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Higher-Order Derivatives of Differential Subordination of Multivalent Functions

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ABSTRACT

The research into theory for analytic univalent as well as multivalent functions is an ancient subject for mathematics, especially in complex analysis, which has attracted a great number for scholars due to utter elegance of the its geometrical characteristics as well as numerous research opportunities. The study of univalent functions is one of most important areas of complex analysis for only one and many variables. Researchers have been interested in the traditional study of this subject since at least 1907. During this time until now many researchers in the field of complex analysis, including as Euler, Gauss, Riemann, Cauchy, and many others, have developed. Geometric function theory is a combination or interplay of geometry and analysis. The main goal of this article is to investigate the principle for dependence as well as add an additional subset for polyvalent functions with a different operator that is related to derivatives of higher order. As a result, the findings were important in terms of various geometric properties, including coefficient estimation, distortion as well as growth borders, radii for starlikeness, convexity, as well as close-to-convexity.

1. INTRODUCTION

The main driving force behind this line of believed is the renowned conjecture known to be the Bieberbach conjecture as well as coefficient problem, and these offered enormous scope over development about 1916 until a positive settlement within 1985 through De Branges, during which innumerable results derived from this problem emerged. Since then, Geometric Function Theory is being studied separately. Geometric Function Theory is a popular topic. Despite this, it continues to find new uses for a variety of fields range for fields, including modern mathematical physics, engineering, medical, as well as others, more traditional physics topics like fluid dynamics, nonlinear integrable systems theory, as well as others partial differential equation theory. In complex analysis, a geometric function is a function whose range describes specific geometries.

The research aim is ability to study a new class for multivalent functions established through the new linear operator and start investigating a new linear operator by using Hadamard product of the basic higher-order derivatives of differential subordination of multivalent functions. Using generalized hypergeometric function and the properties of the generalized derivative operator will have obtaining A number of findings over higher-order derivatives about differential subordination within an open unit disk are presented.

We used the properties of the generalized derivative operator, derive certain subordination and superordination properties, as well as examine the characteristics about variations subordination for analytic univalent functions over an open unit the disc. Furthermore, its findings have shedding illumination on geometric features that include coefficient inequality and Hadamard product characteristics. Certain fascinating findings to feed higher-order derivatives, variations subordination, as well as superordination about analytic univalent functions were recently installed. Following that, employing the convolution formed by two linear operators, specific findings for differential subordination comprising linear operators have been presented. Multiple findings to feed higher-order differential subordination within the open unit disk employing a generalized hypergeometric function are being discussed employing the convolution operator.

Allow $g(w)$ to serve as an anaytic function an open unit disc $\mathcal{L} = \{w \in \mathbb{C} : |w| < 1\}$. In the event the equation $v = g(w)$ possesses fewer compared to p-solutions through $\mathcal{L}$, then $g(w)$ is said to be p-valent through $\mathcal{L}$. 5. "The class of all analytic p-valent functions is denoted by $\mathcal{A}_p$, where g is expressed of the forms"

$$g(w) = w^p - \sum_{\iota=p+1}^{\infty} a_\iota w^\iota, \quad (w \in \mathcal{L}), \tag{1}$$

and $p, \iota \in \mathbb{N} = \{1,2,3, \ldots\}$. The Hadamard product from two functions in $\mathcal{A}_p$, which means

$$k(w) = w^p - \sum_{\iota=p+1}^{\infty} c_\iota w^\iota, \quad (w \in \mathcal{L}) \tag{2}$$

is provided by

$$g(w) * k(w) = w^p - \sum_{\iota=p+1}^{\infty} a_\iota c_\iota w^\iota. \quad (w \in \mathcal{L}) \tag{3}$$

The theory about harmonic as well as analytic [1] univalent functions for (bi or just multi-types) [2-6] constitutes a few about the most significant ideas associated with complex analysis. Thus, a few unique elements are described within this theory to establish novel interesting certain groups or just sub-classes [7-9] for special functions associated to multiple operators [10-14] that could have maximized as well as maximized a number real problem via a certain functional relative that results via the theory of conventional functions by way of a few characteristics for complex functions [15, 16].

Especially, this area of study has piqued the interest for a number of applied science researchers in a variety of situations. Furthermore, these principles play an important role in determining the exact solution for mathematical modeling, especially in the analysis for physical, chemical, as well as building domains [17-19].

2. BASIC PROPERTIES

We allow $g(w)$ as well as $k(w)$ be analytic function in $\mathcal{L}$. The function $g(w)$ is said to be subordinate to a function $k(w)$ or $k(w)$ is said to be superordinate to $g(w)$, if and only if there exists a Schwarz function $z(w)$ analytic in $\mathcal{L}$, with $z(0) = 0$ and $|z(w)| < 1, (w \in \mathcal{L})$, such that,

$$g(w) = k(z(w))$$

written as

$$g \prec k \; or \; g(w) \prec k(w), (w \in \mathcal{L})$$

Furthermore, if the function k is univalent in $\mathcal{L}$, then we get the following equivalence $g(w) \prec k(w)$ if and only if $g(0) = k(0)$ and $g(\mathcal{L}) \subset k(\mathcal{L})$ [20-22]. A function $g(w)$ is called starlike (convex) in $\mathcal{L}$ if satisfies the following condition:

$$\left\{\Re e\left\{\frac{wg'(w)}{g(w)}\right\} > 0, g(w) \neq 0\right\}, \left\{\Re e\left\{1 + \frac{zg''(w)}{g'(w)}\right\} > 0\right\}$$

respectively [20]. The linear multiplier fractional q-differintegral operator $\mathcal{L}_{\gamma,\tau}^{\alpha,n}$ introduced by studies [15, 23] defined as follows:

$$\begin{gathered} \mathcal{L}_{\gamma,\tau}^{\alpha,0} g(w) = g(w), \\ \mathcal{L}_{\gamma,\tau}^{\alpha,1} g(w) = (1-\tau)\mathcal{L}_{\gamma,\tau}^{\alpha} g(w) + \\ \tau w \left(\mathcal{L}_{\gamma,\tau}^{\alpha} g(w)\right)', (\tau \geq 0), \\ \mathcal{L}_{\gamma,\tau}^{\alpha,2} g(w) = \mathcal{L}_{\gamma,\tau}^{\alpha,1}\left(\mathcal{L}_{\gamma,\tau}^{\alpha,1} g(w)\right), \\ \mathcal{L}_{\gamma,\tau}^{\alpha,n} g(w) = \mathcal{L}_{\gamma,\tau}^{\alpha,1} g\left(\mathcal{L}_{\gamma,\tau}^{\alpha,n-1} g(w)\right), \quad (n \in N) \end{gathered} \tag{4}$$

and with the following form:

$$g(w) = w - \sum_{\iota=p+1}^{\infty} |a_\iota| w^\iota \tag{5}$$

then by Eqs. (4) as well as (5), we get:

$$\mathcal{L}_{\gamma,\tau}^{\alpha,n} g(w) = w^p - \sum_{\iota=p+1}^{\infty} \left(\frac{\Gamma_\gamma(2-\alpha)\Gamma_\gamma(\iota+1)}{\Gamma_\gamma(2)\Gamma_\gamma(\iota+1-\alpha)}\left[1-\tau + [\iota]_\gamma \tau\right]\right)^\alpha |a_\iota| w^\iota \tag{6}$$

where,

$$\mu_\iota^\alpha = \left(\frac{\Gamma_\gamma(2-\alpha)\Gamma_\gamma(\iota+1)}{\Gamma_\gamma(2)\Gamma_\gamma(\iota+1-\alpha)}\left[1-\tau+[\iota]_\gamma\tau\right]\right)^\alpha \tag{7}$$

By Eqs. (6)-(7), then we have:

$$\mathcal{L}_{\gamma,\tau}^{\alpha,n} g(w) = w^p - \sum_{\iota=p+1}^{\infty} \mu_\iota^\alpha |a_\iota| w^\iota$$

Note that, if we put $\alpha = 0$ the operator $\mathcal{L}_{\gamma,\tau}^{\alpha,n}$ decreases into the operator studied through Al-Oboudi [24] as well as for $\alpha = 0,\ \tau = 1$, we as a species get the operator introduced through Salagean [25]. As a higher order derivatives q-differintegral operator is described below:

$$\begin{gathered} g^{(1)}(w) = pw^{p-1} - \sum_{\iota=p+1}^{\infty} \iota\mu_\iota^\alpha |a_\iota| w^{\iota-1} \\ g^{(k)}(w) = \frac{p!}{(p-k)!} w^{p-k} - \sum_{\iota=p+1}^{\infty} \frac{\iota!}{(\iota-k)!}\mu_\iota^\alpha |a_\iota| w^{\iota-k} \end{gathered} \tag{8}$$

where, $p \geq k, p \in \mathbb{N}$ and $k \in \mathbb{N} \cup \{0\}$.

Definition 2.1 When a function $g(w)$ compared to $\mathcal{A}_p$ is within the class $K(\alpha, \lambda, n, \gamma, \tau, D, E)$, if it satisfies the following:

$$1 + \frac{w\left(\mathcal{L}_{\gamma,\tau}^{\alpha+\lambda,n} g(w)\right)^{k+1}}{\left(\mathcal{L}_{\gamma,\tau}^{\alpha,n} g(w)\right)^k} - \beta \prec \frac{1+Dw}{1+Ew} \tag{9}$$

$(-1 \leq E < D \leq 1, 0 \leq \beta < 1)$, and let,

$$B(w) = 1 + \frac{w\left(\mathcal{L}_{\gamma,\tau}^{\alpha+\lambda,n} g(w)\right)^{k+1}}{\left(\mathcal{L}_{\gamma,\tau}^{\alpha,n} g(w)\right)^k} - \beta \prec \frac{1+Dw}{1+Ew} \tag{10}$$

then we get:

$$B(w) = \frac{1+DY(w)}{1+EY(w)},$$

where, $Y(w)$ is Schwarz function [26, 27], thus:

$$B(w)\big(1+EY(w)\big) = 1 + DY(w)$$

$$Y(w) = \frac{B(w)-1}{D-EB(w)}, \text{and } |Y(w)| < 1$$

then, we obtain,

$$\left|\frac{\frac{w\left(\mathcal{L}_{\gamma,\tau}^{\alpha+\lambda,n} g(w)\right)^{k+1}}{\left(\mathcal{L}_{\gamma,\tau}^{\alpha,n} g(w)\right)^k} - \beta}{D - E\left(1 + \frac{w\left(\mathcal{L}_{\gamma,\tau}^{\alpha+\lambda,n} g(w)\right)^{k+1}}{\left(\mathcal{L}_{\gamma,\tau}^{\alpha,n} g(w)\right)^k} - \beta\right)}\right| < 1, (w \in \mathcal{L}) \tag{11}$$

3. MAIN RESULTS

This section investigates and demonstrates the necessary conditions for differential subordination for the class $K(\alpha,\lambda,n,\gamma,\tau,D,E)$. A few fascinating findings to feed differential subordination as well as superordination about analytic univalent functions were recently installed. subsequently employing the convolution formed by two linear operators, specific outcomes about differential subordination including linear operators were presented.

Theorem 3.1 When a function $g(w)\in\mathcal{A}_p$ about a given type (1) falls into the class $K(\alpha,\lambda,n,\gamma,\tau,D,E)$ if it satisfies the following condition:

$$\sum_{\iota=p+1}^{\infty}\big((1-D)+(1-E)\beta\big)\left[\frac{\iota!}{(\iota-k)!}-\frac{\iota!}{(\iota-k-1)!}\right]\mu_\iota^\alpha|a_\iota| \le \big((D-1)+(E-1)\beta\big)\left[\frac{p!}{(p-k-1)!}-\frac{p!}{(p-k)!}\right] \quad (12)$$

for $\alpha,\lambda,n\in\mathbb{N}_0,\tau\le n+1,\gamma\ \ge 0$ and $-1\le E<D\le 1$.

Proof. If $g(w)\in K(\alpha,\lambda,n,\gamma,\tau,D,E)$, then by Eq. (11), we obtain:

$$\left|\frac{\dfrac{w\left(\mathcal{L}_{\gamma,\tau}^{\alpha+\lambda,n}g(w)\right)^{k+1}}{\left(\mathcal{L}_{\gamma,\tau}^{\alpha,n}g(w)\right)^{k}}-\beta}{D-E\left(1+\dfrac{w\left(\mathcal{L}_{\gamma,\tau}^{\alpha+\lambda,n}g(w)\right)^{k+1}}{\left(\mathcal{L}_{\gamma,\tau}^{\alpha,n}g(w)\right)^{k}}-\beta\right)}\right|<1$$

$$\left|\frac{w\left(\mathcal{L}_{\gamma,\tau}^{\alpha+\lambda,n}g(w)\right)^{k+1}-\beta\left(\mathcal{L}_{\gamma,\tau}^{\alpha,n}g(w)\right)^{k}}{(D-(1-\beta)E)\left(\mathcal{L}_{\gamma,\tau}^{\alpha,n}g(w)\right)^{k}-wE\left(\mathcal{L}_{\gamma,\tau}^{\alpha+\lambda,n}g(w)\right)^{k+1}}\right|<1$$

$$\left|\frac{w\left(\frac{p!}{(p-k-1)!}w^{p-k-1}-\sum_{\iota=p+1}^{\infty}\frac{\iota!}{(\iota-k-1)!}\mu_\iota^\alpha a_\iota w^{\iota-k-1}\right)}{(D-(1-\beta)E)\left(\frac{p!}{(p-k)!}w^{p-k}-\sum_{\iota=p+1}^{\infty}\frac{\iota!}{(\iota-k)!}\mu_\iota^\alpha a_\iota w^{\iota-k}\right)}\right|-$$

$$\left|\frac{\beta\left(\frac{p!}{(p-k)!}w^{p-k}-\sum_{\iota=p+1}^{\infty}\frac{\iota!}{(\iota-k)!}\mu_\iota^\alpha a_\iota w^{\iota-k}\right)}{wE\left(\frac{p!}{(p-k-1)!}w^{p-k-1}-\sum_{\iota=p+1}^{\infty}\frac{\iota!}{(\iota-k-1)!}\mu_\iota^\alpha a_\iota w^{\iota-k-1}\right)}\right|<1$$

Hence, when $w\to 1$, we obtain:

$$(1+\beta)\left(\frac{p!}{(p-k-1)!}-\frac{p!}{(p-k)!}\right)+\sum_{\iota=p+1}^{\infty}\left(\frac{\iota!}{(\iota-k)!}-\frac{\iota!}{(\iota-k-1)!}\right)(\beta+1)\mu_\iota^\alpha|a_\iota|\le$$

$$(D+\beta E)\left(\frac{p!}{(p-k-1)!}-\frac{p!}{(p-k)!}\right)+\sum_{\iota=p+1}^{\infty}\left(\frac{\iota!}{(\iota-k)!}-\frac{\iota!}{(\iota-k-1)!}\right)(D+\beta E)\mu_\iota^\alpha|a_\iota|.$$

Then,

$$\sum_{\iota=p+1}^{\infty}\big((1-D)+(1-E)\beta\big)\left[\frac{\iota!}{(\iota-k)!}-\frac{\iota!}{(\iota-k-1)!}\right]\mu_\iota^\alpha|a_\iota| \le \big((D-1)+(E-1)\beta\big)\left[\frac{p!}{(p-k-1)!}-\frac{p!}{(p-k)!}\right].$$

Thus, $g(w)\in K(\alpha,\lambda,n,\gamma,\tau,D,E)$. The evidence is therefore complete.

The following result establishes the associated class's distortion and growth theorem [12].

Theorem 3.2 If $g(w)$ about a given type (1) falls into the class$K(\alpha,\lambda,n,\gamma,\tau,D,E)$, then for $|w|=r<1$, we have,

$$|w|^p-\left(\frac{\big((D-1)+(E-1)\beta\big)\left[\frac{p!}{(p-k-1)!}-\frac{p!}{(p-k)!}\right]}{\big((1-D)+(1-E)\beta\big)\left[\frac{\iota!}{(\iota-k)!}-\frac{\iota!}{(\iota-k-1)!}\right]\mu_\iota^\alpha}\right)|w|^{p+1}\le|g(w)| \quad (13)$$

and

$$|g(w)|\le|w|^p+\left(\frac{\big((D-1)+(E-1)\beta\big)\left[\frac{p!}{(p-k-1)!}-\frac{p!}{(p-k)!}\right]}{\big((1-D)+(1-E)\beta\big)\left[\frac{\iota!}{(\iota-k)!}-\frac{\iota!}{(\iota-k-1)!}\right]\mu_\iota^\alpha}\right)|w|^{p+1} \quad (14)$$

The equality in Eqs. (13)-(14) is achieved to feed the function g(w) provided by:

$$g(w)=w^p-\left(\frac{\big((D-1)+(E-1)\beta\big)\left[\frac{p!}{(p-k-1)!}-\frac{p!}{(p-k)!}\right]}{\big((1-D)+(1-E)\beta\big)\left[\frac{\iota!}{(\iota-k)!}-\frac{\iota!}{(\iota-k-1)!}\right]\mu_\iota^\alpha}\right)w^{p+1}. \quad (15)$$

Proof. Via Theorem 3.1, we have,

$$\sum_{\iota=p+1}^{\infty}\big((1-D)+(1-E)\beta\big)\left[\frac{\iota!}{(\iota-k)!}-\frac{\iota!}{(\iota-k-1)!}\right]\mu_\iota^\alpha|a_\iota| \le \big((D-1)+(E-1)\beta\big)\left[\frac{p!}{(p-k-1)!}-\frac{p!}{(p-k)!}\right].$$

Then, for $|w|=r<1$, we get,

$$|g(w)|\ge r^p-\sum_{\iota=p+1}^{\infty}|a_\iota|\,r^{p+1}\ge\ r^p-r^{p+1}\sum_{\iota=p+1}^{\infty}|a_\iota|$$

$$\ge r^p-\left(\frac{\big((D-1)+(E-1)\beta\big)\left[\frac{p!}{(p-k-1)!}-\frac{p!}{(p-k)!}\right]}{\big((1-D)+(1-E)\beta\big)\left[\frac{\iota!}{(\iota-k)!}-\frac{\iota!}{(\iota-k-1)!}\right]\mu_\iota^\alpha}\right)r^{p+1}.$$

also,

$$|g(w)| \le r^p + \sum_{\iota=p+1}^{\infty} |a_\iota| r^{p+1} \le r^p + r^{p+1} \sum_{\iota=p+1}^{\infty} |a_\iota|$$
$$\le r^p + \left(\frac{((D-1)+(E-1)\beta)\left[\frac{p!}{(p-k-1)!} - \frac{p!}{(p-k)!}\right]}{((1-D)+(1-E)\beta)\left[\frac{\iota!}{(\iota-k)!} - \frac{\iota!}{(\iota-k-1)!}\right]\mu_\iota^\alpha} \right) r^{p+1}.$$

The evidence is therefore complete.

Growth theorem for the considered class $K(\alpha,\lambda,n,\gamma,\tau,D,E)$ is given by.

Theorem 3.3 If $g(w)$ about a given type (1) falls into the class $K(\alpha,\lambda,n,\gamma,\tau,D,E)$, then for $|w| = r < 1$, we have,

$$pr^{p-1} - \left(\frac{((D-1)+(E-1)\beta)\left[\frac{p!(p-k-1)}{(p-k-m-1)!} - \frac{p!(p-k)}{(p-k-m)!}\right]}{((1-D)+(1-E)\beta)\left[\frac{\iota!(\iota-k)}{(\iota-k-m)!} - \frac{\iota!(\iota-k-1)}{(\iota-k-m-1)!}\right]\mu_\iota^\alpha} \right) r^p \le |g'(w)|, \quad (16)$$

and

$$|g'(w)| \le pr^{p-1} + \left(\frac{((D-1)+(E-1)\beta)\left[\frac{p!(p-k-1)}{(p-k-m-1)!} - \frac{p!(p-k)}{(p-k-m)!}\right]}{((1-D)+(1-E)\beta)\left[\frac{\iota!(\iota-k)}{(\iota-k-m)!} - \frac{\iota!(\iota-k-1)}{(\iota-k-m-1)!}\right]\mu_\iota^\alpha} \right) r^p. \quad (17)$$

The equality within Eqs. (16)-(17) is achieved to feed the function g(w) provided by,

$$g'(w) = pw^{p-1} - \left(\frac{((D-1)+(E-1)\beta)\left[\frac{p!(p-k-1)}{(p-k-m-1)!} - \frac{p!(p-k)}{(p-k-m)!}\right]}{((1-D)+(1-E)\beta)\left[\frac{\iota!(\iota-k)}{(\iota-k-m)!} - \frac{\iota!(\iota-k-1)}{(\iota-k-m-1)!}\right]\mu_\iota^\alpha} \right) w^p. \quad (18)$$

Proof. Since

$$|g'(w)| \le p|w|^{p-1} - \sum_{\iota=p+1}^{\infty} \iota |a_\iota||w|^{\iota-1}.$$

Form Theorem 3.1, we obtain,

$$\sum_{\iota=p+1}^{\infty} ((1-D)+(1-E)\beta)\left[\frac{\iota!}{(\iota-k)!} - \frac{\iota!}{(\iota-k-1)!}\right]\mu_\iota^\alpha |a_\iota| \le ((D-1)+(E-1)\beta)\left[\frac{p!}{(p-k-1)!} - \frac{p!}{(p-k)!}\right].$$

Then, for $|w| = r < 1$, we obtain,

$$|g'(w)| \ge pr^{p-1} - r^p \sum_{\iota=p+1}^{\infty} \iota |a_\iota|$$
$$\ge pr^{p-1} - \left(\frac{((D-1)+(E-1)\beta)\left[\frac{p!(p-k-1)}{(p-k-m-1)!} - \frac{p!(p-k)}{(p-k-m)!}\right]}{((1-D)+(1-E)\beta)\left[\frac{\iota!(\iota-k)}{(\iota-k-m)!} - \frac{\iota!(\iota-k-1)}{(\iota-k-m-1)!}\right]\mu_\iota^\alpha} \right) r^p$$

We can get in a similar way,

$$|g'(w)| \le p\, r^{p-1} + r^p \sum_{\iota=p+1}^{\infty} \iota |a_\iota|$$
$$\le pr^{p-1} + \left(\frac{((D-1)+(E-1)\beta)\left[\frac{p!(p-k-1)}{(p-k-m-1)!} - \frac{p!(p-k)}{(p-k-m)!}\right]}{((1-D)+(1-E)\beta)\left[\frac{\iota!(\iota-k)}{(\iota-k-m)!} - \frac{\iota!(\iota-k-1)}{(\iota-k-m-1)!}\right]\mu_\iota^\alpha} \right) r^p$$

So, we have finished the proof of the theorem.

The following result shows that, based on the study [2], the function $g(w)$ meets the radii about starlikeness, convexity, as well as close-to-convexity to convexity.

Theorem 3.4 Allow $g(w) \in K(\alpha,\lambda,n,\gamma,\tau,D,E)$. Then the function g is of starlikeness order σ in $|w| < r_1$, where,

$$r_1(\alpha,\lambda,n,\gamma,\tau,D,E,\sigma) = \inf\left\{ \frac{(p+\sigma-2)}{(\iota+\sigma-2)} \left(\frac{((1-D)+(1-E)\beta)\left[\frac{\iota!}{(\iota-k)!} - \frac{\iota!}{(\iota-k-1)!}\right]\mu_\iota^\alpha}{((D-1)+(E-1)\beta)\left[\frac{p!}{(p-k-1)!} - \frac{p!}{(p-k)!}\right]} \right) \right\}^{\frac{1}{\iota-p}}.$$

Proof. We must prove that,

$$\left|\frac{wg'(w)}{g(w)}\right| \le 1-\sigma,$$
$$\frac{(p-1)|w|^p - \sum_{\iota=p+1}^{\infty}(\iota-1)|a_\iota||w^\iota|}{|w|^p - \sum_{\iota=p+1}^{\infty}|a_\iota||w^\iota|} \le 1-\sigma. \quad (19)$$

From Eq. (19) holds if,

$$(p-1)|w|^p - \sum_{\iota=p+1}^{\infty}(\iota-1)|a_\iota||w^\iota| \le (1-\sigma)\left(|w|^p - \sum_{\iota=p+1}^{\infty}|a_\iota||w^\iota|\right).$$

Then,

$$\sum_{\iota=p+1}^{\infty} \frac{(\iota+\sigma-2)}{(p+\sigma-2)} |a_\iota||w|^{\iota-p} \le 1. \quad (20)$$

From Theorem 3.1, we obtain,

$$\sum_{\iota=p+1}^{\infty} \left(\frac{((1-D)+(1-E)\beta)\left[\frac{\iota!}{(\iota-k)!} - \frac{\iota!}{(\iota-k-1)!}\right]\mu_\iota^\alpha}{((D-1)+(E-1)\beta)\left[\frac{p!}{(p-k-1)!} - \frac{p!}{(p-k)!}\right]} \right) |a_\iota| \le 1. \quad (21)$$

Using Eqs. (20) and (21), we have,

$$\frac{(\iota+\sigma-2)}{(p+\sigma-2)}|w|^{\iota-p} \le \left(\frac{((1-D)+(1-E)\beta)\left[\frac{\iota!}{(\iota-k)!} - \frac{\iota!}{(\iota-k-1)!}\right]\mu_\iota^\alpha}{((D-1)+(E-1)\beta)\left[\frac{p!}{(p-k-1)!} - \frac{p!}{(p-k)!}\right]} \right)$$

that is,

$$|w|^{\iota-p} \le \frac{(p+\sigma-2)}{(\iota+\sigma-2)} \left(\frac{((1-D)+(1-E)\beta)\left[\frac{\iota!}{(\iota-k)!} - \frac{\iota!}{(\iota-k-1)!}\right]\mu_\iota^\alpha}{((D-1)+(E-1)\beta)\left[\frac{p!}{(p-k-1)!} - \frac{p!}{(p-k)!}\right]} \right)$$

Therefore,

$$|w| \le \left\{\frac{(p+\sigma-2)}{(\iota+\sigma-2)}\left(\frac{((1-D)+(1-E)\beta)\left[\frac{\iota!}{(\iota-k)!}-\frac{\iota!}{(\iota-k-1)!}\right]\mu_\iota^\alpha}{((D-1)+(E-1)\beta)\left[\frac{p!}{(p-k-1)!}-\frac{p!}{(p-k)!}\right]}\right)\right\}^{\frac{1}{\iota-p}}.$$

The subsequent theorem demonstrates the concave shape property found in the thought about subclass functions.

Theorem 3.5 Allow $g(w) \in K(\alpha,\lambda,n,\gamma,\tau,D,E)$. Afterwards the function g possesses a convex order σ within $|w| < r_2$, where,

$$r_2(\alpha,\lambda,n,\gamma,\tau,D,E,\sigma) = inf\left\{\frac{\frac{p!}{(p-k-1)!}(p-k-\sigma+2)}{\frac{\iota!}{(\iota-k-1)!}(\iota-k-\sigma)}\right\}^{\frac{1}{\iota-p}} \times$$
$$inf\left\{\left(\frac{((1-D)+(1-E)\beta)\left[\frac{\iota!}{(\iota-k)!}-\frac{\iota!}{(\iota-k-1)!}\right]\mu_\iota^\alpha}{((D-1)+(E-1)\beta)\left[\frac{p!}{(p-k-1)!}-\frac{p!}{(p-k)!}\right]}\right)\right\}^{\frac{1}{\iota-p}}$$

Proof. We must prove that,

$$\left|\frac{wh'(w)}{h(w)}\right| \le 1-\sigma$$

where,

$$h(w) = w\left(\mathcal{L}_{\gamma,\tau}^{\alpha+\lambda,n} g(w)\right)^{k+1}, \text{and } h^{(m)}(w) = \frac{p!\,(p-k)}{(p-k-m)!}w^{p-k-m} - \sum_{\iota=p+1}^{\infty}\frac{\iota!\,(\iota-k)}{(\iota-k-m)!}a_\iota w^{\iota-k-m} \quad (22)$$

$$\frac{\frac{p!}{(p-k-1)!}(p-k-1)|w|^{p-k} - \sum_{\iota=p+1}^{\infty}\frac{\iota!}{(\iota-k-1)!}(\iota-k-1)|a_\iota||w|^{\iota-k}}{\frac{p!}{(p-k-1)!}|w|^{p-k} - \sum_{\iota=p+1}^{\infty}\frac{\iota!}{(\iota-k-1)!}|a_\iota||w|^{\iota-k}} \le 1-\sigma$$

From Eq. (22) holds if,

$$\frac{p!}{(p-k-1)!}(p-k-1)|w|^{p-k} - \sum_{\iota=p+1}^{\infty}\frac{\iota!}{(\iota-k-1)!}(\iota-k-1)|a_\iota||w|^{\iota-k} \le (1-\sigma)\left(\frac{p!}{(p-k-1)!}|w|^{p-k} - \sum_{\iota=p+1}^{\infty}\frac{\iota!}{(\iota-k-1)!}|a_\iota||w|^{\iota-k}\right)$$

Then,

$$\sum_{\iota=p+1}^{\infty}\frac{\frac{\iota!}{(\iota-k-1)!}(\iota-k-\sigma)}{\frac{p!}{(p-k-1)!}(p-k-\sigma+2)}|a_\iota||w|^{\iota-p} \le 1 \quad (23)$$

From Theorem 3.1, we obtain,

$$\sum_{\iota=p+1}^{\infty}\left(\frac{((1-D)+(1-E)\beta)\left[\frac{\iota!}{(\iota-k)!}-\frac{\iota!}{(\iota-k-1)!}\right]\mu_\iota^\alpha}{((D-1)+(E-1)\beta)\left[\frac{p!}{(p-k-1)!}-\frac{p!}{(p-k)!}\right]}\right)|a_\iota| \le 1. \quad (24)$$

Using Eqs. (23) and (24), we have

$$\left(\frac{\frac{\iota!}{(\iota-k-1)!}(\iota-k-\sigma)}{\frac{p!}{(p-k-1)!}(p-k-\sigma+2)}\right)|w|^{\iota-p} \le \left(\frac{((1-D)+(1-E)\beta)\left[\frac{\iota!}{(\iota-k)!}-\frac{\iota!}{(\iota-k-1)!}\right]\mu_\iota^\alpha}{((D-1)+(E-1)\beta)\left[\frac{p!}{(p-k-1)!}-\frac{p!}{(p-k)!}\right]}\right)$$

$$|w|^{\iota-p} \le \left(\frac{\frac{p!}{(p-k-1)!}(p-k-\sigma+2)}{\frac{\iota!}{(\iota-k-1)!}(\iota-k-\sigma)}\right) \times \left(\frac{((1-D)+(1-E)\beta)\left[\frac{\iota!}{(\iota-k)!}-\frac{\iota!}{(\iota-k-1)!}\right]\mu_\iota^\alpha}{((D-1)+(E-1)\beta)\left[\frac{p!}{(p-k-1)!}-\frac{p!}{(p-k)!}\right]}\right)$$

Hence,

$$|w| \le \left\{\frac{\frac{p!}{(p-k-1)!}(p-k-\sigma+2)}{\frac{\iota!}{(\iota-k-1)!}(\iota-k-\sigma)}\right\}^{\frac{1}{\iota-p}} \times \left\{\left(\frac{((1-D)+(1-E)\beta)\left[\frac{\iota!}{(\iota-k)!}-\frac{\iota!}{(\iota-k-1)!}\right]\mu_\iota^\alpha}{((D-1)+(E-1)\beta)\left[\frac{p!}{(p-k-1)!}-\frac{p!}{(p-k)!}\right]}\right)\right\}^{\frac{1}{\iota-p}}$$

since a consequence, the proof has been complete.

Close-to-convexity property that characterizes the thought about subclass functions can be seen within the paragraphs that follow theorem.

Theorem 3.6 Allow $g(w) \in K(\alpha,\lambda,n,\gamma,\tau,D,E)$. Afterwards the function g possesses a close-to-convex order σ within $|w| < r_3$, where,

$$r_3(\alpha,\lambda,n,\gamma,\tau,D,E,\sigma) = inf\left\{\frac{\left(p|w|^{p-1}-(2-\sigma)\right)}{\iota|w|^{p-2}}\right\}^{\frac{1}{\iota-p}} \times$$
$$inf\left\{\left(\frac{((1-D)+(1-E)\beta)\left[\frac{\iota!}{(\iota-k)!}-\frac{\iota!}{(\iota-k-1)!}\right]\mu_\iota^\alpha}{((D-1)+(E-1)\beta)\left[\frac{p!}{(p-k-1)!}-\frac{p!}{(p-k)!}\right]}\right)\right\}^{\frac{1}{\iota-p}}$$

Proof. We must prove that,

$$|g'(w)-1| \le 1-\sigma$$

that is,

$$|g'(w)-1| \le p|w|^{p-1} - \sum_{\iota=p+1}^{\infty}\iota|a_\iota||w|^{\iota-1} - 1 < 1-\sigma$$

$$|g'(w)-1| \le p|w|^{p-1} - \sum_{\iota=p+1}^{\infty}\iota|a_\iota||w|^{\iota-1} < 2-\sigma$$

From Theorem 3.1, we obtain,

$$\sum_{\iota=p+1}^{\infty}|a_\iota| \le \frac{((D-1)+(E-1)\beta)\left[\frac{p!}{(p-k-1)!}-\frac{p!}{(p-k)!}\right]}{((1-D)+(1-E)\beta)\left[\frac{\iota!}{(\iota-k)!}-\frac{\iota!}{(\iota-k-1)!}\right]\mu_\iota^\alpha}$$

where, $\iota \ge p+1$, then,

$$\sum_{\iota=p+1}^{\infty}\left(\frac{((1-D)+(1-E)\beta)\left[\frac{\iota!}{(\iota-k)!}-\frac{\iota!}{(\iota-k-1)!}\right]\mu_\iota^\alpha}{((D-1)+(E-1)\beta)\left[\frac{p!}{(p-k-1)!}-\frac{p!}{(p-k)!}\right]}\right)|a_\iota| \le 1 \qquad (25)$$

Observe that Eq. (25) is true if,

$$\frac{\iota|w|^{\iota-2-p+p}}{p|w|^{p-1}-(2-\sigma)} \le \left(\frac{((1-D)+(1-E)\beta)\left[\frac{\iota!}{(\iota-k)!}-\frac{\iota!}{(\iota-k-1)!}\right]\mu_\iota^\alpha}{((D-1)+(E-1)\beta)\left[\frac{p!}{(p-k-1)!}-\frac{p!}{(p-k)!}\right]}\right)$$

that is,

$$|w|^{\iota-p} \le \frac{(p|w|^{p-1}-(2-\sigma))}{\iota|w|^{p-2}}\left(\frac{((1-D)+(1-E)\beta)\left[\frac{\iota!}{(\iota-k)!}-\frac{\iota!}{(\iota-k-1)!}\right]\mu_\iota^\alpha}{((D-1)+(E-1)\beta)\left[\frac{p!}{(p-k-1)!}-\frac{p!}{(p-k)!}\right]}\right)$$

Hence,

$$|w| \le \left\{\frac{(p|w|^{p-1}-(2-\sigma))}{\iota|w|^{p-2}}\right\}^{\frac{1}{\iota-p}} \times \left\{\left(\frac{((1-D)+(1-E)\beta)\left[\frac{\iota!}{(\iota-k)!}-\frac{\iota!}{(\iota-k-1)!}\right]\mu_\iota^\alpha}{((D-1)+(E-1)\beta)\left[\frac{p!}{(p-k-1)!}-\frac{p!}{(p-k)!}\right]}\right)\right\}^{\frac{1}{\iota-p}}$$

since a consequence, the proof has been complete.

4. CONCLUSION

We've shown that higher-order derivatives of multivalent functions are correlated with subclass. There are a lot of fascinating findings about harmonic multivalent functions defined by differential operators. The investigation focused on a subclass for analytical univalent function linked to the notion differential subordination. We investigated a few differential subordination along with superordination results including a specific class defined upon the dimension for univalent meromorphic functions within the open unit disc.

Gain geometric properties such as coefficient border, coefficient disparities, distortion theorem, closing theorem, severe points, starlikeness radii, convexity, near-perfect convexity, as well as integration principles. We studied neighbourhood property by using differential subordination.

We obtained a few differential subordination leads to including a linear operator, as well as certain ones sandwich theorems. As a several convolution operators, we presented a few uses of the differential subordination notion upon subclasses about univalent functions. That was obtained some important results differential subordination and differential superordination of second order of meromorphic analytical univalent function by using linear operator. Finally, by using the convolution operator, we give some results for Second order differential subordination within the open unit disk including a general hypergeometric function.

5. FUTURE STUDY

The following is a breakdown of the future study:

- Using generalized hypergeometric function and the properties of the generalized derivative operator will have obtaining multiple findings to feed fourth order different subordination within the open unit disk
- Can develop two novel bi-univalent function subclasses and obtain estimates for the concepts for class of functions.
- Ability to study a new class of multivalent functions characterized by a fresh linear operator and start investigating a new linear operator by using Hadamard product of the basic hypergeometric function as well as the Mittag-Leffler function of meromorphic functions.

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