion, along with minor drawbacks, managerial implications, and innovative future suggestions.

2. LITERATURE REVIEW

The primary objective of this research is to present a comprehensive model for inventory control and management, with a specific emphasis on perishable products susceptible to degradation. The literature review delves into previous studies, examining various issues such as the impact of trade credit programs, preservation technologies, pollution restrictions, and different inventory control systems on overall supply chain performance.

The literature review highlights various research studies that have contributed to the understanding of inventory management, degradation, and emissions. Inventory control for perishable materials such as vegetables, flowers, chemical compounds, and gasoline is affected by deterioration. Material degradation and depletion can result from anthropogenic activities, technological factors, and environmental conditions.

In an inventory model developed by Chen and Teng [7], demand is time-dependent and influenced by credit. They introduced a constant rate of material deterioration in their approach. Another study by Bakker et al. [8] examined material deterioration from 1990 to 2011 and considered the credit rate. Taghizadeh-Yazdi et al. [9] investigated an integrated multi-tier supply chain inventory system dealing with deteriorating items. According to their findings, demand is influenced by trade credit policy and price. This approach also takes into account-controlled material deterioration and emissions rates in the supply chain inventory model.

A crucial factor in determining the least cost/maximum profit for producers is the manageable deterioration of materials. In the study [10], a scenario involving seasonal materials was investigated, managed through investments in high-tech technology preservation costs. Hsieh and Dye [11] analyzed how preservation investments impact overall total inventory costs and assessed the influence of investment in preservation on supply chain risks using the Dye [12] model. The study also established the maximum profit during the supply chain process.

Building on this, the approach of the study [13], developed by Liu et al. [14], considers dynamic pricing and a preservation investment approach for perishable goods sensitive to both price and quality. Mishra [15] examined a manufacturing system of degrading items and estimated the effectiveness of preservation investment. Furthermore, Bardhan et al. [16] expanded on a model for preservation investment, replenishment policies, and the deterioration of materials.

The literature review also discusses studies that have considered financial aspects during trading for decaying materials. For instance, Mohanty et al. [17] examined the financial approach during trading for a system of decaying materials with preservation technologies but neglected to consider an emissions regulation policy. In contrast, Kumar et al. [18] studied a manufacturing framework incorporating a trade credit system, preservation of advanced technologies, and a market regulation approach for emissions. Although they included trade credit, they did not consider that demand depends on credit. The use of technology to slow down the rate of deterioration of materials has been examined in several recent supply chain management studies.

Making it profitable in the current environment is challenging without considering sustainability. In numerous sectors of economics and business, supply chain processes significantly contribute to environmental emissions. Companies focusing on sustainability need to invest in systems to make improvements that will reduce both emissions and the deterioration of materials. As environmental carbon emissions are brought under control, many researchers and practitioners are actively involved in sustainable inventory control and management.

Dye and Yang [19] investigated a sustainable instantaneous stock considering credit terms, cycles, and emission restrictions. They conducted extensive analyses of their models, incorporating various environmental regulations to evaluate the consequences of emissions.

Lou et al. [20] focused on a carbon emissions trading scheme with investments in environmental sustainability technology for emissions reduction.

In a specific case, Qin et al. [21] developed a sustainable inventory control and supply chain management approach incorporating cap-and-trade, carbon tax, and credit terms. Their research discussed both endogenous and exogenous credit terms in two separate cases. The studies [22, 23] developed a replenishing inventory approach that calculates optimum profit by reducing carbon emissions, factoring in the cost of technology equipment with carbon taxation. Additionally, Ahmed and Sarkar [24] created an eco-sustainable supply chain approach incorporating carbon emissions.

Tiwari et al. [25] studied sustainable depreciating inventory systems and emission rates for optimality. In the study [3], the focus was on reducing investment costs for sustainable inventory systems that also aim to reduce transportation emissions through back-ordering. Furthermore, Tiwari et al. [26] created a sustainable production model for multi-item scenarios under trade credit and shortages.

While previous research has focused on degradation and emissions, the current study aims to address knowledge gaps. Notably, few studies analyze inventory models with a limited planning horizon (FPH). Kumar et al. [18] examined production technology with credit terms, preservation approaches, and an emissions regulatory framework. In the study [17], credit terms in an approach to decaying products using preservation techniques were explored, but the study neglected to consider an emissions regulation policy.

The use of technology to slow down the pace of deterioration and emissions has been explored in recent supply chain management studies. Numerous studies have introduced research on material deterioration by emissions or material degradation by emissions. Mishra et al. [4] considered an emissions regulatory approach and preservation approach with inventory dependent on demand but did not address the inventory model under FPH (finite planning horizon).

Furthermore, Mishra et al. [6] extended preservation technology and an emissions regulatory policy by linking credit to demand, but all parameters were not considered under a finite planning horizon. Wu and Zhao [27, 28], and Singh et al. [29] explained the inventory model with trade credit for a finite planning horizon but did not consider preservation technology and emissions regulations. Similarly, the studies [30, 31] considered a model with deterioration and trade credit under FPH but did not explore preservation technology and emissions regulations.

There are limited studies on a finite planning horizon with carbon emission preservation techniques, along with trade credit. On the other hand, [32] addresses carbon emissions but does not incorporate trade credit and a finite planning horizon. While there is research on a finite planning horizon with carbon emission regulations, credit has not been considered [33]. The study employs a finite planning horizon (FPH) technique, a relatively unexplored approach in previous studies. The unique contribution lies in the development of a model that considers preservation and carbon emission reduction policies within a constrained planning horizon featuring different replenishment periods. This aspect sets the study apart from prior research. In summary, the paper proposes a model that integrates preservation and carbon emission reduction programs across a finite planning horizon, addressing gaps in the current literature. It serves as research motivation for determining the direction of inventory management for constantly deteriorating materials and time-dependent demand using preservation techniques and emissions regulatory methods. A comparison of this research study with other studies is presented in Table 1.

Table 1. Summary of the literature referenced above

Article	Time Demand	Deterioration	Preservation Technology	Green Technology	Carbon Emissions Cost	Credit Time	Finite Planning Horizon
Toptal et al. [1]	×	×	×	×	√	×	×
Lin [3]	×	×	×	×	√	×	×
Chen et al. [2]	×	×	×	×	√	×	×
Dye [12]	×	√	√	×	×	×	×
Mishra et al. [6]	×	√	√	×	√	√	×
Mishra et al. [4]	×	√	√	×	√	×	×
Liu et al. [14]	×	×	√	×	√	×	×
Bardhan et al. [16]	×	√	√	×	√	×	×
Mohanty et al. [17]	×	√	√	×	√	√	×
Hovelaque and Bironneau [22]	×		×	√	√		×
Tiwari et al. [26]	×	√	×	√	×	√	
Datta [23]	×	√	×	√	×	√	×
Shi et al. [31]	×	×	×	×	√	√	×
Wu and Zhao [27]	√	×	×	×	×	√	√
Wu and Zhao [28]	√	×	×	×	×	√	√
Singh et al. [29]	×	√	×	×	×	√	√
Singh et al. [30]	×	√	×	×	×	×	√
Mishra and Ranu [34]	×	√	×	×	×	√	√
This Paper	√	√	√	√	√	√	√

2.1 Assumptions

1. There are no shortages or back-ordering. since in the case of routine products, clients have so many alternatives. they can buy their substitute item. The assumption simplifies the model by excluding this scenario.
2. Stock/inventory replenishment is instantaneous. This assumption simplifies the model by assuming immediate replenishment, which allows the model to focus on other parts of the supply chain.
3. The planning horizon under consideration is finite, with variable replenishment cycle lengths. The finite planning horizon specifies a time, and variable replenishment cycle lengths allow the model flexibility.
4. The model is designed for one supplier and one retailer. The model is simplified by focusing on a single supplier and retailer, allowing for a more comprehensive analysis of their interactions with each other.
5. Two cases are discussed in which trade-credit periods are considered: (a) once the credit period is longer than the

replenishment cycle length, and (b) second, the credit period is less than the replenishment cycle length that provides an extensive analysis of how trade-credit periods affect the proposed model.

6. The lead time has been set to zero. This implies that inventory is replenished immediately.
7. The equation for the increase in emissions is $E = \emptyset\,(1 - e^{-mG})$, where G represents the GRA charge of carbon emission per unit of time decreasing and Φ represents the carbon emission proportion following GRA investment (0 < Φ < 1). The parameter m, where (m>0), represents investment (carbon offset) sensitivity concerning carbon emission rates. Various methods of utilizing green technology exist, including renewable energy, green transportation technologies, and energy efficiency, that can be used to achieve carbon offset goals, particularly in the context of inventory control and management. The notations used in the research study are mentioned independently after the equation.
8. The cost of carbon emissions comprises three components: replenishment, handling, and environmental deterioration. The formula used to calculate the cost is $P(\Psi) = (1 - e^{-\lambda\Psi})$, where λ represents the preservation investment cost and the rate of deterioration is determined by the effectiveness of the preservation approach investment, represented by the variable Ψ. The function P(Ψ) is continuous, twice differentiable, and concave, representing the retailer's expenditure related to greenhouse emissions. The first derivative of the function, $P'(\Psi) = \lambda e^{-\lambda\Psi}$, indicates that the retailer should invest, while the second derivative, $P''(\Psi) = -\lambda^2 e^{-\lambda\Psi}$ <0, shows that the function is concave. These concepts were explored in studies by Mishra [16] and Bardhan et al. [17].

3. THE MODEL'S MATHEMATICAL FORMULATION AND ANALYSIS

Before the preliminary stock level reaches zero, the retailer places an order for replenishment stock with the supplier. The retailer's purchase will be instantly replenished by the supplier. Therefore, no shortages or lost sales are considered in this study. The following differential equation will represent the change in stock levels during the (i+1)th cycle:

$$\frac{IL_{i+1}(t)}{dt} = -D(t) - (1 - P(\Psi))\theta \tag{1}$$

where, $t_i \le t \le t_{i+1}$

$$IL_{i+1}(t) = e^{-\theta(1-P(\Psi))t} \int_{t}^{t_{i+1}} D(t)\, e^{\theta(1-P(\Psi))u} du \tag{2}$$

where, Boundary conditions are given below.

$IL_{i+1}(t_{i+1}) = 0$ and $IL_{i+1}(t_i) = OQ_{i+1}$

$$Q_{i+1} = IL_{i+1}(t_i) = e^{-\theta(1-P(\Psi))t_i} \int_{t_i}^{t_{i+1}} D(t, M)\, e^{\theta(1-P(\Psi))t} dt \tag{3}$$

$$Q_{i+1} = IL_{i+1}(t_i) = \int_{t_i}^{t_{i+1}} (a + bt) e^{\theta(1-P(\Psi))(t-t_i)} dt$$

The cost of replenishment of an order is

$$\mathrm{n} \times O_r \tag{4}$$

Purchasing cost: $\sum_{i=0}^{n-1} P_r \times Q_{i+1}$

$$\sum_{i=0}^{n-1} P_r \int_{t_i}^{t_{i+1}} (a + bt)\, e^{\theta(1-P(\Psi))(t-t_i)} dt \tag{5}$$

Cost of hold and stock:

$$\sum_{i=0}^{n-1} h_r \int_{t_i}^{t_{i+1}} \int_{t}^{t_{i+1}} D(t) e^{\theta(1-P(\Psi))(u-t)} du\, dt$$
$$\sum_{i=0}^{n-1} h_r \int_{t_i}^{t_{i+1}} \int_{t}^{t_{i+1}} (a + bu) e^{\theta(1-P(\Psi))(u-t)} du\, dt \tag{6}$$

Cost of deteriorating inventory:

$$(1 - P(\Psi)) \sum_{i=0}^{n-1} \theta\, d_r \int_{t_i}^{t_{i+1}} \int_{t}^{t_{i+1}} (a + bt) e^{\theta(1-P(\Psi))(u-t)} du\, dt \tag{7}$$
$$e^{-\lambda\Psi} \sum_{i=0}^{n-1} \theta\, d_r \int_{t_i}^{t_{i+1}} \int_{t}^{t_{i+1}} (a + bt) e^{\theta(1-P(\Psi))(u-t)} du\, dt$$

The implementation of green technology is crucial for achieving sustainability and reducing carbon emissions, as emphasized by the United Nations Environment Programme (UNEP). Carbon offsets serve as compensation for emissions rather than a replacement. The variable ĉ represents the fixed carbon emissions related to order placement, including transportation emissions. Meanwhile, the dynamic carbon emissions for each unit ordered are denoted by $\hat{P}_r$, and the carbon emissions associated with refrigeration during warehousing are represented by $\widehat{h_r}$. These concepts have been explored in various studies [31-35]. Thus, the following equation calculates the total carbon emissions for each replenishment cycle:

Amount of carbon emission during holding, placing an order and transportation:

$$Ce = \sum_{i=0}^{n-1} \hat{c} + \hat{P}_r * Q_{i+1} + \widehat{h_r} \int_{t_i}^{t_{i+1}} \int_{t}^{t_{i+1}} (a + bt) e^{\theta(1-P(\Psi))(u-t)} du\, dt$$

The following presented for investment in the green approach.

$$(1 - \emptyset\,(1 - e^{-mG}))\, Ce$$

The per year carbon emission cost for a cycle is $CT = \tau\left(1 - \emptyset\left(1 - e^{-mG}\right)\right) Ce$

$$= \tau\left(1 - \emptyset\left(1 - e^{-mG}\right)\right) \sum_{i=0}^{n-1} c\hat{} + \hat{P}_r * Q_{i+1} + \widehat{h_r} \int_{t_i}^{t_{i+1}} \int_{t}^{t_{i+1}} (a + bt) e^{\theta(1 - P(\Psi))(u-t)} du\ dt \tag{8}$$

In general, buyers adhere to market norms by acquiring goods or materials from suppliers and, in return, make payments. Trade Credit periods are commonly provided by a significant number of businesses. In such arrangements, the supplier extends a period to the retailer, allowing a permissible delay for payment. This practice ensures that neither the supplier nor the retailer incurs losses. Additionally, suppliers may incentivize early payments by offering various discounts. Consequently, the retailer is granted a credit period $M_{i+1} = \delta(t_{i+1})$ by the seller. The supplier approves trade credit for the retailer based on multiple criteria, including the order amount placed, creditworthiness, previous records, the nature of the items, and the history of cooperation between the supplier and retailer. The duration of credit is often extended based on the order amount, with larger orders resulting in a more extended credit period.

3.1 First case

Now, we consider two cases: first, where M_{i+1} lies within the cycle length (t_i, t_{i+1}), indicating that the credit duration provided by the seller does not exceed the inventory replenishment length T_{i+1}. In this scenario, the retailer accrues interest on their sales revenue up to the credit duration length, while also incurring interest charges on items already stocked.

Referring to Figure 1, the interest charges M_{i+1} should be less than or equal to T_{i+1} like the $M_{i+1} \le T_{i+1}$. In this case $M_{i+1} = \delta(t_{i+1})$ lies into the interval $t_i \le t \le t_{i+1}$.

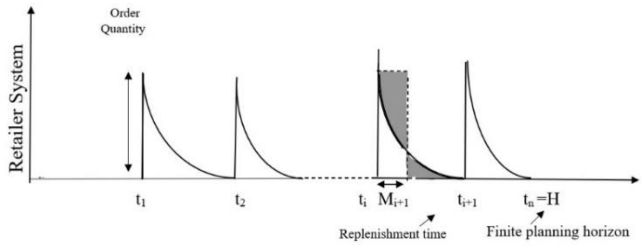

Figure 1. Inventory model diagram for $M_{i+1} \le T_{i+1}$

So, interest earned by the retailer is as follows:

$$\sum_{i=0}^{n-1} I_e * s \int_{t_i}^{t_i + \delta(t_{i+1} - t_i)} (a + bt)[t_i + \delta(t_{i+1} - t_i) - t] dt \tag{9}$$

Interest payable by the retailer is given by:

$$\sum_{i=0}^{n-1} I_c * W \int_{t_i + (t_{i+1} - t_i)\delta}^{t_{i+1}} (a + bt)[t - t_i - (t_{i+1} - t_i)\delta] dt \tag{10}$$

The total cost of the retailer is given below:

Cost of placing an order + holding cost + purchasing cost + carbon preservation cost + deterioration preservation technology cost + Interest charges - Interest Earned.

$$\begin{aligned} T_{Ret} = {} & \mathrm{n} * O_r + \sum_{i=0}^{n-1} h_r \int_{t_i}^{t_{i+1}} \int_{t}^{t_{i+1}} (a + bu) e^{\theta(1 - P(\Psi))(u-t)} du\ dt + \sum_{i=0}^{n-1} P_r * Q_{i+1} \\ & + \tau\left(1 - \emptyset\left(1 - e^{-mG}\right)\right) \sum_{i=0}^{n-1} \left(c\hat{} + \hat{P}_r * Q_{i+1} + \widehat{h_r} \int_{t_i}^{t_{i+1}} \int_{t}^{t_{i+1}} (a + bu) e^{\theta(1 - P(\Psi))(u-t)} du\ dt \right) \\ & + e^{-\lambda\Psi} \sum_{i=0}^{n-1} \theta\, d_r \int_{t_i}^{t_{i+1}} \int_{t}^{t_{i+1}} (a + bu) e^{\theta(1 - P(\Psi))(u-t)} du\ dt \\ & + \sum_{i=0}^{n-1} I_e * s \int_{t_i}^{t_i + (t_{i+1} - t_i)\delta} (a + bt)[t_i + (t_{i+1} - t_i)\delta - t] dt \\ & - \sum_{i=0}^{n-1} I_c * W \int_{t_i + \delta(t_{i+1} - t_i)}^{t_{i+1}} (a + bt)[t - t_i - (t_{i+1} - t_i)\delta] dt \end{aligned}$$

$$\begin{aligned} T_{Ret} = {} & \mathrm{n} * O_r + \sum_{i=0}^{n-1} h_r \int_{t_i}^{t_{i+1}} \int_{t}^{t_{i+1}} (a + bu) e^{\theta(1 - P(\Psi))(u-t)} du\ dt \tau\left(1 - \emptyset\left(1 - e^{-mG}\right)\right) \sum_{i=0}^{n-1} c\hat{} + \hat{P}_r * Q_{i+1} \\ & + \widehat{h_r} \int_{t_i}^{t_{i+1}} \int_{t}^{t_{i+1}} (a + bu) e^{\theta(1 - P(\Psi))(u-t)} du\ dt + e^{-\lambda\Psi} \sum_{i=0}^{n-1} \theta\, d_r \int_{t_i}^{t_{i+1}} \int_{t}^{t_{i+1}} (a + bu) e^{\theta(1 - P(\Psi))(u-t)} du\ dt \\ & + \sum_{i=0}^{n-1} s * I_e \int_{t_i}^{t_i + (t_{i+1} - t_i)\delta} (a + bt)\,[t_i + (t_{i+1} - t_i)\delta - t] dt - \sum_{i=0}^{n-1} I_c * W \int_{t_i + (t_{i+1} - t_i)\delta}^{t_{i+1}} (a + bt)[t - t_i - (t_{i+1} - t_i)\delta] dt \end{aligned}$$

$$T_{Ret} = \mathrm{n} * O_r + \sum_{i=0}^{n-1} h_r \int_{t_i}^{t_{i+1}} \int_{t}^{t_{i+1}} (a+bu) e^{\theta(1-P(\Psi))(u-t)} du\ dt + \sum_{i=0}^{n-1} P_r \int_{t_i}^{t_{i+1}} (a+bt)\, e^{\theta(1-P(\Psi))(t-t_i)} dt$$
$$+ \tau\,(1-\emptyset\,(1-e^{-mG})) \sum_{i=0}^{n-1} \hat{c} + \hat{P}_r * \int_{t_i}^{t_{i+1}} (a+bt)\, e^{\theta(1-P(\Psi))(t-t_i)} dt$$
$$+ \widehat{h_r} \int_{t_i}^{t_{i+1}} \int_{t}^{t_{i+1}} (a+bu) e^{\theta(1-P(\Psi))(u-t)} du\ dt + e^{-\lambda\Psi} \sum_{i=0}^{n-1} \theta\, d_r \int_{t_i}^{t_{i+1}} \int_{t}^{t_{i+1}} (a+bu) e^{\theta(1-P(\Psi))(u-t)} du\ dt$$
$$+ \sum_{i=0}^{n-1} s * I_e \int_{t_i}^{t_i+(\delta*t_{i+1}-\delta*t_i)} (a+bt)\,[t_i + (\delta*t_{i+1}-\delta*t_i) - t] dt$$
$$- \sum_{i=0}^{n-1} I_c * W \int_{t_i+(\delta*t_{i+1}-\delta*t_i)}^{t_{i+1}} (a+bt)[t - t_i - (\delta*t_{i+1}-\delta*t_i)\,] dt$$

$$T_{Ret} = \mathrm{n} * O_r + \sum_{i=0}^{n-1} \left\{ h_r + \tau\,(1-\emptyset\,(1-e^{-mG}))\widehat{h_r} + \theta\, d_r\, e^{-\lambda\Psi} \right\} \int_{t_i}^{t_{i+1}} \int_{t}^{t_{i+1}} (a+b*u)\, e^{\theta(P(\Psi)-1)(t-u)} du\ dt$$
$$+ \sum_{i=0}^{n-1} \left(\left\{ P_r + \hat{P}_r * \tau(1-\emptyset\,(1-e^{-mG})) \right\} \int_{t_i}^{t_{i+1}} (a+bt)\, e^{\theta(1-P(\Psi))(t-t_i)} dt \right) + \hat{c} * \tau\,(1-\emptyset\,(1-e^{-mG}))$$
$$+ \sum_{i=0}^{n-1} s * I_e \int_{t_i}^{t_i+(\delta*t_{i+1}-\delta*t_i)} (a+bt)[t_i + (\delta*t_{i+1}-\delta*t_i) - t] dt$$
$$- \sum_{i=0}^{n-1} I_c * W \int_{t_i+(\delta*t_{i+1}-\delta*t_i)}^{t_{i+1}} (a+bt)\,[t - t_i - (\delta*t_{i+1}-\delta*t_i)\,] dt$$

$$T_{Ret} = \mathrm{n} * O_r + \sum_{i=0}^{n} \left\{ \frac{h_r}{\theta(1-P(\Psi))} + \frac{\tau(1-\emptyset\,(1-e^{-mG}))\widehat{h_r}}{\theta(1-P(\Psi))} + d_r\, e^{-\lambda\Psi} \right\} \int_{t_i}^{t_{i+1}} (a+bt) e^{\theta(1-P(\Psi))(t-t_i)} dt$$
$$+ \sum_{i=0}^{n} \left\{ P_r + \hat{P}_r * \tau\,(1-\emptyset\,(1-e^{-mG})) \right\} \int_{t_i}^{t_{i+1}} (a+bt)\, e^{\theta(1-P(\Psi))(t-t_i)} dt + \hat{c} * \tau\,(1-\emptyset\,(1-e^{-mG}))$$
$$- \left\{ \frac{h_r}{\theta(1-P(\Psi))} + \frac{\tau(1-\emptyset\,(1-e^{-mG}))\widehat{h_r}}{\theta(1-P(\Psi))} + d_r\, e^{-\lambda\Psi} \right\} (a\,H + 0.5*b*H^2)$$
$$+ \sum_{i=0}^{n-1} s * I_e \int_{t_i}^{t_i+(\delta*t_{i+1}-\delta*t_i)} (a+bt)\,[t_i + (\delta*t_{i+1}-\delta*t_i) - t] dt$$
$$- \sum_{i=0}^{n-1} I_c * W \int_{t_i+(\delta*t_{i+1}-\delta*t_i)}^{t_{i+1}} (a+bt)[t - t_i - (\delta*t_{i+1}-\delta*t_i)\,] dt$$

$$\Bigg[T_{Ret} = \mathrm{n} * O_r + \sum_{i=0}^{n} \left(\left\{ \frac{h_r}{\theta(1-P(\Psi))} + \frac{\tau(1-\emptyset\,(1-e^{-mG}))\widehat{h_r}}{\theta(1-P(\Psi))} + d_r\, e^{-\lambda\Psi} \right\} + \left\{ P_r + \hat{P}_r * € \,(1-\emptyset\,(1-e^{-mG})) \right\} \right) \int_{t_i}^{t_{i+1}} (a$$
$$+ bt)\, e^{\theta(1-P(\Psi))(t-t_i)} dt + \hat{c} * \tau\,(1-\emptyset\,(1-e^{-mG}))$$
$$- \left\{ \frac{h_r}{\theta(1-P(\Psi))} + \frac{\tau(1-\emptyset\,(1-e^{-mG}))\widehat{h_r}}{\theta(1-P(\Psi))} + d_r\, e^{-\lambda\Psi} \right\} (a\,H + 0.5*b*H^2)$$
$$+ \sum_{i=0}^{n-1} s * I_e \int_{t_i}^{t_i+(\delta*t_{i+1}-\delta*t_i)} (a+bt)\,[t_i + (\delta*t_{i+1}-\delta*t_i) - t] dt$$
$$- \sum_{i=0}^{n-1} I_c * W \int_{t_i+(\delta*t_{i+1}-\delta*t_i)}^{t_{i+1}} (a+bt)\,[t - t_i - (\delta*t_{i+1}-\delta*t_i)\,] dt \Bigg]$$

$$\frac{\partial(T_{Ret})}{\partial t_i} = \left(\left\{\frac{h_r}{\theta(1-P(\Psi))} + \frac{\tau(1-\emptyset(1-e^{-mG}))\widehat{h_r}}{\theta(1-P(\Psi))} + d_r\, e^{-\lambda\Psi}\right\} + \left\{P_r + \hat{P}_r * \tau(1-\emptyset(1-e^{-mG}))\right\}\right)\left((a+b\,t_i)\left(e^{\theta(1-P(\Psi))(t_i-t_{i-1})}-1\right) - \theta(1 - P(\Psi))\int_{t_i}^{t_{i+1}}(a+bt)e^{\theta(1-P(\Psi))(t-t_i)}\,dt\right) + s * I_e\int_{t_i}^{t_i+(\delta * t_{i+1}-\delta * t_i)}(a+bt)(1-\delta)\,dt - (a+b\,t_i)(\delta * t_{i+1} - \delta * t_i) + * W\int_{t_i+(\delta * t_{i+1}-\delta * t_i)}^{t_{i+1}}(a+bt)(1-\delta)dt \quad (11)$$

$$\frac{\partial(T_{Ret})}{\partial t_i} = \left(\left\{\frac{h_r}{\theta(1-P(\Psi))} + \frac{\tau(1-\emptyset(1-e^{-mG}))\widehat{h_r}}{\theta(1-P(\Psi))} + d_r\, e^{-\lambda\Psi}\right\} + \left\{P_r + \hat{P}_r * \tau(1-\emptyset(1-e^{-mG}))\right\}\right)\left((a+b\,t_i)\left(e^{\theta(t_i-t_{i-1})(1-P(\Psi))}-1\right) + \theta(P(\Psi) - 1)\int_{t_i}^{t_{i+1}}(a+bt)e^{\theta(1-P(\Psi))(t-t_i)}\,dt\right) + s * I_e\left\{\int_{t_{i-1}}^{t_{i-1}+(\delta * t_{i+1}-\delta * t_i)}(a+bt)\,\delta\,dt + \int_{t_i}^{t_i+(\delta * t_{i+1}-\delta * t_i)}(a+bt)(1-\delta)\,dt - (a + b\,t_i)(\delta * t_{i+1} - \delta * t_i)\right\} - I_c * W\left\{\int_{t_{i-1}+(\delta * t_i-\delta * t_{i-1})}^{t_i}(a+bt)\,\delta\,dt + \int_{t_i+(\delta * t_{i+1}-\delta * t_i)}^{t_{i+1}}(a+bt)(1-\delta)dt - (a+bt_i)(1 - \delta)[t_i - t_{i-1}]\right\} \quad (12)$$

By taking a partial derivative of T_{Ret} w.r.t to t_i equal to zero, we can find the value of t_i.

$$\frac{\partial(T_{Ret})}{\partial t_i} = 0$$

3.2 Second case

When the credit duration is higher than the inventory replenishment length T_{i+1}. M_{i+1} lies outside the cycle length (t_i, t_{i+1}). In the second case, we assume, $M_{i+1} \geq T_{i+1}$. In this scenario, M_{i+1} lies outside the interval as illustrated in Figure 2, where $t_i \leq t \leq t_{i+1}$ represents the time interval. Therefore, the interest earned by the retailer is as follows:

$$\sum_{i=0}^{n-1} I_e * s\int_{t_i}^{t_{i+1}}(a+bt)\,[t_i + \delta(t_{i+1} - t_i) - t]dt \quad (13)$$

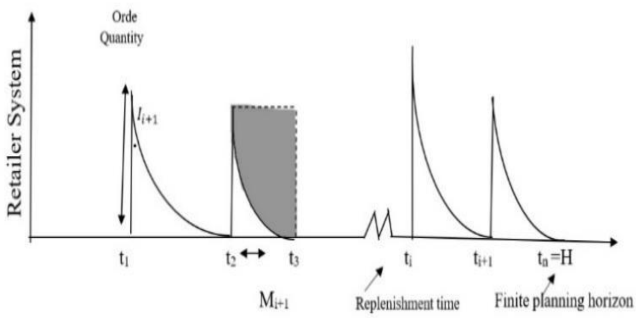

Figure 2. Inventory model diagram for $M_{i+1} \geq T_{i+1}$

Interest payable by the retailer is zero. Then:

$$T_{Ret} = n * O_r + \sum_{i=0}^{n}\left(\left\{\frac{h_r}{\theta(1-P(\Psi))} + \frac{\tau(1-\emptyset(1-e^{-mG}))\widehat{h_r}}{\theta(1-P(\Psi))} + d_r\, e^{-\lambda\Psi}\right\} + \left\{P_r + \hat{P}_r * \tau(1-\emptyset(1-e^{-mG}))\right\}\right)\int_{t_i}^{t_{i+1}}(a + bt)\, e^{\theta(1-P(\Psi))(t-t_i)}dt + + c\hat{} * \tau(1-\emptyset(1-e^{-mG})) - \left\{\frac{h_r}{\theta(1-P(\Psi))} + \frac{\tau(1-\emptyset(1-e^{-mG}))\widehat{h_r}}{\theta(1-P(\Psi))} + d_r\, e^{-\lambda\Psi}\right\}(a\,H + 0.5 * b * H^2) + \sum_{i=0}^{n-1} s * I_e\int_{t_i}^{t_{i+1}}(a + b * t)\,[t_i + (\delta * t_{i+1} - t_i * \delta) - t]dt$$

$$\frac{\partial(T_{Ret})}{\partial t_i} = \left(\left\{\frac{h_r}{\theta(1-P(\Psi))} + \frac{\tau\,(1-\emptyset\,(1-e^{-mG}))\widehat{h_r}}{\theta(1-P(\Psi))} + d_r\, e^{-\lambda\Psi}\right\} + \left\{P_r + \widehat{P_r} * \tau\,(1-\emptyset\,(1-e^{-mG}))\right\}\right)\left((a+b\,t_i)\left(e^{\theta(1-P(\Psi))(t_i-t_{i-1})}-1\right) - \theta(1 - P(\Psi))\int_{t_i}^{t_{i+1}}(a+bt)e^{\theta(1-P(\Psi))(t-t_i)}\,dt\right) + s * I_e\left\{\int_{t_{i-1}}^{t_i}(a+bt)\,\delta\,dt + \int_{t_i}^{t_{i+1}}(a+bt)(1-\delta)\,dt + (a+b\,t_i)\,(t_i - t_{i-1} - \delta * t_i - \delta * t_{i-1}) - (a+b\,t_i)\,(t_{i+1} - t_i - \delta * t_{i+1} + \delta * t_i)\right\} \quad (14)$$

By taking a partial derivative of T_{Ret} w.r.t to t_i equal to zero, we can find the value of t_i.

$$\frac{\partial(T_{Ret})}{\partial t_i} = 0$$

$$T_{Sup} = n^* * S_r + \sum_{i=0}^{n^*-1} C_p\delta(t_{i+1} - t_i)I_c * Q_i^* \quad (15)$$

$$Q_i = \sum_{i=0}^{n^*-1} Q_i^* \quad (16)$$

4. NUMERICAL ILLUSTRATION FOR THE PROPOSED MODEL

Let us consider parametric values such as a = 0.0001 unt., or= 20$/setup/year, b = 1000 units, M = 0.5year, hr = 0.4 $/unt. /Annually, τ = 0.6$/kg/ annually, m = 0.5 unt., ∅=0.4 unt., θ=4, α= 0.02 unt., Ψ=0.5, λ=0.8 unt., Pr=2$/unt. /Year, $\hat{c}$ = 10kg/year, $\widehat{P_r}$= 40 kg/order/ annually, $(\widehat{h_r})$ =8 kg/ annually, Ie= 0.08 $/unt. / Annually, Ic= 0.1 $/unt. /Year, s= 50 $/unt., Ss= 120$/setup/ annually. An iterative method is implemented using Mathematica software to ensure the validity and reliability of the numerical solution. An iterative method is an approach to solving problems for optimization and convexity.

5. ALGORITHM

Step 1: First of all, set the starting value for all parameters. a = 0.0001 unt., $O_r = 20$$/setup/year, b = 1000 units, M = 0.5year, h_r = 0.4 $/unt. /annually, τ = 0.6$/kg/ annually, m = 0.5 unt., $\emptyset$ =0.4 unt. , $\theta = 4$, $\alpha = 0.02$ unt. , $\Psi = 0.5$, λ =0.8 unt. , P_r =2$/unt. /Year, c^ = 10kg/year, $\widehat{P_r}$ = 40 kg/order/ annually, $\widehat{h_r}$ =8 kg/ annually, $I_e = 0.08$ $/unt. / Annually, $I_c = 0.1$ $/unt. /Year, s= 50 $/unt., S_s= 120$/setup/ annually.

Step 2: Find the Root $\left[\frac{\partial(T_{Ret})}{\partial t_i} = 0\right]$.

Step 3: The final results are optimal solutions t_i.

Step 4: Insert optimal values of t_i into the equations to find the value of total cost and order quantity.

5.1 Tables and figures

Figure 3 shows the optimal level of the total cost of the retailer. at n=5 replenishment cycle, the total optimal value of the total cost of the retailer is 49743.41, similarly, for different values of 'a', we can see in Table 2 and Table 3. Optimal level of retailer total cost. For the different values of 'a' optimal value arrived at the 5th replenishment cycle. Table 3 shows the replenishment time, optimal value of retailer, supplier and order quantity.

Table 2. Total cost for the retailer when $M_{i+1} \le T_{i+1}$ for five different combinations of 'a'

↓a →n	1	2	3	4	5	6	7	8
0.001	807408.58	212851.20	101532.77	65367.62	**49743.41**	74762.86	155208.69	516831.09
0.0008	807408.73	212851.23	101532.79	65367.62	**55494.86**	93719.13	216591.70	942996.74
0.0009	807408.65	212851.22	101532.78	65367.62	**50832.96**	83209.24	181501.34	675694.18
0.0011	807408.50	212851.19	101532.76	65367.61	**49743.41**	67850.95	134956.30	409364.16
0.0012	807408.42	212851.17	101532.76	65367.61	**49743.41**	62106.30	118989.62	333520.31

Table 3. Replenishment time for T_{Ret}, T_{sup}, and Q_{nt} when $M_{i+1} \le T_{i+1}$

↓a → t_i	t_0	t_1	t_2	t_3	t_4	t_5	n	T_{Ret}	T_{sup}	Tc	Q_{nt}
0.001	0	1.4259	2.0827	2.6329	3.1244	4	5	49743.41	23977.87	73721.28	18092.3
0.0008	0	1.4719	2.1499	2.7178	3.2251	4	5	55494.86	29425.50	84920.36	20168.3
0.0009	0	1.4350	2.0960	2.6496	3.1443	4	5	50832.96	25604.07	76437.03	18486.1
0.0011	0	1.4259	2.0827	2.6329	3.1244	4	5	49743.41	23085.97	72829.38	18092.3
0.0012	0	1.4259	2.0827	2.6329	3.1244	4	5	49743.41	22324.44	72067.85	18092.3

Table 4. Total cost for retailer for different values of ‘a’ when $M_{i+1} > T_{i+1}$

↓a →n	1	2	3	4	5	6	7	8
0.001	762631.00	179074.16	74765.77	43266.14	**30942.01**	46862.71	102241.42	388303.80
0.0008	762631.16	179074.19	74765.79	43266.15	**34815.61**	60016.82	148908.53	744379.65
0.0009	762631.08	179074.18	74765.78	43266.15	**31670.30**	52660.24	121929.16	525295.82
0.0011	762630.92	179074.15	74765.76	43266.14	30942.01	42201.82	87445.92	298127.06
0.0012	762630.85	179074.13	74765.75	43266.14	30942.01	38389.75	76045.01	236187.31

Table 5. Replenishment time for T_{Ret}, T_{sup}, and Q_{nt} when $M_{i+1} > T_{i+1}$

↓a →t_i	t_0	t_1	t_2	t_3	t_4	t_5	n	T_{Ret}	T_{sup}	Tc	Q_{nt}
0.001	0	1.4259	2.0827	2.6329	3.1244	4	**5**	**30942.01**	15428.31	46370.32	18092.29
0.0008	0	1.4719	2.1499	2.7178	3.2251	4	5	**34815.61**	18896.41	53722.02	20168.36
0.0009	0	1.4350	2.0960	2.6496	3.1443	4	5	**31670.30**	16463.58	48133.88	18486.1
0.0011	0	1.4259	2.0827	2.6329	3.1244	4	5	30942.01	14860.51	45802.52	18092.3
0.0012	0	1.4259	2.0827	2.6329	3.1244	4	5	30942.01	14375.68	45317.69	18092.3

Table 6. Represent the following facts of sensitivity analysis for main parameters

Parameters	%Changes	Optimal Replenish Cycle	Total Order Quantity Q_{nt}	Retailer's Total Cost T_{Ret}	Supplier's Total Cost T_{sup}
	+20	5	18092.3	49743.41	22324.44
	+10	5	18092.3	49743.41	23977.87
a	0	5	18092.3	49743.41	23085.97
	−10	5	20168.3	55494.86	29425.50
	−20	5	18486.1	50832.9	25604.07
	+20	4	28206.8	78286.61	3218901.1
	+10	4	25856.2	71789.39	2950659.02
b	0	4	23505.6	65292.17	2682416.88
	−10	4	21155.1	58794.94	2794658.59
	−20	4	18804.5	52297.7	2575774.22
	+20	5	24276.9	68347.01	4267545.71
	+10	5	20870.7	58057.29	3583507.45
θ	0	5	18092.2	49668.03	3033108.27
	−10	5	15812.2	42789.90	2588149.43
	−20	5	13929.1	37118.60	2226602.26
	+20	5	211.42	859.78	1860.18
	+10	6	211.42	834.78	1860.18
O_b	0	6	211.42	809.78	1860.18
	−10	5	211.42	781.85	1860.18
	−20	5	211.42	751.85	1860.18
	+20	4	211.42	809.78	723.10
	+10	4	211.42	809.78	663.10
	0	4	211.42	809.78	603.10
S_s	−10	4	211.42	809.78	543.10
	−20	4	211.42	809.78	483.10

Table 7. Represent the following facts of sensitivity analysis for the main parameters

Parameters	%Changes	Optimal Replenish Cycle	Total Order Quantity Q_{nt}	Retailer's Total Cost T_{Ret}	Supplier's Total Cost T_{sup}
	+20	6	246961.56	290818.70	37108.96
	+10	6	229788.07	271067.58	32219.83
a	0	6	214125.64	253278.14	28003.36
	−10	6	199814.24	237205.04	24358.73
	−20	6	186711.34	222635.75	21201.25
	+20	4	627510476	186533.33	723033.14
	+10	4	575278343	171006.99	662817.30
b	0	4	523043328	155479.80	602686.98
	−10	4	470804689	1399515.29	603342.10
	−20	5	418864929	124512.83	542508.00
	+20	4	221117363	662621004	254786691720
	+10	4	107218244	320461137	123541769573
θ	0	4	52304332	155479805	60268698111
	−10	4	25694820	75525839	29607391885
	−20	4	12726308	36553824	14664173720

O_b	+20	6	214125.64	253374.14	14830218.5
	+10	6	214125.64	253278.14	14830218.5
	0	6	214125.64	253230.14	14830218.5
	−10	6	214125.64	253326.14	14830218.5
	−20	6	214125.64	253182.14	14830218.5
S_s	+20	6	214125.64	253278.14	28111.36
	+10	6	214125.64	253182.14	28057.36
	0	6	214125.64	253230.14	28003.36
	−10	6	214125.64	253326.14	27949.36
	−20	6	214125.64	253374.14	27895.36

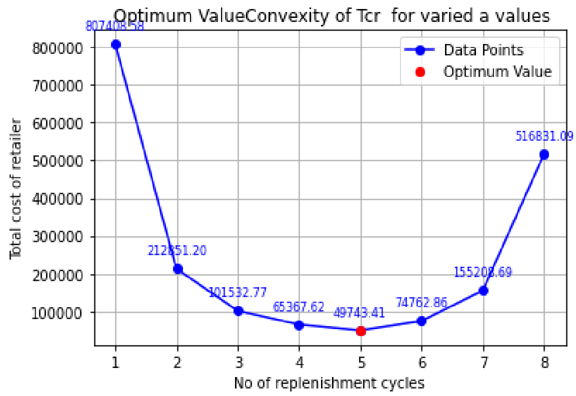

Figure 3. Optimal level of total cost of retailer $M_{i+1} < T_{i+1}$

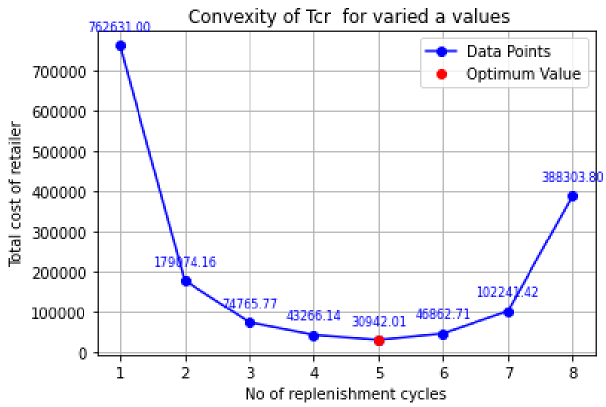

Figure 4. Optimal level of the total cost of the retailer when $M_{i+1} > T_{i+1}$

Figure 4 shows the optimal level of the total cost of retailer for $M_{i+1} > T_{i+1}$. At n=5 replenishment cycle, the total optimal value of the total cost of the retailer is 30942.01. Similarly, as seen in Tables 4 and 5, the impact of varying the parameter 'a' is illustrated. Optimal level of retailer total cost. For the different values of 'a' optimal value arrived at the 5th replenishment cycle. Table 5 shows the replenishment time, optimal value of retailer, supplier, and order quantity.

Lemma1: t_i increase where i=1,2,3………… n-1 strictly monotonic increase function of last replenishment cycle t_n.

$$T_{i+1} = t_{i+1} - t_i \text{ and } t_n = \text{H-}T_n$$

Theorem 1: The unique solution only exists for the non-linear system of Eq. (11) is the optimal replenishment period for a fixed replenishment cycle n.

The Hessian matrix of T_{Ret} must be positive definite for t_i to be minimum for a fixed n.

Therefore, in Appendix, the theorem establishes that T_{Ret} is positive definite. As a result, the optimum value of t_i for a given fixed n +ve integer can be computed by using the numerical iterative technique and Mathematica programs version 12.0. On the basis, of the optimal value of t_i, the total cost function also will be optimal.

Theorem 2: If t_i satisfy in equation s(i) $\frac{\partial^2 T_{Ret}}{\partial t_i^2} \geq 0$ (ii) $\frac{\partial^2 T_{Ret}}{\partial t_i^2} \geq \left|\frac{\partial^2 T_{Ret}}{\partial t_i t_{i-1}}\right| + \left|\frac{\partial^2 T_{Ret}}{\partial t_i t_{i+1}}\right|$ for all i= 1, 2, 3 ……… n_1 then $\nabla^2 T_{Ret}$ is positive definite.

6. SENSITIVITY ANALYSIS

To identify the solution for optimization of sustainability in the inventory replenishment approach, the retailer and supplier need this study to understand how the inventory system's parameters are operated. To determine the minimum total cost and maximum profit, retailers need to increase or reduce parameters. Therefore, the purpose of this study is to identify significant management implications and to demonstrate the usefulness and applicability of the model's solutions. To analyse the effect of all parameters, one parameter changed at one time while other parameters remained unchanged. Some main parameter analyses are presented in Table 6 and Table 7.

1. Table 4 reveals that if the initial market demand parameter 'a' increases then the replenishment quantity, retailer's total cost and supplier's total cost and practically insensitive to increases in 'a'. If the initial market demand parameter 'a' decreases then moderately reactive with order quantity, retailer's, and supplier's total cost. But it is insensitive to the replenishment cycles per year. Thus, we can say that initial market demand has a negligible effect on order quantity, retailer, and supplier cost. But 'b' (demand that is dependent upon time) is very sensitive to the order quantity, and the total cost of the retailer and supplier. Table 7 gives the detail that the replenishment quantity, retailer's and supplier's costs and very sensitive to increases with 'a' (initial market demand) and 'b' (demand that is dependent upon time).
2. The analysis in Table 6 and Table 7 shows that the retailer's total cost function is moderately sensitive to changes in ordering cost. But, insensitive to supplier total cost and ordering quantity. As ordering cost increase or decrease, the retailer's total cost increases or decrease in every replenishment cycle, giving a similar relationship between ordering cost and retailer total cost.
3. Table 6 and Table 7 show a brief description of the relationship between setup cost and supplier total cost function that is moderately sensitive to changes in setup cost. But, insensitive to retailer total cost and ordering quantity. As setup increases or decreases, the supplier's total cost increases or decreases in every replenishment cycle, giving a similar relationship between setup cost and supplier total cost.

7. COMPARISION AND CONCLUSION

Now, we will discuss about the comparison between existing literature and proposed model:

Upon validation of our comparative study against the benchmarks set forth in study [28], we have rigorously evaluated and contrasted our research outcomes with those posited in the referenced work. As depicted in Figure 5, our analysis entailed a thorough comparison with the existing body of literature. The results indicate that our study not only aligns with but also surpasses the solutions presented in study [28]. The graphical representation in Figure 5 clearly illustrates the enhancements and progressive strides our research has made over the foundational work of study [28].

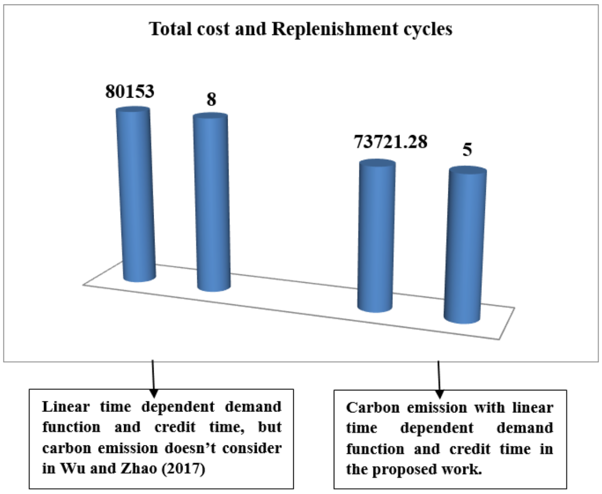

Figure 5. Comparison between current study and existing literature

Precisely evaluating the unique trade credit period supplied by suppliers to retailers, which considerably impacts the overall findings and conclusions, is an important aspect of our research. This unique feature extensively affects inventory management, financial strategy, and supply chain sustainability.

The demand rate in this article is influenced by the time and edit period. We analyse, how variations in demand over time have a direct influence on holding costs, ordering patterns, and total replenishment techniques. To identify the optimality of the supply chain inventory problem, an algorithm is prepared, and sensitivity analysis is conducted. This transparency enhances the credibility of our findings.

The following aspects are contained in the proposed research study, which distinguishes it from most other inventory models.

- Demand is dependent on time and has a linear relationship with time.
- Preservation technology to reduce the deterioration of materials.
- Green technology investment.
- Supplier offers the retailer a special trade credit period.
- The finite planning horizon.
- Discussed two different trade credit phases.

According to the findings, continuous technology investments that reduce the deterioration of materials and preserve the environment can significantly enhance inventory management and control accuracy. To prevent the rate of deterioration of materials and environmental pollutants, both retailers and suppliers need to make a preservation investment.

The following outcomes of this research study were achieved:

- The research yields a unique solution set, demonstrating that replenishment time cycles are convex functions that provide a single optimal solution, thereby achieving optimum cost.

The proposed approach may be modified for future study by various approaches. For some examples, it may extend into a new approach by considering some other kinds of demand functions, including exponential and quadratic, time-varying demand, carbon, and inventory-dependent demand, etc. In addition, we may further consider the instantaneous deterioration of materials, shortages, and some carbon regulations, Therefore, future research can appraise the effects of each of these extra aspects.

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NOMENCLATURE

The following assumptions underpin the paper. The notations used in the research study are also mentioned independently after the assumptions.

IL_{i+1}	Level of stock at the duration ti with the (i+1) th cycle.
Q_{i+1}	represents the order quantity within (i+1) th cycle at time ti.
T_{i+1}	Length of Each replenishment cycle.
D(t)	Time-dependent and a function of time t are given below. $D(t) = (a + bt)$, a>0, and b>0 where a is the initial demand of the market, and b is time-dependent demand.
O_r	Denotes the cost per unit for placing an order.
d_r	The cost of deterioration of materials INR / unit.
C_e	The total amount of CO2 released during a replenishing cycle.
S_s	The supplier's setup service charges each cycle.
h_S	h_S: The cost of holding the stocks for suppliers per unit per year.
W	symbolizes the per unit wholesale cost for the retailer (W > P_r)
P_r	acquisition/purchasing cost for the supplier.
I_e	interest earned for each unit of time.
I_c	I_c= interest charge for each unit of time.
δ	δ =The supplier's specified trade credit period coefficient.
M_{i+1}	The trade-credit period's length for $(i + 1)$th cycle, $M_{i+1} = \delta(t_{i+1})$
n	the number of replenishment cycles.
$\hat{c}$	fixed amount of carbon emissions per unit.
$\widehat{P_r}$	The quantity of carbon dioxide emissions during refrigerating each unit.
s	selling price for each unit.
C_p	opportunity cost for each unit.
τ	tax paid to slow the rate of emitting per unit of carbon.
$\widehat{h_r}$	The variable amount of carbon emissions associated with placing an order.
h_r	The holding cost of the stocks per unit/ year.
θ	Rate of deterioration.
H	The planning horizon.
T_{Ret}	Total Retailer Cost Under the FPH.
T_{sup}	Total supplier Cost Under the FPH

Decision-making variables

ti	ti symbolizes replenishment time, where $t_0 = 0$ and tn = H.
T_{i+1}	$T_{i+1} = t_{i+1} - t_i$ is the length of the (i+1) th replenishment cycle.
M	M= Supplier provides a credit period to the retailer.

APPENDIX

Proof of the Theorem 1.

To calculate the total variable cost T_{Ret} of the system by computing the values of t_i. The first of all, the find ti by putting the $\frac{\partial\,(T_{Ret}\,(t_i,n))}{\partial t_i} = 0$

$$\frac{\partial(T_{Ret})}{\partial t_i} = \left(\left\{\frac{h_r}{\theta(1-P(\Psi))} + \frac{\tau(1-\emptyset(1-e^{-mG}))\widehat{h_r}}{\theta(1-P(\Psi))} + d_r\, e^{-\lambda\Psi}\right\} + \left\{P_r + \hat{P}_r * \tau\,(1-\emptyset(1-e^{-mG}))\right\}\right)\Bigg((a + b\,t_i)\left(e^{\theta(1-P(\Psi))(t_i-t_{i-1})}-1\right) - \theta(1-P(\Psi))\int_{t_i}^{t_{i+1}}(a+bt)e^{\theta(1-P(\Psi))(t-t_i)}\,dt\Bigg) + s * I_e\left\{\int_{t_{i-1}}^{t_{i-1}+(\delta * t_i - \delta * t_{i-1})}(a+bt)\,\delta\,dt + \int_{t_i}^{t_i+(\delta * t_{i+1} - \delta * t_i)}(a+bt-\delta(a+bt))\,dt - \delta(t_{i+1}(a+b\,t_i) - t_i(a+b\,t_i))\right\} - I_c * W\left\{\int_{t_{i-1}+(\delta * t_i - \delta * t_{i-1})}^{t_i}(a+bt)\,\delta\,dt + \int_{t_i+(\delta * t_{i+1} - \delta * t_i)}^{t_{i+1}}(a+bt-\delta(a+bt))dt - (a+bt_i-\delta(a+bt_i))[t_i - t_{i-1}]\right\}$$

$$\frac{\partial^2 T_{Ret}}{\partial {t_i}^2} = \left(\left\{\frac{h_r}{\theta(1-P(\Psi))} + \frac{\tau(1-\emptyset(1-e^{-mG}))\widehat{h_r}}{\theta(1-P(\Psi))} + d_r\, e^{-\lambda\Psi}\right\} + \left\{P_r + \hat{P}_r * \tau\,(1-\emptyset(1-e^{-mG}))\right\}\right)\left(b\left(e^{\theta(1-P(\Psi))(t_i-t_{i-1})}-1\right) + \theta(1-P(\Psi))(a + b\,t_i)e^{\theta(1-P(\Psi))(t_i-t_{i-1})} + (\theta(1-P(\Psi)))^2\,(a+b\,t_i)\right) + I_e * s\{(a+b(t_{i-1}+(\delta * t_i - \delta * t_{i-1})))\,\delta^2 + (a+b(t_i+(\delta * t_{i+1} - \delta * t_i)))\,(1-\delta) - (t_{i+1} * b * \delta - b * \delta * t_i) + (\delta * a + \delta * b * t_i)\} - I_c * W\{(\delta * a + \delta * b * t_i) + (a + b(t_{i-1}+(\delta * t_i - \delta * t_{i-1})))\delta^2 + \left(a + b * t_i + (\delta * t_{i+1} - \delta * t_i)\right)(1-\delta)^2 - (b - b * \delta)[t_i - t_{i-1}] - \left(a+bt_i-\delta(a+bt_i)\right)\} \tag{A1}$$

$$\frac{\partial^2 T_{Ret}}{\partial t_i\,\partial t_{i-1}} = \left(\left\{\frac{h_r}{\theta\big(1-P(\Psi)\big)} + \frac{\tau\big(1-\emptyset\,(1-e^{-mG})\big)\widehat{h_r}}{\theta\big(1-P(\Psi)\big)} + d_r\,e^{-\lambda\Psi}\right\} + \left\{P_r + \hat{P}_r * \tau\left(1-\emptyset\,(1-e^{-mG})\right)\right\}\right)\left(-\theta\big(1-P(\Psi)\big)(a+b*t_i)\big(e^{\theta(1-P(\Psi))(t_i-t_{i-1})}\big)\right) + I_e * s\{(a+b(t_{i-1}+(\delta*t_i-\delta*t_{i-1})))\,(\delta-\delta*\delta)-(a*\delta+b*\delta*t_{i-1})\} - I_c * W\{(a+b(t_{i-1}+(\delta*t_i-\delta*t_{i-1})))\,(\delta-\delta*\delta)+(a+bt_i-\delta(a+b*t_i))\} \quad \text{(A2)}$$

Similarly

$$\frac{\partial^2 T_{Ret}}{\partial t_i\,\partial t_{i+1}} = -\left(\left\{\frac{h_r}{\theta(1-P(\Psi))} + \frac{\tau(1-\emptyset\,(1-e^{-mG}))\widehat{h_r}}{\theta(1-P(\Psi))} + d_r\,e^{-\lambda\Psi}\right\} + \left\{P_r + \hat{P}_r * \tau\,(1-\emptyset\,(1-e^{-mG}))\right\}\right)\left(\theta(1-P(\Psi))(a+b\,t_{i+1})e^{\theta(1-P(\Psi))(t_{i+1}-t_i)}\right) + I_e * s\{(a+(b*t_i+(\delta*b*t_{i+1}-\delta*b*t_i)))\,(\delta-\delta*\delta)-(a*\delta+b*\delta*t_i)\,\delta\} - I_c * W\{(a*\delta+b*\delta*t_{i+1})(1-\delta)-(a+(b*t_i+\delta*b(t_{i+1}-t_i)))\,(\delta-\delta*\delta)\} \quad \text{(A3)}$$

$$\frac{\partial^2 T_{Ret}}{\partial t_i\,\partial t_n}=0 \quad \text{(A4)}$$

for all n ≠ i, i+1, i-1

Moreover, the Hessian matrix had to be positive definite since it contains positive diagonal members and has strictly diagonal dominating features. As a result, the optimal replenishment interval to the nonlinear system of Eq. (11) is obtained. now we need to show that the optimal solution of the non-linear Eq. (11) is unique and also $T_{Ret}(t_i, n)$ is optimal function throughout the optimal value of t_i in a finite horizon planning H.

Furthermore, because it had strictly diagonal dominating characteristics and positive diagonal members, the Hessian matrix required to be positive definite. As a result, the optimum replenishment interval for nonlinear system Eq. (12) is established. Now we need to demonstrate the convexity of $T_{Ret}(t_i, n)$ throughout the optimal value of ti in the finite horizon planning H.

T_{Ret} is positive definite if Eqs. (A1), (A2), (A3) and (A4) satisfy the given inequality (A).

$$\nabla^2 T_{Ret} = \begin{bmatrix} \frac{\partial^2 T_{Ret}}{\partial t_1^2} & \frac{\partial^2 T_{Ret}}{\partial t_1\partial t_2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{\partial^2 T_{Ret}}{\partial t_2\partial t_1} & \frac{\partial^2 T_{Ret}}{\partial t_2^2} & \frac{\partial^2 T_{Ret}}{\partial t_2\partial t_3} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\partial^2 T_{Ret}}{\partial t_3\partial t_2} & \frac{\partial^2 T_{Ret}}{\partial t_3^2} & \frac{\partial^2 T_{Ret}}{\partial t_3\partial t_4} & 0 & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{\partial^2 T_{Ret}}{\partial t_{n-1}\partial t_{n-2}} & \frac{\partial^2 T_{Ret}}{\partial t_{n-1}^2} & \frac{\partial^2 T_{Ret}}{\partial t_{n-1}\partial t_n} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\partial^2 T_{Ret}}{\partial t_n\partial t_{n-1}} & \frac{\partial^2 T_{Ret}}{\partial t_n^2} \end{bmatrix}$$

$$\frac{\partial^2 T_{Ret}}{\partial t_i^2} \geq \left|\frac{\partial^2 T_{Ret}}{\partial t_i t_{i-1}}\right| + \left|\frac{\partial^2 T_{Ret}}{\partial t_i t_{i+1}}\right| \text{ or } \frac{\partial^2 T_{Ret}}{\partial t_i^2} - \left|\frac{\partial^2 T_{Ret}}{\partial t_i t_{i-1}}\right| - \left|\frac{\partial^2 T_{Ret}}{\partial t_i t_{i+1}}\right| \geq 0 \quad \text{(A)}$$

$$\left(\left\{\frac{h_r}{\theta\big(1-P(\Psi)\big)} + \frac{\tau\big(1-\emptyset\,(1-e^{-mG})\big)\widehat{h_r}}{\theta\big(1-P(\Psi)\big)} + d_r\,e^{-\lambda\Psi}\right\} + \left\{P_r + \hat{P}_r * \tau\left(1-\emptyset\,(1-e^{-mG})\right)\right\}\right)\left((a+b\,t_i)\big(e^{\theta(1-P(\Psi))(t_i-t_{i-1})}-1\big) - \theta\big(1-P(\Psi)\big)\int_{t_i}^{t_{i+1}}(a+bt)e^{\theta(1-P(\Psi))(t-t_i)}\,\mathrm{d}t\right) + I_e$$

$$
* \; s\left\{\int_{t_{i-1}}^{t_{i-1}+\delta(t_i - t_{i-1})} (a+bt)\,\delta\,dt + \int_{t_i}^{t_i+(\delta * t_{i+1} - \delta * t_i)} (a + b * t - \delta(a+bt))\,dt - (a + b\,t_i)\,(\delta * t_{i+1} - \delta * t_i)\right\} - I_c
$$

$$
* \; W\left\{\begin{array}{c} \int_{t_{i-1}+(\delta * t_i - \delta * t_{i-1})}^{t_i} (a * \delta + b * \delta * t)\delta\,dt \\ + \int_{t_i+(\delta * t_{i+1} - \delta * t_i)}^{t_{i+1}} (a+bt)\,(1-\delta)dt - (a+bt_i)\,(1-\delta)[t_i - t_{i-1}] \end{array}\right\}
$$

$$
- \left(\left\{\frac{h_r}{\theta(1 - P(\Psi))} + \frac{\tau(1 - \emptyset\,(1 - e^{-mG}))\widehat{h_r}}{\theta(1 - P(\Psi))} + d_r\, e^{-\lambda\Psi}\right\} + \left\{P_r + \hat{P}_r * \tau\,(1 - \emptyset\,(1 - e^{-mG}))\right\}\right)\Big(-\theta(1
$$

$$
- P(\Psi))(a + b\,t_i)\left(e^{\theta(1 - P(\Psi))(t_i - t_{i-1})}\right)\Big) + I_e
$$

$$
* \; s\{(a + (b * t_{i-1} + (\delta * b * t_i - \delta * b * t_{i-1}))\,)\,(\delta - \delta * \delta) - (a * \delta + b * \delta * t_{i-1})\,\} - I_c
$$

$$
* \; W\{(a + (b * t_{i-1} + (b * \delta * t_i - b * \delta * t_{i-1})))\,(\delta - \delta * \delta) + (a + bt_i - \delta(a + bt_i))\}
$$

$$
- \left(\left\{-\frac{h_r}{\theta(P(\Psi) - 1)} + \frac{\tau(1 - \emptyset\,(1 - e^{-mG}))\widehat{h_r}}{\theta(1 - P(\Psi))} + d_r\, e^{-\lambda\Psi}\right\} + \left\{P_r + \hat{P}_r * \tau\,(1 - \emptyset\,(1 - e^{-mG}))\right\}\right)\Big(\theta(1
$$

$$
- P(\Psi))(a + b\,t_{i+1})e^{\theta(1 - P(\Psi))(t_{i+1} - t_i)}\Big) + I_e
$$

$$
* \; s\{(a + (b * t_i + (b * \delta * t_{i+1} - b * \delta * t_i)))\,(\delta - \delta * \delta) - (a * \delta + \delta * b * t_i)\} - I_c
$$

$$
* \; W\left\{(a * \delta + b * \delta * t_{i+1})\,(\delta - \delta * \delta) - (a + (b * t_i + (\delta * b * t_{i+1} - \delta * b * t_i)))\,(\delta - \delta * \delta)\right\} \geq 0
$$

that is true for all $i = 1, 2, \ldots, n$.