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Quotient and Product of Center Topological Groups

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ABSTRACT

The main objective of the research is to link the structure of the proximity space with the topological group structure to establish the central topological group structure. Our basic definition depends on the concept of the central set, which depends on the proximity space, which was used with the central continuous functions in constructing the central topological group. Also, we explained a set of properties related to this structure. We defined the center product of center set which used to establish the central product topological group. Also, we established the central product structure and the central quotient structure to define the central product topological group and the central quotient topological group, and we identified a set of properties associated with these concepts.

1. INTRODUCTION

Mathematicians developed the "fuzzy sets," "intuitionist fuzzy sets," "vague sets," "soft sets" [1] and "fuzzy soft sets" as methods of coping with uncertainty in their field.

The concept of proximity space was first proposed by Efremovic. Kandil et al. [2] presented a subjective topology on the concealed set using clearer proverbs than those found in the surrounding region of Efremovic. Additionally, proximity is a crucial factor for tackling issues that require human observation, such image analysis [3] and facial recognition [4]. Cyclic compression and the optimal proximity point are two of the key concepts in the fixed point hypothesis, and various findings, like the reference [5], have been made. Kandil et al. [1] most recently developed a new method to proximity structures that is based on the ideal and soft set ideas [6]. By utilizing proximity space, the center set was first introduced in reference [7]. In an analogous application with groups, Rosenfeld the notion of fuzzy topological groups was introduced in and properties of fuzzy topological groups were studied in references [8, 9].

The purpose of this work is to introduce the concept of center topological group and the center continuous functions are used in setting up a center topological group. Also, we introduce the product and quotient center topological groups. And study the properties of these concepts.

1.1 Basic definitions and notations

We begin by introducing some fundamental concepts that we will use in our study.

Definition 1.1

A binary relation δ on the power set of X is called an Efremovi ĉ proximity on X if and only if it satisfies the following axioms for each $A, B, C, E \subseteq X$:

P1. $A\delta B$ implies $B\delta A$;
P2. $(A \cup B)\delta C$ if and only if $A\delta C$ or $B\delta C$;
P3. $A\delta B$ implies $A \neq \emptyset, B \neq \emptyset$;
P4. $A\bar{\delta}B$ implies that there exists a subset E such that $A\bar{\delta}E$ and $X - E\bar{\delta}B$;
P5. $A \cap B \neq \emptyset$ implies $A\delta B$.

The pair (X, δ) is called a proximity space [10].

Definition 1.2

Let $(X,\boldsymbol{\delta_X})$, $(Y,\boldsymbol{\delta_Y})$ be proximity spaces the mapping f:$X \to Y$ is said to be a proximally mapping if A $\boldsymbol{\delta_X}$ B implies f(A) $\boldsymbol{\delta_Y}$ f(B) for every two sets A, $B \subseteq X$ [7].

Definition 1.3

Let (X, δ) be a proximity space and $A \subseteq X$. A center set of A is defined by $\mathcal{C}_A := \{\langle A, B\rangle : B \subseteq X \text{ and } A\delta B\}$ [7].

Definition 1.4

For two center sets $\mathcal{C}_A$ and $\mathcal{C}_B$ over a proximity space (X, δ), we say that $\mathcal{C}_A$ is a center subset of $\mathcal{C}_B$ if and only if for each $\langle A, C\rangle \in \mathcal{C}_A$, implies $\langle B, C\rangle \in \mathcal{C}_B$. We write $\mathcal{C}_A \preccurlyeq_{\mathcal{C}} \mathcal{C}_B$ [7].

Definition 1.5

Center union of two center sets of $\mathcal{C}_A$ and $\mathcal{C}_B$ over a proximity space (X, δ) is define by [6]:

$$\mathcal{C}_A \curlyvee_{\mathcal{C}} \mathcal{C}_B = \{\langle A \cup B, C\rangle : \langle A, C\rangle \in \mathcal{C}_A \text{ or } \langle B, C\rangle \in \mathcal{C}_B\}$$

Definition 1.6

Let (X, δ) be a proximity space and $\{x\}, B \subseteq X$, such that $\{x\}\delta B$. Then $x_B = \{\langle\{x\}, B\rangle\}$ is called a center point in X [6].

Definition 1.7 [6]

Let x_B be a center point in (X, δ) and $\mathcal{C}_A$ center set in (X, δ). Then $x_B \in \mathcal{C}_A$ if and only if $\langle A, B\rangle \in \mathcal{C}_A$ and $x \in A$ [6].

Definition 1.8

Let (X, δ_X) and (Y, δ_Y) be two proximity spaces and $A \subseteq X, B \subseteq Y$ and let $\mathcal{C}_A$, $\mathcal{C}_B$ be two center sets. The center product of these center sets defined by [11]:

$\mathcal{C}_A \, X_{\mathcal{C}} \, \mathcal{C}_B = \{\langle A \times B, C \times D\rangle : \langle A, C\rangle \in \mathcal{C}_A$ and $\langle B, D\rangle \in \mathcal{C}_B\}$.

Definition 1.9

Let (X, $\boldsymbol{\delta_X}$) be a proximity space and $\boldsymbol{\tau_{cent(X)}} \subseteq \boldsymbol{cent(X)}$, then $\boldsymbol{\tau_{cent(X)}}$ said to be a center topology if

1. $\boldsymbol{\mathcal{C}_\emptyset}, \boldsymbol{\mathcal{C}_X} \in \boldsymbol{\tau_{cent(X)}}$.
2. $\{\boldsymbol{\mathcal{C}_{A_i}} : \boldsymbol{i} \in \boldsymbol{I}\} \in \boldsymbol{\tau_{cent(X)}} \Rightarrow \curlyvee_{\boldsymbol{\mathcal{C}}} \{\boldsymbol{\mathcal{C}_{A_i}} : \boldsymbol{i} \in \boldsymbol{I}\} \in \boldsymbol{\tau_{cent(X)}}$.
3. $\boldsymbol{\mathcal{C}_{A_1}}, \boldsymbol{\mathcal{C}_{A_2}} \in \boldsymbol{\tau_{cent(X)}} \Rightarrow \boldsymbol{\mathcal{C}_{A_1}} \curlywedge_{\boldsymbol{\mathcal{C}}} \boldsymbol{\mathcal{C}_{A_2}} \in \boldsymbol{\tau_{cent(X)}}$.

The triplet $(\boldsymbol{X}, \boldsymbol{\delta_X}, \boldsymbol{\tau_{cent(X)}})$ is called a center topological space and the members of $\boldsymbol{\tau_{cent(X)}}$ are said to be $\boldsymbol{\mathcal{C}}$ - open. $\boldsymbol{\tau_{cent(X)}}$ is called indiscrete center topology if $\boldsymbol{\tau_{cent(X)}} = \{\boldsymbol{\mathcal{C}_X}, \boldsymbol{\mathcal{C}_\emptyset}\}$ and called discrete **center** toplogy if $\boldsymbol{\tau_{cent(X)}} = \boldsymbol{cent(X)}$ [6].

Definition 1.10

Let $(\boldsymbol{X}, \boldsymbol{\delta_X}, \boldsymbol{\tau_{cen(X)}})$, $(\boldsymbol{Y}, \boldsymbol{\delta_Y}, \boldsymbol{\tau_{cent(Y)}})$ be center topological spaces. The function $\mathbf{cent(f)} : (\boldsymbol{X}, \boldsymbol{\delta_X}, \boldsymbol{\tau_{cen(X)}}) \rightarrow (\boldsymbol{Y}, \boldsymbol{\delta_Y}, \boldsymbol{\tau_{cent(Y)}})$ is said to be center continuous function if $(\mathbf{cent(f)})^{-1}(\boldsymbol{\mathcal{C}_A})$ is $\boldsymbol{\mathcal{C}}$- open set in X for every $\boldsymbol{\mathcal{C}}$- open set $\boldsymbol{\mathcal{C}_A}$ in Y [10].

Definition 1.11

A topological group is a set G with two structures:

(i) (G, μ) is a group.

(ii)(G, τ) is a topological space, such that the multiplication map $\mu : G \, X \, G \rightarrow G, \mu(x, y) = x.y$ and the inversion map ν: $G \rightarrow G, \nu(x) = x^{-1}$ are both continuous. In this definition the set $G \times G$ carries the product topology, and denote to it by (G, μ, τ) [11].

2. CENTER TOPOLOGICAL GROUP

In this section we begin by introducing some concepts that we will use in building a central topological group.

Theorem 2.1

Let (G, μ, τ) be a topological group. Then there exists a proximity relation δ such that (G, δ) is a proximity space.

Proof:

Suppose that $\mho$ be the system of symmetric neighborhoods at e, for every A, $B \subseteq G$ and $V \in \mho$. We define $A \, \delta \, B$ iff $A \cap B.V \neq \emptyset$.

Now, we show that δ is a proximity relation.

(1) Let $A \, \delta \, B$. Then, there exists $a \in A$ and $b \in B$ s.t $a \in b.V$ then $\exists v \in V$ s.t $a = b.v \Rightarrow a^{-1} = v^{-1}.b^{-1} \Rightarrow a^{-1}.b = v^{-1}.b^{-1}.b \Rightarrow a^{-1}.b = v^{-1}.e \Rightarrow a.a^{-1}.b = a.v^{-1}.e \Rightarrow$

$b = a.v^{-1} \in a.V^{-1} = a.V \subseteq A \cdot V$. Thus, $B \cap A.V \neq \emptyset \Rightarrow B \, \delta \, A$

(2) Let $A \, \delta \, (BUC)$ There exist $a \in A$ and $b \in BUC$ s.t $A \cap ((BUC).V) \neq \emptyset$. Thus, $a \in (BUC).V$ there exists $v \in V$ s.t $a = b.v$. Since $b \in BUC \Rightarrow b \in B$ or $b \in C$.

If $b \in B$ then $a = b.v$ by the same way in (1) we have $b = a.v^{-1} \in a.V^{-1} = a.V \subseteq A \cdot V$.

Thus, $B \cap A.V \neq \emptyset \Rightarrow B \, \delta \, A$ by (1) we have $A \, \delta \, B$.

If $b \in C$ then $a = b.v$ thus $b = a.v^{-1} \in a.V$.

Thus, $C \cap A.V \neq \emptyset \Rightarrow C \, \delta \, A$ by (1) we have $A \, \delta \, C$.

Conversely:

Let $A \, \delta \, B$ or $A \, \delta \, C \rightarrow \exists \, a \in A$ and $b \in B$ s.t $a \in B.V$ or $\exists \, a \in A$ and $c \in C$ s.t $a \in C.V$.

If $a \in B.V$ then $\exists \, v \in V$ s.t $a = b.v \rightarrow b = a.v^{-1} \in A.V^{-1} = A.V$. Thus, $B \cap A.V \neq \emptyset$.

If $a \in C.V$ then $\exists \, v \in V$ s.t $a = c.v \rightarrow c = a.v^{-1} \in A.V^{-1} = A.V$. Thus,$C \cap A.V \neq \emptyset$.

If $B \cap A.V \neq \emptyset \rightarrow (B \cap A.V)U(C \cap A.V) \neq \emptyset \rightarrow (BUC) \cap A.V \neq \emptyset \rightarrow (BUC)\delta \, A$. By (1) we have $A \, \delta(BUC)$.

Thus, $A \, \delta \, (BUC)$ iff $A \, \delta \, B$ or $A \, \delta \, C$.

(3) Let $A \cap B \neq \emptyset$, then there exists $x \in A \cap B \subseteq A \cap B.V$. Therefore, $x \in A \cap B.V \neq \emptyset$ thus $A \, \delta \, B$.

(4) Let $A \, \delta \, B$ Then there exist $a \in A$ and $b \in B$ such that $A \cap B.V \neq \emptyset$ thus $A \neq \emptyset$ and $B \neq \emptyset$.

(5) Let $A \, \bar{\delta} \, B$ and $E = B.V$. if $A \, \delta \, B.V$, Then

$A \cap (B.V).V \neq \emptyset$ Therefore $A \cap B.V \neq \emptyset$

Thus, $A \, \delta \, B$ that it is a contradiction. Hence $A \, \bar{\delta} \, E$.

Also, if $B \, \delta \, E^c$, it follows that $B \cap (B.V)^c . V \neq \emptyset$.

Hence there exit $b \in B; x \in (B.V)^c$ and $v \in V$ s.t $b \in x.V$ by the same way in (1), we have $x \in b.V^{-1} = b.V$ thus $x \in (b.V)^c$ and $x \in b.V$ and it is a contradiction. Therefore, $B \, \bar{\delta} \, E^c$.

Now, we introduce the concept of center topological group.

Definition 2.2

Let (G, μ, τ) be a topological group and (G, δ) be the proximity Space which is defined in **(Theorem 2.1)** and let μ and ν be a proximity map then the four fold $(G, \mu, \tau_{cent(G)}, \delta)$ be center topological group if:

(1) The center function cent (μ): $Cp(G) \, X_c Cp(G) \rightarrow Cp(G)$ is center continuous.

(2) The center inverse function cent (ν) : $Cp(G) \rightarrow Cp(G)$is center continuous.

Where $Cp(G)$ denote the set of all center point in G.

Remark 2.3

(1) cent (μ) $(x_{B_o} X_c \, y_{B_1}) =$ cent (μ) ({ $\langle\{x\} \times \{y\}, B_o \times B_1\rangle$: $x\delta B_o$ and $y\delta B_1$ })={ $\langle\mu(x, y), \mu(B_o, B_1)\rangle$: $x\delta B_o$ and $y\delta B_1$ }= $\{\langle\{x.y\}, B_o.B_1\rangle: x\delta B_o$ and $y\delta B_1\}$=$(x.y)_{B_o.B_1}$and is denoted by $x_{B_o} \cdot_c \, y_{B_1}$.

(2) The center inverse function cent (ν) $(x_{B_o}) = (x_{B_o})^{-1} = \{\langle\{x^{-1}\}, {B_o}^{-1}\rangle: x\delta B_o\} = {x^{-1}}_{{B_o}^{-1}}$.

Proposition 2.4

Let $(G, \mu, \tau_{cent(G)}, \delta)$ be a center topological group and let g_{B_o} be a center fixed element of $(G, \mu, \tau_{cent(G)}, \delta)$. The constant map $x_{B_1} \to g_{B_o}$ is center continuous map.

Proof:

Let $\mathcal{C}_A$ be a center open set in G and g_{B_o} be a center fixed element in $G$and cent (f) be the constant map $x_{B_1} \to g_{B_o}$. If $g_{B_o} \in \mathcal{C}_A$ then $(\text{cent}\,(f)\,)^{-1}\;(\;\mathcal{C}_A\;) = \{x_{B_1} \in Cp(G): \text{cent}\,(f)(x_{B_1}) \in \mathcal{C}_A\} = \{x_{B_1} \in Cp(G): g_{B_o} \in \mathcal{C}_A\} = \curlyvee_c \{x_{B_1}: x_{B_1} \in Cp(G)\} = \mathcal{C}_G$. Which is center open set. If $g_{B_o} \notin \mathcal{C}_A$ then $(\text{cent}\,(f)\,)^{-1}(\mathcal{C}_A)$= $\mathcal{C}_\emptyset$. Which is center open set.

Proposition 2.5

Let $(G, \mu, \tau_{cent(G)}, \delta)$ be a center topological group then the identity map $x_{B_1} \to x_{B_1}$ is center continuous map.

Proof:

Let $\mathcal{C}_A$ be a center open set in G and cent (f) be the identity map $x_{B_1} \to x_{B_1}$. Then $(\text{cent}\,(f)\,)^{-1}\;(\;\mathcal{C}_A\;) = \{x_{B_1} \in Cp(G): \text{cent}\,(f)(x_{B_1}) \in \mathcal{C}_A\} = \{x_{B_1} \in Cp(G): x_{B_1} \in \mathcal{C}_A\} = \mathcal{C}_A$. Which is center open set.

Remark 2.6

Let $(G, \mu, \tau_{cent(G)}, \delta)$ be a center topological group and let g_{B_o} be a center fixed element of $(G, \mu, \tau_{cent(G)}, \delta)$. The constant map $x_{B_1} \to g_{B_o}$ and the identity map $x_{B_1} \to x_{B_1}$ are center continuous maps from $Cp(G) \to Cp(G)$, so they induce a center continuous map $x_{B_1} \to (g_{B_o}, x_{B_1})$ from $Cp(G)$ to $Cp(G)\, X_c Cp(G)$. Composing this with the center continuous multiplication $Cp(G)\, X_c Cp(G) \to Cp(G)$ we get a center continuous map cent (L_g): $Cp(G) \to Cp(G)$ defined by $x_{B_1} \to g_{B_o} \cdot_c x_{B_1}$ called center left multiplication (or center left translation) by g_{B_o} . This center map has inverse cent $(L_{g^{-1}})$ which is also center continuous, so cent (L_g) is a center homeomorphism from,

$Cp(G)$ to $Cp(G)$. cent (L_g) o cent $(L_{g^{-1}})$ (x_{B_1})=$\{\langle L_g \text{o}\, L_{g^{-1}}(\{x\}),\, L_g \text{o}\, L_{g^{-1}}(B_1)\rangle$: $x\delta B_1\} = \{\langle\{g.(g^{-1}.x)\}, g.(g^{-1}.B_1)\;\rangle$: $x\delta B_1\} = \{\langle\{(g.g^{-1}).x\}, (g.g^{-1}).B_1\;\rangle : x\delta B_1\} = \{\langle\{e.x\}, e.B_1\;\rangle : x\delta B_1\} = \{\langle\{x\}, B_1\;\rangle$: $x\delta B_1\}$=x_{B_1}= cent (I_G) (x_{B_1}).

The center identity map cent (I_G) (x_{B_1})= $\{\langle\, I_G(\{x\}),\, I_G(\{B_1\})\;\rangle$: $x\delta B_1\}$=$\{\langle\{x\}, B_1\;\rangle$: $x\delta B_1\}$=x_{B_1}.

Similarly, all center right translations cent (r_g): $x_{B_1} \to x_{B_1} \cdot_c g_{B_o}$ are center homeomorphisms from $Cp(G)$ to $Cp(G)$.

Note: When we say G is a center topological group we mean the fourfold $(G, \mu, \tau_{cent(G)}, \delta)$.

Remark 2.7

If $\mathcal{C}_A$, $\mathcal{C}_B \preccurlyeq_c \mathcal{C}_G$ and $g_{B_o} \in Cp(G)$ where G is a center topological group.

(1) $\mathcal{C}_A \cdot_c g_{B_o}$= cent $(r_g)(\mathcal{C}_A) = \{a_{B_1} \cdot_c g_{B_o}\,;\, a_{B_1} \in \mathcal{C}_A\}$
$\mathcal{C}_A \cdot_c g_{B_o}$ is called the center right translate of $\mathcal{C}_A$ by g_{B_o}.

(2) $g_{B_o} \cdot_c \mathcal{C}_A$ = cent $(L_g)(\mathcal{C}_A) = \{g_{B_o} \cdot_c a_{B_1};\, a_{B_1} \in \mathcal{C}_A\}$

(3) $\mathcal{C}_A \cdot_c \mathcal{C}_B$= $\curlyvee_{c_{b_{B_1} \in \mathcal{C}_B}} \mathcal{C}_A \cdot_c b_{B_1}$= $\curlyvee_{c_{a_{A_1} \in \mathcal{C}_A}} a_{A_1}.\mathcal{C}_B$.

(4) $(\mathcal{C}_A)^{-1}$ =$\{\langle A^{-1}, B^{-1}\rangle: A\delta B\}$ or $\{(a_{A_1})^{-1};\, a_{A_1} \in \mathcal{C}_A\}$.

Proposition 2.8

Let G be a center topological group $\mathcal{C}_A$, $\mathcal{C}_B \preccurlyeq_c \mathcal{C}_G$, $g_{B_o} \in Cp(G)$ then

(1) $\mathcal{C}_A$ center open implies $\mathcal{C}_A \cdot_c g_{B_o}$ and $g_{B_o} \cdot_c \mathcal{C}_A$, center open.

(2) $\mathcal{C}_A$ center closed implies $\mathcal{C}_A \cdot_c g_{B_o}$ and $g_{B_o} \cdot_c \mathcal{C}_A$, center closed.

(3) $\mathcal{C}_A$ center open implies $\mathcal{C}_A \cdot_c \mathcal{C}_B$ and $\mathcal{C}_B \cdot_c \mathcal{C}_A$ center open.

(4) $\mathcal{C}_A$ center closed and $\mathcal{C}_B$ finite implies $\mathcal{C}_A \cdot_c \mathcal{C}_B$ and $\mathcal{C}_B \cdot_c \mathcal{C}_A$ center closed.

Proof:

(1), (2) cent (L_g), cent (r_g) being center homeomorphisms are both center open and center closed.

(3) $\mathcal{C}_A \cdot_c \mathcal{C}_B = \curlyvee_{c_{b_{B_1}}} \mathcal{C}_A \cdot_c b_{B_1} = \curlyvee_c$ cent $(r_b)(\mathcal{C}_A)$ is a center union of center open sets and hence center open. Similarly, for $\mathcal{C}_B \cdot_c \mathcal{C}_A$.

(4) $\mathcal{C}_A \cdot_c \mathcal{C}_B$ and $\mathcal{C}_B \cdot_c \mathcal{C}_A$ are each the center union of a finite number of center closed sets and hence center closed.

Definition 2.9

Let $(G, \mu, \tau_{cent(G)}, \delta)$ be a center topological group where (G, δ) is the proximity space which is defined in (**Theorem 2.1**). We defined identity center point as follows let $\{e\}\,\delta B_o$ then $e_{B_o} = \{\langle e\}, B_o\rangle\;\}$ and B_o satisfies $\mu(B, B_o)$= $\mu(B_o, B) = B$ and $B_o{}^{-1} = B_o$ for any $B \subseteq G$.

Definition 2.10

A fundamental system of center neighborhoods of x_{B_o} in $Cp(X)$ is a collection $\mathcal{F}_c$ of center neighborhoods of x_{B_o} such that every center neighborhood of x_{B_o} contains a center member of $\mathcal{F}_c$. If each member of $\mathcal{F}_c$ is center open, we speak of a fundamental system of open center neighborhoods of x_{B_o}.

Proposition 2.11

Any fundamental system $\mathcal{F}_c$ of center open neighborhoods of e_{B_o} in $Cp(G)$ has the following properties:

CFN1: If $\mathcal{C}_U$, $\mathcal{C}_V \in \mathcal{F}_c$, then $\exists\; \mathcal{C}_W \in \mathcal{F}_c$ such that $\mathcal{C}_W \preccurlyeq_c \mathcal{C}_U \curlywedge_c \mathcal{C}_V$.

CFN2: If $a_{B_1} \in \mathcal{C}_U \in \mathcal{F}_c$ then $\exists\; \mathcal{C}_V \in \mathcal{F}_c$ such that $\mathcal{C}_V \cdot_c a_{B_1} \preccurlyeq_c \mathcal{C}_U$.

CFN3: If $\mathcal{C}_U \in \mathcal{F}_c$, then $\exists\; \mathcal{C}_V \in \mathcal{F}_c$ such that $(\mathcal{C}_V)^{-1} \cdot_c \mathcal{C}_V \preccurlyeq_c \mathcal{C}_U$.

CFN4: If $\mathcal{C}_U \in \mathcal{F}_c$, $x_{B_2} \in Cp(G)$, then $\exists\, \mathcal{C}_V \in \mathcal{F}_c$ such that $x^{-1}{}_{B_2{}^{-1}} \cdot_c \mathcal{C}_V \cdot_c x_{B_2} \preccurlyeq_c \mathcal{C}_U$.

Proof:

In (1) $\mathcal{C}_U \curlywedge_c \mathcal{C}_V$, in (2) $\mathcal{C}_U \cdot_c a^{-1}{}_{B_1{}^{-1}}$, are center open neighborhood of e_{B_o}. So, contain a center element of $\mathcal{F}_c$.

(3): The center map cent(f) : $Cp(G) X_c Cp(G) \to Cp(G)$ given by $(a_{B_1}, b_{B_2}) \to a^{-1}{}_{B_1{}^{-1}} \cdot_c b_{B_2}$ is center continuous. Thus $(\text{cent}(f))^{-1}(\mathcal{C}_U)$ is center open, contains (e_{B_o}, e_{B_o}), and hence contains a center set of the form $\mathcal{C}_A X_c \mathcal{C}_B$, where $\mathcal{C}_A$, $\mathcal{C}_B$ are center open and contain e_{B_o}. Since $\mathcal{C}_A \curlywedge_c \mathcal{C}_B$ is a center open neighborhood of e_{B_o}, $\exists\, \mathcal{C}_V \in \mathcal{F}_c$ so that $\mathcal{C}_V \preccurlyeq_c \mathcal{C}_A \curlywedge_c \mathcal{C}_B$. For this $\mathcal{C}_V$ we have $\mathcal{C}_V X_c \mathcal{C}_V$

$\preccurlyeq_c \left(\text{cent}(f)\right)^{-1}(\mathcal{C}_U)$, that is $(\mathcal{C}_V)^{-1}\cdot_c \mathcal{C}_V \preccurlyeq_c \mathcal{C}_U$.

(4): The center map $\text{cent}(f): Cp(G) \to Cp(G)$ given by $a_{B_1} \to x^{-1}{}_{B_2{}^{-1}} \cdot_c a_{B_1} \cdot_c x_{B_2}$ is center continuous, so $\left(\text{cent}(f)\right)^{-1}(\mathcal{C}_U)$ is center open and contains e_{B_o}, hence contains some $\mathcal{C}_V \in \mathcal{F}_c$. For this $\mathcal{C}_V$ we have $\text{cent}(f)(\mathcal{C}_V) = x^{-1}{}_{B_2{}^{-1}} \cdot_c \mathcal{C}_V \cdot_c x_{B_2} \preccurlyeq_c \mathcal{C}_U$.

3. CENTER QUOTIENT TOPOLOGICAL GROUP AND PRODUCT TOPOLOGICAL GROUP

This section will contain the concepts of center product topological group and center quotient topological group.

Theorem 3.1

Let (G, μ, τ) be a topological group and H be a normal sub group then G/H with group structure and quotient topology is a topological group. We define δ^* on G/H by $AH\ \delta^* BH$ iff $A\ \delta\ B$ where δ is the proximity relation which is defined in (**Theorem 2.1**). Then $(G/H, \delta^*)$ is a proximity space.

Proof:

(1) Let $AH\ \delta^* BH$ then $A\ \delta\ B$ thus $B\ \delta\ A$ then $BH\ \delta^* AH$.

(2) Let $AH\ \delta^*(BH \cup CH)$ then $AH\ \delta^*(B \cup C)\,H \leftrightarrow A\ \delta\ B \cup C \leftrightarrow A\ \delta\ B$ or $A\ \delta\ C \leftrightarrow AH\ \delta^*\ BH$ or $AH\ \delta^*\ CH$.

(3) Let $AH\ \delta^*\ BH \leftrightarrow A\ \delta\ B \to A \neq \emptyset$ and $B \neq \emptyset \to AH \neq \emptyset$ and $BH \neq \emptyset$.

(4) Let $AH\ \overline{\delta^*}\ BH \leftrightarrow A\ \overline{\delta}\ B \to A \cap B = \emptyset \to A \subseteq B^c \to AH \subseteq B^cH \to AH \cap BH \subseteq B^cH \cap BH = \emptyset \to AH \cap BH = \emptyset$.

(5) Let $AH\ \overline{\delta^*}\ BH \leftrightarrow A\ \overline{\delta}\ B$ then there exists $E = B.V$ if $AH\ \delta^*\ (B.V)H$

$A\ \delta\ B.V \to A\ \delta\ E$ that is a contradiction. Hence $AH\ \overline{\delta^*}\ EH$. Also, if $BH\ \delta^* E^cH$ thus $B\ \delta\ E^c$ and we have by (**Theorem 2.1**) (5) contradiction. Therefore $B\ \overline{\delta}\ E^c$ thus $BH\ \overline{\delta^*}\ E^cH$. Thus $(G/H, \delta^*)$ be a proximity space.

Definition 3.2

Let : $(G, \mu, \tau_{cent(G)}, \delta)$ be a center topological group and let $\mathcal{C}_H$ be a center subset of G we say that $\mathcal{C}_H$ be a center subgroup iff for each $x_{B_o}, y_{B_1} \in \mathcal{C}_H$ then $x_{B_o} \cdot_c y^{-1}{}_{B_1{}^{-1}} \in \mathcal{C}_H$.

Definition 3.3

Let: $(G, \mu, \tau_{cent(G)}, \delta)$ be a center topological group and let $\mathcal{C}_H$ be a center subset of G we say that $\mathcal{C}_H$ be a center normal subgroup iff

$$x_{B_1} \cdot_c \mathcal{C}_H \cdot_c x^{-1}{}_{B_1{}^{-1}} = \mathcal{C}_H.$$

Theorem 3.4

Let G be a center topological group and H be a center normal sub group then $\text{cent}(\mu')$ and $\text{cent}(v')$ are center continuous, i.e $(G/H, \mu', \tau_{cent(G/H)}, \delta^*)$ be center quotient topological group.

Proof:

Let G be a center topological group and H be a center normal sub group and let $\text{cent}(q): Cp(G) \to \text{cent}(G/H)$ be a center quotient map defined by $x_{B_o} \cdot_c \mathcal{C}_H$ define a center quotient topology on G/H ($\mathcal{C}_A$ is center open in G/H iff $(\text{cent}(q))^{-1}(\mathcal{C}_A) \in \tau_{cent(G)}$).

The center quotient map is center open map for if $\mathcal{C}_A$ is center open subset of G then $(\text{cent}(q))^{-1}(\text{cent}(q))(\mathcal{C}_A)$ =the center union of all center left cosets of $\mathcal{C}_H$ which center meet $\mathcal{C}_A = \mathcal{C}_A \cdot_c \mathcal{C}_H$ which is center open by **(Proposition 2.8**) (3). And it follows that $\text{cent}(q)(\mathcal{C}_A)$ is center open in G/H by the definition of the center quotient topology.

Now, let μ' and v' be proximally maps. If $\text{cent}(\mu)$ and $\text{cent}(\mu')$ are the Center multiplication in G and G/H and $\text{cent}(v)$, $\text{cent}(v')$ the center inversion in G and G/H respectively, then $\text{cent}(\mu')$, $\text{cent}(v')$ are uniquely defined by Commutatively of the following diagrams as shown in Figure 1 and Figure 2.

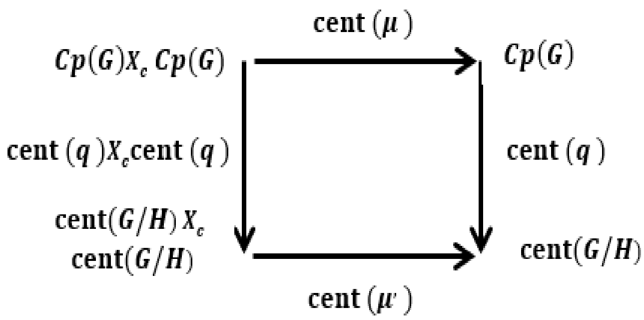

Figure 1. Diagram of center product of center quotient map

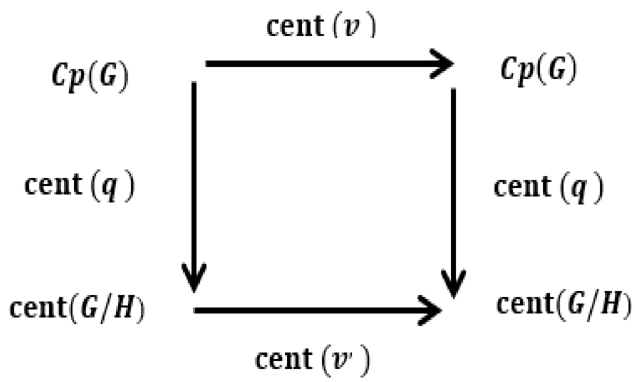

Figure 2. Diagram of center inverse map of center quotient map

$\text{Cent}(q)\ o\ \text{cent}(v)$ is center continuous and so by the center universal property of $\text{cent}(q)$. There exists a center unique map $\text{cent}(\theta): \text{cent}(G/H) \to \text{cent}(G/H)$ making $\text{cent}(\theta)\ o\ \text{cent}(q) = \text{cent}(q)\ o\ \text{cent}(v)$. However, $\text{cent}(v')$ satisfies the condition $\text{cent}(v')\ o\ \text{cent}(q) = \text{cent}(q) o\ \text{cent}(v)$ and so $\text{cent}(v') = \text{cent}(\theta)$. Also, $\text{cent}(q)\ o\ \text{cent}(\mu)$ is center continuous and so by the same way $\text{cent}(\mu')$ is center continuous provided that $\text{cent}(q)\ X_c\ \text{cent}(q)$ is center quotient map. In our case $\text{cent}(q)$ is center open and so $\text{cent}(q)\ X_c\ \text{cent}(q)$ is center open. In addition, $\text{cent}(q)\ X_c\ \text{cent}(q)$ is center continuous and a surjection and hence a center quotient map. Thus, $\text{cent}(\mu')$ is center continuous and (G/H) is center quotient topological group.

Theorem 3.5

Let $(G_1, \mu_1, \tau_{cent(G_1)}, \delta_1)$ and $(G_2, \mu_2, \tau_{cent(G_2)}, \delta_2)$ be two centre topological groups where δ_1 and δ_2 are the proximity relation which is defined in **(Theorem 2.1)** then $G_1 \times G_2$ be a topological group has a natural structure (product of groups) and a natural topology (product of topological spaces) then there exists δ^{**} on $G_1 \times G_2$ which is defined as follows $A_1 \times A_2\ \delta^{**} B_1 \times B_2$ iff $A_1\delta_1 B_1$ and $A_2\delta_2 B_2$. Then $(G_1 \times G_2, \delta^{**})$ be a proximity space.

Proof:

(1) Let $A_1 \times A_2\ \delta^{**} B_1 \times B_2$ then $A_1\delta_1 B_1$ and $A_2\delta_2 B_2$. Since δ_1, δ_2 be two proximity relations thus $B_1\ \delta_1 A_1$ and $B_2\ \delta_2 A_2$ thus $B_1 \times B_2\ \delta^{**}\ A_1 \times A_2$.

(2) Let $A_1 \times A_2\ \delta^{**}(B_1 \times B_2) \cup (C_1 \times C_2) \leftrightarrow A_1 \times A_2\ \delta^{**}(B_1 \cup C_1) \times (B_2 \cup C_2) \leftrightarrow A_1\delta_1(B_1 \cup C_1)$ and $A_2\delta_2(B_2 \cup C_2) \leftrightarrow (A_1\delta_1 B_1$ or $A_1\delta_1 C_1)$ and $(A_2\delta_2 B_2$ or $A_2\delta_2 C_2) \leftrightarrow A_1\delta_1 B_1$ and $A_2\delta_2 B_2$ or $A_1\delta_1 C_1$ and $A_2\delta_2 C_2 \leftrightarrow A_1 \times A_2\ \delta^{**} B_1 \times B_2$ or $A_1 \times A_2\ \delta^{**} C_1 \times C_2$.

(3) Let $A_1 \times A_2\ \delta^{**} B_1 \times B_2 \leftrightarrow A_1\delta_1 B_1$ and $A_2\delta_2 B_2$ thus $A_1 \neq \emptyset$, $B_1 \neq \emptyset$, $A_2 \neq \emptyset$ and $B_2 \neq \emptyset$. Thus $A_1 \times A_2 \neq \emptyset$ and $B_1 \times B_2 \neq \emptyset$.

(4) Let $(A_1 \times A_2) \cap (B_1 \times B_2) \neq \emptyset \rightarrow (A_1 \cap B_1) \times (A_2 \cap B_2) \neq \emptyset \rightarrow A_1 \cap B_1 \neq \emptyset$ and $A_2 \cap B_2 \neq \emptyset$. Since δ_1, δ_2 be two proximity relations. Then, $A_1\delta_1 B_1$ and $A_2\delta_2 B_2$. Thus, $A_1 \times A_2\ \delta^{**} B_1 \times B_2$.

(5) Let $A_1 \times A_2\ \overline{\delta^{**}}\ B_1 \times B_2$. Thus, either $A_1\overline{\delta_1}B_1$ or $A_2\overline{\delta_2}B_2$. If $A_1\overline{\delta_1}B_1$ then $\exists\ E_1 = B_1.V_1$ such that $A_1\overline{\delta_1}E_1$ and $B_1\overline{\delta_1}{E_1}^c$. If $A_2\overline{\delta_2}B_2$ then $\exists\ E_2 = B_2.V_2$ such that $A_2\overline{\delta_2}E_2$ and $B_2\overline{\delta_2}{E_2}^c$. If $A_1 \times A_2\ \delta^{**} E_1 \times E_2$. Then, $A_1\delta_1 E_1$ and $A_2\delta_2 E_2$ (that's a contradiction). Thus, $A_1 \times A_2\ \overline{\delta^{**}}\ E_1 \times E_2$. If $B_1 \times B_2\ \delta^{**}\ {E_1}^c \times {E_2}^c$. Then, $B_1\delta_1\ {E_1}^c$ and $B_2\delta_2\ {E_2}^c$ (that's a contradiction). Hence, $B_1 \times B_2\ \overline{\delta^{**}}\ {E_1}^c \times {E_2}^c$.

From (1) to (5) we have $(G_1 \times G_2, \delta^{**})$ be proximity space.

Theorem 3.6

Let $\{(G_i, \mu_i, \tau_{cent(G_i)}, \delta_i) : i \in I\}$ be a family of centre topological groups where δ_i be the proximity relation which is defined in **(Theorem 2.1).** Then there exists a proximity relation on $\prod_{i\in I} G_i$ which is defined as follows $\prod_{i\in I} A_i\ \delta^{***} \prod_{i\in I} B_i$ iff $A_i\delta_i B_i$ for each $i \in I$.

Proof:

(1) Let $\prod_{i\in I} A_i\ \delta^{***} \prod_{i\in I} B_i$ then $A_i\delta_i B_i$ for each $i \in I$. Since δ_i is a proximity relations for each $i \in I$ thus $B_i\ \delta_i A_i$ for each $i \in I$ thus $\prod_{i\in I} B_i\ \delta^{***} \prod_{i\in I} A_i$.

(2) Let $\prod_{i\in I} A_i\ \delta^{***}(\prod_{i\in I} B_i) \cup (\prod_{i\in I} C_i) \leftrightarrow \prod_{i\in I} A_i\ \delta^{***} \prod_{i\in I}(B_i \cup C_i) \leftrightarrow A_i\delta_i(B_i \cup C_i)$ for each $i \in I \leftrightarrow A_i\delta_i B_i$ or $A_i\delta_i C_i$ for each $i \in I$ (Since δ_i is a proximity relations for each $i \in I$) $\leftrightarrow \prod_{i\in I} A_i\ \delta^{***} \prod_{i\in I} B_i$ or $\prod_{i\in I} A_i\ \delta^{***} \prod_{i\in I} C_i$.

(3) Let $\prod_{i\in I} A_i\ \delta^{***} \prod_{i\in I} B_i \leftrightarrow A_i\delta_i B_i$ for each $i \in I$ (Since δ_i is a proximity relation for each $i \in I$) thus $A_i \neq \emptyset$, $B_i \neq \emptyset$ for each $i \in I$. Thus, $\prod_{i\in I} A_i \neq \emptyset$ and $\prod_{i\in I} B_i \neq \emptyset$.

(4) Let $(\prod_{i\in I} A_i) \cap (\prod_{i\in I} B_i) \neq \emptyset \leftrightarrow \prod_{i\in I}\ (A_i \cap B_i) \neq \emptyset \leftrightarrow A_i \cap B_i \neq \emptyset$ for each $i \in I$ (Since δ_i is a proximity relations for each $i \in I$) $\rightarrow A_i\delta_i B_i$ for each $i \in I$. Thus, $\prod_{i\in I} A_i\ \delta^{***} \prod_{i\in I} B_i$.

(5) Let $\prod_{i\in I} A_i\ \overline{\delta^{***}}\ \prod_{i\in I} B_i$. Thus, either $\exists\ i \in I$ such that $A_i\overline{\delta_i}B_i$.

Since δ_i is a proximity relation thus, $\exists\ E_i = B_i.V_i$ such that $A_i\overline{\delta_i}E_i$ and $B_i\overline{\delta_i}\ {E_i}^c$. If $\prod_{i\in I} A_i\ \ \delta^{***} \prod_{i\in I} E_i$ then $A_i\delta_i E_i$ for each $i \in I$ (that's a contradiction). If $\prod_{i\in I} B_i\ \ \delta^{***} \prod_{i\in I} {E_i}^c$ then $B_i\delta_i {E_i}^c$ for each $i \in I$ (that's a contradiction).

Thus, $\prod_{i\in I} A_i\ \overline{\delta^{***}}\ \prod_{i\in I} E_i$ and $\prod_{i\in I} B_i\ \overline{\delta^{***}}\ \prod_{i\in I} {E_i}^c$. From (1) to (5) we have $(\prod_{i\in I} G_i, \delta^{***})$ be proximity space.

Theorem 3.7

Let cent(f) be center function from a center space $cent(Y)$ into a center product space $cent(X_1)\ X_{\mathcal{C}}\ cent(X_2)$. Then cent(f) is center continuous if the composition $P_{X_i}\ o$ cent(f)$: cent(Y) \rightarrow cent(X_i)$ is center continuous, where $i = 1{,}2$.

Proof:

Let cent(f) be center continuous. Since P_{X_1} and P_{X_2} are center continuous then $P_{X_i}\ o$ cent(f) is center continuous, where $i = 1{,}2$. Conversely, let $P_{X_i}\ o$ cent(f) is center continuous, where $i = 1{,}2$ and let $\mathcal{C}_U$ be any center member of the defining sub base be ${B^*}_{\mathcal{C}}$ of the center product space $cent(X_1)\ X_{\mathcal{C}}\ cent(X_2)$ then $\mathcal{C}_U = {P^{-1}}_{X_i}(\mathcal{C}_G)$ for some $i = 1{,}2$ and some $\mathcal{C}_G \in \tau_{cent(X_i)}$ also, $(\text{cent(f)})^{-1}\ (\ \mathcal{C}_U) = (\text{cent(f)})^{-1}$.

$({P^{-1}}_{X_i}(\mathcal{C}_G)) = (P_{X_i}\ o\ \text{cent(f)})^{-1}(\mathcal{C}_G)$. Since $P_{X_i}\ o$ cent(f) is center continuous. It follows that $\left(P_{X_i}\ o\ \text{cent(f)}\right)^{-1}\ (\mathcal{C}_G) = (\text{cent(f)})^{-1}\ (\mathcal{C}_U)$ is $\mathcal{C}$- open in $cent(Y)$. Thus, we shown that the invers image under cent(f) of every sub basic $\mathcal{C}$- open set in the center product $cent(X_1)\ X_{\mathcal{C}}\ cent(X_2)$ is $\mathcal{C}$- open in $cent(Y)$. Thus $F$is center continuous.

Theorem 3.8

Let $(G_1, \mu_1, \tau_{cent(G_1)}, \delta_1)$ and $(G_2, \mu_2, \tau_{cent(G_2)}, \delta_2)$ be two centre topological groups where δ_1 and δ_2 are the proximity relation which is defined in **(Theorem 2.1)** then $(G_1 \times G_2, \mu_{G_1 \times G_2}, \tau_{cent(G_1)}\ X_c.$

$\tau_{cent(G_2)}, \delta^{**})$ be the center product topological group with this center topology on $G_1 \times G_2$. A center map cent (f): cent $(X) \rightarrow$ cent $(G_1 \times G_2)$ is center continuous iff each P_{G_i} o cent (f) : cent $(X) \rightarrow$ cent(G_i) is center continuous $\forall i = 1{,}2$.

Proof:

Let $G = G_1 \times G_2$. The center group operation cent (μ), cent (v) are defined on $Cp(G)$, so that the following diagrams Commute for each $i = 1{,}2$ as shown in Figures 3 and 4. Whence, as P_{G_i}, cent (μ_i), cent (v_i) are center continuous for all i, so are P_{G_i}o cent $(\mu\ )$ and P_{G_i}o cent $(v\ )$. Thus, cent $(\mu\ )$,cent $(v\ )$ are center continuous (**by Theorem 3.7**) and G a center topological group.

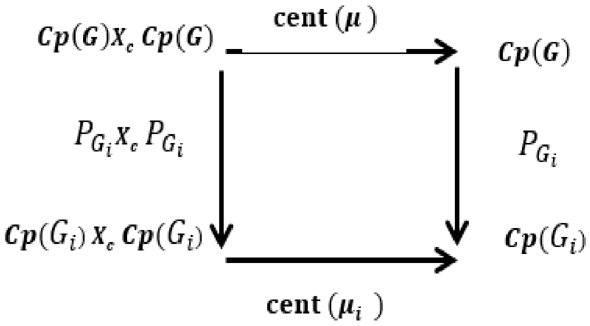

Figure 3. Diagram of center product maps

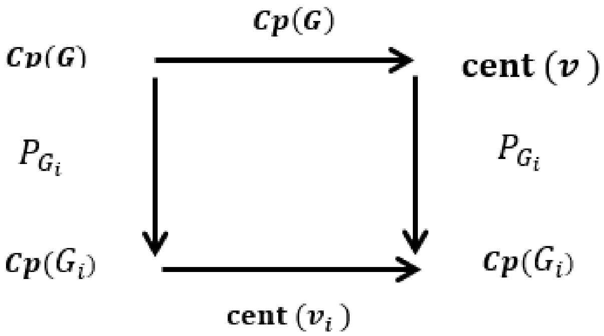

Figure 4. Diagram of center inverse map of center product maps

4. CONCLUSION

In our investigation of central set theory applied to topological groups, we have made significant strides in understanding the intricate relationship between proximity spaces and topological group structures. By focusing on the novel concept of central topological groups, we have laid the groundwork for a deeper exploration of the topological properties that emerge from central sets and central continuous functions.

Throughout our research, we have meticulously defined the central set and utilized it as a fundamental tool in constructing the central topological group. By doing so, we have not only categorized a unique class of topological groups but have also shed light on the underlying structural properties that characterize such groups.

The introduction of the center product of center sets has been a pivotal advancement in our work, providing the necessary framework to establish the central product topological group. This concept has bridged a critical gap in the literature, offering a robust structure for combining topological groups in a manner that respects the centrality of their components.

Moreover, the establishment of the central product and central quotient structures has been instrumental in defining both the central product topological group and the central quotient topological group. Our research has unveiled a collection of properties that are inherent to these constructions, contributing to the theoretical development of central topological groups. Among these properties, we have identified and proven several that are pivotal to understanding the behavior of center sets under the operations of product and quotient.

The implications of our findings are vast and suggest numerous avenues for future research. The properties we have proven provide a solid foundation for further exploratory studies into the dynamics of central topological groups, particularly in relation to their applications in various branches of mathematics and physics.

In conclusion, our study has not only addressed a gap in the current mathematical literature regarding the structure of central topological groups but has also laid the groundwork for subsequent theoretical advancements. We anticipate that the concepts and structures we have introduced will be pivotal in the continued evolution of topological group theory and its applications.

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