

Ninth step block method for numerical solution of a fourth order ordinary differential equation

Saumya R. Jena*, Minakshi Mohanty, Satya K. Mishra

Department of Mathematics, School of Applied Sciences, KIIT, DT University, Bhubaneswar 751024, Odisha, India

Corresponding Author Email: saumyafma@kiit.ac.in

https://doi.org/10.18280/ama_a.550202

ABSTRACT

Received: 12 April 2018

Accepted: 1 June 2018

Keywords:

nine-step block method, power series, interpolation, collocation, ordinary differential equation, stability, order of the method

In this study a unique style of collocation and interpolation have been used to get a nine step block method for the numerical solution of linear or nonlinear initial value problems of fourth order ordinary differential equations. The present technique has been implemented at the selected mesh points to generate a direct nine step block method. In this paper zero stability, order consistency and convergence have been incorporated as the basic properties and two numerical examples have been considered and compared with ODE45 as well as continuous Linear Multistep Method (LMM) for the numerical results with exact results.

1. INTRODUCTION

Fourth order differential equations of initial value problems have many applications in the field of science and technology such as fluid dynamics [1], beam theory [2-3], electric circuits [4]. An attempt has been made to solve such problems numerically by adopting block method instead of reduction method. Our method has given a more accurate result as compared with predictor and corrector method [5] and direct block method [19]. The general fourth-order ordinary differential equations with initial conditions of the form

$$y^{(iv)} = f(z, y, y', y'', y'''), \quad y^{(n)}(x_0) = y_n, \quad n = 0(1)3 \quad (1)$$

$$z \in [a, b]$$

The present method is a better approximation eq. (1) which avoids difficulties of reduction of order [11-13], direct method [9-10, 18] and [14-15], continuous LMM method [6, 16] and occurrence of errors in the process of integration of higher order differential equations [7]. The organization of this paper is as follows:

In section-2 the construction of the nine step block method has been made. Section-3 carries assumptions of the nine step block method. In section-4 numerical examples have been taken and in section-5 conclusions have been drawn.

2. CONSTRUCTION OF THE NEW METHOD

Assume that the power series of the form

$$y(z) = \sum_{n=0}^{t+k-1} a_n z^n \quad (2)$$

is an approximate solution to eq. (1) where t and k are the number of interpolation and collocation points respectively. The first, second, third and fourth derivative of eq. (2) are

$$y'(z) = \sum_{n=0}^{t+k-1} n a_n z^{(n-1)}$$

$$y''(z) = \sum_{n=0}^{t+k-1} n(n-1) a_n z^{(n-2)} \quad (3)$$

$$y'''(z) = \sum_{n=0}^{t+k-1} n(n-1)(n-2) a_n z^{(n-3)}$$

$$y^{(iv)}(z) = \sum_{n=0}^{t+k-1} n(n-1)(n-2)(n-3) a_n z^{(n-4)}$$

By substituting eq. (3) into eq(1), we have

$$f(z, y, y', y'', y''') = \sum_{n=0}^{t+k-1} n(n-1)(n-2)(n-3) a_n z^{n-4} \quad (4)$$

Interpolating eq(2) at $z_{n+i}, i=1(1)4$ and collocating eq(4) at $z_{n+k}, k=0(1)9$ we get the following matrix form

$$BC = D \quad (5)$$

where $C = [a_0, a_1, a_2, \dots, a_{13}]^T$

$$D = [y_{n+i}, f_{n+j}]^T, \quad i = 1(1)4, \quad j = 0(1)9$$

Solving the values of a_n 's where $n = 0, 1, 2, 3, \dots, 13$ from eq. (5) using inverse matrix method and substituting the results into eq. (2), we have obtained a system of linear equations of the form

$$Y(z) = \sum_{i=1}^4 \beta_n(z) y_{n+i} + h^4 \sum_{i=0}^9 \mu_n(z) f_{n+i} \quad (6)$$

The coefficients $\beta_n(z)$ and $\mu_n(z)$ are computed & expressed as a function of $x = \frac{z - z_{n+8}}{h}$ as follows;

$$\beta_1 = \frac{-z^3}{6} - \frac{5x^2}{2} - \frac{37x}{3} - 20, \quad \beta_2 = \frac{x^3}{2} + 8x^2 + 83x + 70$$

$$\beta_3 = \frac{-x^3}{2} - \frac{17x^2}{2} - 47x - 84, \quad \beta_4 = \frac{x^3}{6} + 3x^2 + \frac{107x}{6} + 35$$

$$\mu_0 = \frac{-1}{261534873600} \begin{pmatrix} 42x^{13} + 1638x^{12} + 26754x^{11} + 234234x^{10} \\ + 1146145x^9 + 2729727x^8 - 54912x^7 \\ -16072056x^6 - 30270240x^5 \\ +40766453x^3 - 85992543x^2 \\ -1005764760 \end{pmatrix}$$

$$\mu_1 = \frac{1}{87178291200} \begin{pmatrix} 126x^{13} + 5096x^{12} + 85722x^{11} + 768768x^{10} \\ +3838835x^9 + 9333324x^8 + 195624x^7 \\ -54486432x^6 - 103783680x^5 \\ +140147189x^3 - 304269056x^2 \\ -2147088316x - 359326968 \end{pmatrix}$$

$$\mu_2 = \frac{-1}{21794572800} \begin{pmatrix} 126x^{13} + 5278x^{12} + 91728x^{11} \\ +846846x^{10} + 4339335x^9 + 10837827x^8 \\ + 808236x^7 - 62606544x^6 - 121080960x^5 \\ -421882279x^3 - 9157002127x^2 \\ -45955803266x - 74698904280 \end{pmatrix}$$

$$\mu_3 = \frac{1}{3113510400} \begin{pmatrix} 42x^{13} + 1820x^{12} + 32760x^{11} + 313170x^{10} \\ + 1659515x^9 + 4298580x^8 + 634920x^7 \\ -24504480x^6 - 48432384x^5 \\ + 511558775x^3 + 6280039090x^2 \\ + 29636403872x + 4660992336 \end{pmatrix}$$

$$\mu_4 = \frac{-1}{6227020800} \begin{pmatrix} 126x^{13} + 5642x^{12} + 105378x^{11} \\ + 1049334x^{10} + 5812235x^9 \\ + 15854553x^8 + 4025736x^7 - 88864776x^6 \\ -181621440x^5 - 732439669x^3 \\ -12147800873x^2 \\ -49273301086x - 5558973480 \end{pmatrix}$$

$$\mu_5 = \frac{1}{6227020800} \begin{pmatrix} 126x^{13} + 5824x^{12} + 113022x^{11} + 1178892x^{10} \\ + 6913335x^9 + 20370636x^8 + 8638344x^7 \\ -111759648x^6 - 242161920x^5 \\ + 1448528549x^3 + 9720953996x^2 \\ + 28712344444x + 29259550320 \end{pmatrix}$$

$$\mu_6 = \frac{-1}{3113510400} \begin{pmatrix} 42x^{13} + 2002x^{12} + 40404x^{11} + 443586x^{10} \\ + 2790645x^9 + 9187893x^8 + 6869148x^7 \\ -49153104x^6 - 121080960x^5 \\ -311421097x^3 - 2966397577x^2 \\ -6042142382x - 3772128360 \end{pmatrix}$$

$$\mu_7 = \frac{1}{21794572800} \begin{pmatrix} 126x^{13} + 6188x^{12} + 129948x^{11} \\ + 1507506x^{10} + 10295285x^9 \\ + 39171132x^8 + 52519896x^7 \\ - 173837664x^6 - 726485760x^5 \\ + 4892526821x^3 + 11396384818x^2 \\ + 11137184504x + 4604920320 \end{pmatrix}$$

(7)

$$\mu_8 = \frac{-1}{87178291200} \begin{pmatrix} 126x^{13} + 6370x^{12} + 139230x^{11} \\ + 1711710x^{10} + 12807795x^9 \\ + 57702645x^8 + 130862160x^7 \\ -51291240x^6 - 1247998752x^5 \\ - 3632428800x^4 - 5050171945x^3 \\ -3273351445x^2 - 215602214x + 971650680 \end{pmatrix}$$

$$\mu_9 = \frac{1}{261534873600} \begin{pmatrix} 42x^{13} + 2184x^{12} + 49686x^{11} + 648648x^{10} \\ + 5350345x^9 + 28864836x^8 + 101350392x^7 \\ + 219387168x^6 + 242161920x^5 \\ -293773753x^3 - 208001976x^2 \\ + 146122428x + 293333040 \end{pmatrix}$$

By solving eq. (7) at non interpolating points $x = -8, -3, -2, -1, 0, 1$, and using eq. (6), we obtain the following results.

$$y_n = 4y_{n+1} - 6y_{n+2} + 4y_{n+3} - y_{n+4}$$

$$+ h^4 \begin{pmatrix} \frac{-19}{80640} f_n + \frac{311}{1899} f_{n+1} + \frac{515}{751} f_{n+2} + \frac{405}{3364} f_{n+3} \\ + \frac{904}{14635} f_{n+4} - \frac{531}{10268} f_{n+5} - \frac{106}{3705} f_{n+6} \\ - \frac{47}{4536} f_{n+7} + \frac{76}{34027} f_{n+8} - \frac{12}{55121} f_{n+9} \end{pmatrix}$$

$$y_{n+5} = -y_{n+1} + 2y_{n+2} - 6y_{n+3} + y_{n+4}$$

$$+ h^4 \begin{pmatrix} \frac{79}{362880} f_n - \frac{1751}{725760} f_{n+1} + \frac{7873}{45360} f_{n+2} \\ + \frac{119683}{181440} f_{n+3} + \frac{30139}{181440} f_{n+4} + \frac{2507}{362880} f_{n+5} \\ - \frac{17}{2835} f_{n+6} + \frac{451}{181440} f_{n+7} \\ + \frac{41}{72576} f_{n+8} + \frac{41}{725760} f_{n+9} \end{pmatrix}$$

$$y_{n+6} = 15y_{n+1} - 4y_{n+2} - 20y_{n+3} + 10y_{n+4}$$

$$+ h^4 \begin{pmatrix} \frac{197}{241920} f_n - \frac{1609}{181440} f_{n+1} \\ + \frac{17867}{25920} f_{n+2} - \frac{255727}{90720} f_{n+3} \\ + \frac{476173}{362880} f_{n+4} + \frac{4717}{22680} f_{n+5} - \frac{5251}{181440} f_{n+6} \\ + \frac{139}{12960} f_{n+7} - \frac{1681}{725760} f_{n+8} + \frac{41}{181440} f_{n+9} \end{pmatrix}$$

$$y_{n+7} = -10y_{n+1} + 36y_{n+2} - 45y_{n+3} + 20y_{n+4}$$

$$+ h^4 \begin{pmatrix} \frac{59}{30240} f_n - \frac{5093}{241920} f_{n+1} + \frac{51911}{30240} f_{n+2} \\ + \frac{442273}{60480} f_{n+3} + \frac{134039}{30240} f_{n+4} \\ + \frac{173969}{120960} f_{n+5} + \frac{3041}{30240} f_{n+6} + \frac{199}{8640} f_{n+7} \\ - \frac{11}{2160} f_{n+8} + \frac{41}{80640} f_{n+9} \end{pmatrix}$$

$$\begin{aligned}
y_{n+8} &= -20y_{n+1} + 70y_{n+2} - 84y_{n+3} + 35y_{n+4} \\
&+ h^4 \left(\begin{aligned} &\left(\frac{2791}{725760} f_n - \frac{14957}{362880} f_{n+1} + \frac{621869}{181440} f_{n+2} \right) \\ &+ \frac{97007}{6480} f_{n+3} + \frac{545779}{51840} f_{n+4} + \frac{121793}{25920} f_{n+5} \\ &+ \frac{31403}{25920} f_{n+6} + \frac{599}{2835} f_{n+7} - \frac{8089}{725760} f_{n+8} \\ &+ \frac{407}{362880} f_{n+9} \end{aligned} \right) \\
y_{n+9} &= -35y_{n+1} + 120y_{n+2} - 140y_{n+3} + 56y_{n+4} \\
&+ h^4 \left(\begin{aligned} &\left(\frac{131}{20160} f_n - \frac{25177}{362880} f_{n+1} + \frac{67849}{181440} f_{n+2} \right) \\ &+ \frac{345371}{12960} f_{n+3} + \frac{266309}{12960} f_{n+4} + \frac{286483}{25920} f_{n+5} \\ &+ \frac{13781}{3240} f_{n+6} + \frac{130013}{90720} f_{n+7} \\ &+ \frac{25591}{181440} f_{n+8} + \frac{743}{362880} f_{n+9} \end{aligned} \right)
\end{aligned} \tag{8}$$

Evaluation of first, second and third order derivatives of $x = -8(1)1$ yields set of equations

$$\begin{aligned}
hy'_n &= -\frac{13}{3}y_{n+1} + \frac{19}{2}y_{n+2} - 7y_{n+3} + \frac{11}{6}y_{n+4} \\
&+ h^4 \left(\begin{aligned} &\left(-\frac{395}{52253} f_n - \frac{781}{1591} f_{n+1} - \frac{907}{706} f_{n+2} - \frac{173}{610} f_{n+3} \right) \\ &- \frac{2104}{59473} f_{n+4} + \frac{1401}{46033} f_{n+5} - \frac{145}{8971} f_{n+6} \\ &+ \frac{241}{43320} f_{n+7} - \frac{139}{121597} f_{n+8} + \frac{6}{56195} f_{n+9} \end{aligned} \right) \\
hy'_{n+1} &= -\frac{11}{6}y_{n+1} + 3y_{n+2} - \frac{3}{2}y_{n+3} + \frac{1}{3}y_{n+4} \\
&+ h^4 \left(\begin{aligned} &\left(\frac{11}{37626} f_n - \frac{76}{6963} f_{n+1} - \frac{175}{986} f_{n+2} - \frac{386}{6169} f_{n+3} \right) \\ &- \frac{106}{70985} f_{n+4} + \frac{48}{11443} f_{n+5} - \frac{33}{11071} f_{n+6} \\ &+ \frac{47}{38959} f_{n+7} - \frac{27}{97423} f_{n+8} + \frac{7}{248583} f_{n+9} \end{aligned} \right) \\
hy'_{n+2} &= -\frac{1}{3}y_{n+1} - \frac{1}{2}y_{n+2} + y_{n+3} - \frac{1}{6}y_{n+4} \\
&+ h^4 \left(\begin{aligned} &\left(\frac{13}{292740} f_n - \frac{13}{56456} f_{n+1} + \frac{199}{4357} f_{n+2} \right) \\ &+ \frac{362}{7909} f_{n+3} - \frac{131}{9989} f_{n+4} + \frac{92}{11637} f_{n+5} \\ &- \frac{84}{22595} f_{n+6} + \frac{32}{26081} f_{n+7} - \frac{11}{44160} f_{n+8} \\ &+ \frac{3}{128839} f_{n+9} \end{aligned} \right) \\
hy'_{n+2} &= -\frac{1}{3}y_{n+1} - \frac{1}{2}y_{n+2} + y_{n+3} - \frac{1}{6}y_{n+4} \\
&+ h^4 \left(\begin{aligned} &\left(\frac{13}{292740} f_n - \frac{13}{56456} f_{n+1} + \frac{199}{4357} f_{n+2} \right) \\ &+ \frac{362}{7909} f_{n+3} - \frac{131}{9989} f_{n+4} + \frac{92}{11637} f_{n+5} \\ &- \frac{84}{22595} f_{n+6} + \frac{32}{26081} f_{n+7} - \frac{11}{44160} f_{n+8} \\ &+ \frac{3}{128839} f_{n+9} \end{aligned} \right)
\end{aligned}$$

$$\begin{aligned}
hy'_{n+3} &= -\frac{1}{6}y_{n+1} - y_{n+2} + \frac{1}{2}y_{n+3} + \frac{1}{3}y_{n+4} \\
&+ h^4 \left(\begin{aligned} &\left(-\frac{3}{50362} f_n - \frac{29}{42687} f_{n+1} - \frac{66}{2185} f_{n+2} \right) \\ &- \frac{97}{1578} f_{n+3} + \frac{87}{6593} f_{n+4} - \frac{40}{4763} f_{n+5} \\ &+ \frac{49}{12203} f_{n+6} - \frac{28}{209031} f_{n+7} + \frac{9}{32933} f_{n+8} \\ &- \frac{8}{311761} f_{n+9} \end{aligned} \right) \\
hy'_{n+4} &= -\frac{1}{3}y_{n+1} + \frac{3}{2}y_{n+2} - y_{n+3} + \frac{11}{6}y_{n+4} \\
&+ h^4 \left(\begin{aligned} &\left(\frac{11}{111984} f_n - \frac{97}{86577} f_{n+1} + \frac{304}{5093} f_{n+2} \right) \\ &+ \frac{526}{2819} f_{n+3} - \frac{13}{18271} f_{n+4} - \frac{214}{23693} f_{n+5} \\ &+ \frac{43}{8586} f_{n+6} + \frac{46}{26067} f_{n+7} \\ &- \frac{26}{70023} f_{n+8} + \frac{7}{197149} f_{n+9} \end{aligned} \right) \\
hy'_{n+5} &= -\frac{11}{6}y_{n+1} + 7y_{n+2} - \frac{19}{2}y_{n+3} + \frac{13}{3}y_{n+4} \\
&+ h^4 \left(\begin{aligned} &\left(\frac{13}{35752} f_n - \frac{77}{19396} f_{n+1} + \frac{1267}{4016} f_{n+2} \right) \\ &+ \frac{1067}{838} f_{n+3} + \frac{1318}{2725} f_{n+4} + \frac{89}{4258} f_{n+5} \\ &- \frac{125}{13273} f_{n+6} + \frac{61}{16012} f_{n+7} + \frac{37}{42588} f_{n+8} \\ &+ \frac{9}{103064} f_{n+9} \end{aligned} \right) \\
hy'_{n+6} &= -\frac{13}{3}y_{n+1} + \frac{31}{2}y_{n+2} - 19y_{n+3} + \frac{47}{6}y_{n+4} \\
&+ h^4 \left(\begin{aligned} &\left(\frac{20}{23363} f_n - \frac{95}{10306} f_{n+1} + \frac{451}{606} f_{n+2} \right) \\ &+ \frac{570}{179} f_{n+3} + \frac{1720}{871} f_{n+4} + \frac{188}{351} f_{n+5} \\ &- \frac{155}{6618} f_{n+6} + \frac{236}{19945} f_{n+7} - \frac{41}{15966} f_{n+8} \\ &+ \frac{23}{9275} f_{n+9} \end{aligned} \right) \\
hy'_{n+7} &= -\frac{47}{6}y_{n+1} + 27y_{n+2} - \frac{63}{2}y_{n+3} + \frac{37}{3}y_{n+4} \\
&+ h^4 \left(\begin{aligned} &\left(\frac{31}{21290} f_n - \frac{101}{6495} f_{n+1} + \frac{1070}{799} f_{n+2} \right) \\ &+ \frac{7177}{1208} f_{n+3} + \frac{1228}{277} f_{n+4} + \frac{3993}{1909} f_{n+5} \\ &+ \frac{476}{1091} f_{n+6} + \frac{177}{6818} f_{n+7} - \frac{21}{5513} f_{n+8} \\ &+ \frac{18}{49031} f_{n+9} \end{aligned} \right)
\end{aligned} \tag{9}$$

$$\begin{aligned}
hy'_{n+8} &= -\frac{37}{3}y_{n+1} + \frac{83}{2}y_{n+2} - 47y_{n+3} + \frac{107}{6}y_{n+4} \\
&+ h^4 \left(\begin{aligned} &\frac{124}{53497}f_n - \frac{199}{8080}f_{n+1} + \frac{1301}{617}f_{n+2} \\ &+ \frac{2808}{295}f_{n+3} + \frac{1543}{195}f_{n+4} + \frac{1434}{311}f_{n+5} \\ &+ \frac{1438}{741}f_{n+6} + \frac{766}{1499}f_{n+7} + \frac{23}{9300}f_{n+8} \\ &+ \frac{6}{10739}f_{n+9} \end{aligned} \right) \\
hy'_{n+9} &= -\frac{107}{6}y_{n+1} + 59y_{n+2} - \frac{131}{2}y_{n+3} + \frac{73}{3}y_{n+4} \\
&+ h^4 \left(\begin{aligned} &-\frac{87}{26176}f_n - \frac{207}{5894}f_{n+1} + \frac{2610}{857}f_{n+2} \\ &+ \frac{12113}{869}f_{n+3} + \frac{46141}{3732}f_{n+4} + \frac{5503}{673}f_{n+5} \\ &+ \frac{11379}{2594}f_{n+6} + \frac{1184}{577}f_{n+7} + \frac{737}{1549}f_{n+8} \\ &+ \frac{110}{12131}f_{n+9} \end{aligned} \right) \\
h^2y''_n &= 3y_{n+1} - 8y_{n+2} + 7y_{n+3} - 2y_{n+4} \\
&+ h^4 \left(\begin{aligned} &\frac{169}{2876}f_n + \frac{1345}{1199}f_{n+1} + \frac{640}{529}f_{n+2} \\ &+ \frac{1150}{1473}f_{n+3} - \frac{557}{1136}f_{n+4} + \frac{1573}{4063}f_{n+5} \\ &- \frac{802}{3739}f_{n+6} + \frac{394}{5001}f_{n+7} - \frac{894}{51853}f_{n+8} \\ &+ \frac{89}{52242}f_{n+9} \end{aligned} \right) \\
h^2y''_{n+1} &= 2y_{n+1} - 5y_{n+2} + 4y_{n+3} - y_{n+4} \\
&+ h^4 \left(\begin{aligned} &-\frac{105}{49088}f_n + \frac{181}{2245}f_{n+1} + \frac{929}{1331}f_{n+2} \\ &+ \frac{293}{3084}f_{n+3} + \frac{197}{2171}f_{n+4} - \frac{683}{9128}f_{n+5} \\ &+ \frac{327}{7901}f_{n+6} - \frac{31}{2062}f_{n+7} + \frac{41}{12609}f_{n+8} \\ &- \frac{26}{81779}f_{n+9} \end{aligned} \right) \\
h^2y''_{n+2} &= y_{n+1} - 2y_{n+2} + y_{n+3} \\
&+ h^4 \left(\begin{aligned} &\frac{3}{29932}f_n - \frac{149}{51279}f_{n+1} - \frac{293}{3726}f_{n+2} \\ &+ \frac{3}{15655}f_{n+3} - \frac{43}{9912}f_{n+4} + \frac{73}{19652}f_{n+5} \\ &- \frac{79}{38295}f_{n+6} + \frac{21}{28004}f_{n+7} - \frac{19}{117150}f_{n+8} \\ &+ \frac{3}{189229}f_{n+9} \end{aligned} \right)
\end{aligned}$$

$$\begin{aligned}
h^2y''_{n+3} &= y_{n+2} - 2y_{n+3} + y_{n+4} \\
&+ h^4 \left(\begin{aligned} &-\frac{3}{189229}f_n + \frac{89}{343941}f_{n+1} - \frac{221}{61065}f_{n+2} \\ &- \frac{125}{1629}f_{n+3} - \frac{24}{7649}f_{n+4} - \frac{10}{29153}f_{n+5} \\ &+ \frac{55}{142733}f_{n+6} - \frac{29}{180713}f_{n+7} + \frac{2}{54837}f_{n+8} \\ &- \frac{1}{274185}f_{n+9} \end{aligned} \right) \\
h^2y''_{n+4} &= -y_{n+1} + 4y_{n+2} - 5y_{n+3} + 2y_{n+4} \\
&+ h^4 \left(\begin{aligned} &\frac{15}{67766}f_n - \frac{111}{45031}f_{n+1} + \frac{689}{3960}f_{n+2} \\ &+ \frac{453}{691}f_{n+3} + \frac{267}{2962}f_{n+4} + \frac{88}{30857}f_{n+5} \\ &- \frac{79}{14174}f_{n+6} + \frac{93}{38219}f_{n+7} - \frac{20}{35633}f_{n+8} \\ &+ \frac{13}{230119}f_{n+9} \end{aligned} \right) \\
h^2y''_{n+5} &= -2y_{n+1} + 7y_{n+2} - 8y_{n+3} + 3y_{n+4} \\
&+ h^4 \left(\begin{aligned} &\frac{8}{21113}f_n - \frac{41}{10151}f_{n+1} + \frac{415}{1213}f_{n+2} \\ &+ \frac{28778}{19185}f_{n+3} + \frac{1419}{1454}f_{n+4} + \frac{241}{2039}f_{n+5} \\ &- \frac{317}{15092}f_{n+6} + \frac{19}{3076}f_{n+7} - \frac{22}{17761}f_{n+8} \\ &+ \frac{27}{231497}f_{n+9} \end{aligned} \right) \\
h^2y''_{n+6} &= -3y_{n+1} + 108y_{n+2} - 117y_{n+3} + 4y_{n+4} \\
&+ h^4 \left(\begin{aligned} &\frac{65}{121161}f_n - \frac{113}{19850}f_{n+1} + \frac{1165}{2278}f_{n+2} \\ &+ \frac{1394}{597}f_{n+3} + \frac{1737}{880}f_{n+4} + \frac{1300}{1267}f_{n+5} \\ &+ \frac{77}{1017}f_{n+6} + \frac{28}{62449}f_{n+7} - \frac{7}{9134}f_{n+8} \\ &+ \frac{17}{175021}f_{n+9} \end{aligned} \right) \\
h^2y''_{n+7} &= -4y_{n+1} + 13y_{n+2} - 14y_{n+3} + 5y_{n+4} \\
&+ h^4 \left(\begin{aligned} &\frac{21}{27143}f_n - \frac{99}{12158}f_{n+1} + \frac{5956}{8705}f_{n+2} \\ &+ \frac{1037}{328}f_{n+3} + \frac{2133}{716}f_{n+4} + \frac{156}{77}f_{n+5} \\ &- \frac{803}{818}f_{n+6} + \frac{561}{5765}f_{n+7} - \frac{69}{11161}f_{n+8} \\ &+ \frac{14}{32489}f_{n+9} \end{aligned} \right)
\end{aligned} \tag{10}$$

$$h^2 y_{n+8}'' = -5y_{n+1} + 16y_{n+2} - 17y_{n+3} + 6y_{n+4}$$

$$+h^4 \left(\begin{array}{l} \frac{19}{28893} f_n - \frac{58}{8309} f_{n+1} + \frac{1005}{1196} f_{n+2} \\ + \frac{1303}{323} f_{n+3} + \frac{238}{61} f_{n+4} + \frac{971}{311} f_{n+5} \\ + \frac{1351}{709} f_{n+6} + \frac{6188}{5917} f_{n+7} + \frac{275}{3662} f_{n+8} \\ - \frac{19}{11945} f_{n+9} \end{array} \right)$$

$$h^2 y_{n+9}'' = -6y_{n+1} + 19y_{n+2} - 20y_{n+3} + 7y_{n+4}$$

$$+h^4 \left(\begin{array}{l} \frac{59}{20367} f_n - \frac{123}{4138} f_{n+1} + \frac{553}{500} f_{n+2} \\ + \frac{3073}{667} f_{n+3} + \frac{1309}{244} f_{n+4} + \frac{503}{142} f_{n+5} \\ + \frac{3752}{1097} f_{n+6} + \frac{3191}{1845} f_{n+7} + \frac{215}{193} f_{n+8} \\ + \frac{159}{2671} f_{n+9} \end{array} \right)$$

$$h^3 y_{n+10}''' = -y_{n+1} + 3y_{n+2} - 3y_{n+3} + y_{n+4}$$

$$+h^4 \left(\begin{array}{l} -\frac{618f_n}{2207} - \frac{1898f_{n+1}}{1149} + \frac{643f_{n+2}}{1701} - \frac{1071f_{n+3}}{542} \\ + \frac{1059f_{n+4}}{533} - \frac{379f_{n+5}}{242} + \frac{456f_{n+6}}{529} - \frac{465f_{n+7}}{1478} \\ + \frac{1518f_{n+8}}{22163} - \frac{77f_{n+9}}{11425} \end{array} \right)$$

$$h^3 y_{n+11}''' = -y_{n+1} + 3y_{n+2} - 3y_{n+3} + y_{n+4}$$

$$+h^4 \left(\begin{array}{l} \frac{184f_n}{26447} - \frac{707f_{n+1}}{2021} - \frac{1336f_{n+2}}{1137} + \frac{1011f_{n+3}}{4417} \\ - \frac{423f_{n+4}}{1072} + \frac{109f_{n+5}}{369} - \frac{1297f_{n+6}}{8272} + \frac{375f_{n+7}}{6728} \\ - \frac{73f_{n+8}}{6136} + \frac{35f_{n+9}}{30357} \end{array} \right)$$

$$h^3 y_{n+12}''' = -y_{n+1} + 3y_{n+2} - 3y_{n+3} + y_{n+4}$$

$$+h^4 \left(\begin{array}{l} -\frac{25f_n}{26731} + \frac{85f_{n+1}}{5288} - \frac{942f_{n+2}}{4129} - \frac{647f_{n+3}}{1716} \\ + \frac{146f_{n+4}}{955} - \frac{353f_{n+5}}{3634} + \frac{904f_{n+6}}{19121} - \frac{161f_{n+7}}{10091} \\ + \frac{46f_{n+8}}{13983} - \frac{18f_{n+9}}{57871} \end{array} \right)$$

$$h^3 y_{n+13}''' = -y_{n+1} + 3y_{n+2} - 3y_{n+3} + y_{n+4}$$

$$+h^4 \left(\begin{array}{l} \frac{24f_n}{45391} - \frac{69f_{n+1}}{10684} + \frac{515f_{n+2}}{2529} + \frac{2525f_{n+3}}{6406} \\ - \frac{521f_{n+4}}{3578} + \frac{81f_{n+5}}{995} - \frac{88f_{n+6}}{2327} + \frac{23f_{n+7}}{1849} \\ - \frac{59f_{n+8}}{23384} + \frac{21f_{n+9}}{89044} \end{array} \right)$$

$$h^3 y_{n+4}''' = -y_{n+1} + 3y_{n+2} - 3y_{n+3} + y_{n+4}$$

$$+h^4 \left(\begin{array}{l} -\frac{f_n}{55140} + \frac{37f_{n+1}}{77980} + \frac{159f_{n+2}}{1016} + \frac{74f_{n+3}}{83} \\ + \frac{404f_{n+4}}{791} - \frac{623f_{n+5}}{7859} + \frac{109f_{n+6}}{4211} - \frac{64f_{n+7}}{9103} \\ + \frac{52f_{n+8}}{41221} - \frac{9f_{n+9}}{83168} \end{array} \right)$$

$$h^3 y_{n+5}''' = -y_{n+1} + 3y_{n+2} - 3y_{n+3} + y_{n+4}$$

$$+h^4 \left(\begin{array}{l} \frac{23f_n}{70570} - \frac{92f_{n+1}}{26189} + \frac{207f_{n+2}}{1157} + \frac{563f_{n+3}}{701} \\ + \frac{215f_{n+4}}{199} + \frac{51f_{n+5}}{104} - \frac{359f_{n+6}}{5740} + \frac{1682f_{n+7}}{109331} \\ - \frac{93f_{n+8}}{34117} + \frac{21f_{n+9}}{89044} \end{array} \right)$$

$$h^3 y_{n+5}''' = -y_{n+1} + 3y_{n+2} - 3y_{n+3} + y_{n+4}$$

$$+h^4 \left(\begin{array}{l} \frac{23f_n}{70570} - \frac{92f_{n+1}}{26189} + \frac{207f_{n+2}}{1157} + \frac{563f_{n+3}}{701} \\ + \frac{215f_{n+4}}{199} + \frac{51f_{n+5}}{104} - \frac{359f_{n+6}}{5740} + \frac{1682f_{n+7}}{109331} \\ - \frac{93f_{n+8}}{34117} + \frac{21f_{n+9}}{89044} \end{array} \right)$$

$$h^3 y_{n+6}''' = -y_{n+1} + 3y_{n+2} - 3y_{n+3} + y_{n+4}$$

$$+h^4 \left(\begin{array}{l} -\frac{f_n}{55140} + \frac{11f_{n+1}}{40492} + \frac{331f_{n+2}}{2076} + \frac{332f_{n+3}}{383} \\ + \frac{928f_{n+4}}{1009} + \frac{2235f_{n+5}}{1949} + \frac{958f_{n+6}}{2203} - \frac{493f_{n+7}}{15524} \\ + \frac{86f_{n+8}}{20443} - \frac{18f_{n+9}}{57871} \end{array} \right)$$

$$h^3 y_{n+7}''' = -y_{n+1} + 3y_{n+2} - 3y_{n+3} + y_{n+4}$$

$$+h^4 \left(\begin{array}{l} \frac{24f_n}{45391} - \frac{47f_{n+1}}{8482} + \frac{105f_{n+2}}{559} + \frac{197f_{n+3}}{252} \\ + \frac{3107f_{n+4}}{2829} + \frac{2957f_{n+5}}{3486} + \frac{2944f_{n+6}}{2441} \\ + \frac{3491f_{n+7}}{8727} - \frac{51f_{n+8}}{2783} + \frac{35f_{n+9}}{30357} \end{array} \right)$$

$$h^3 y_{n+8}''' = -y_{n+1} + 3y_{n+2} - 3y_{n+3} + y_{n+4}$$

$$+h^4 \left(\begin{array}{l} -\frac{25f_n}{26731} + \frac{83f_{n+1}}{8605} + \frac{577f_{n+2}}{4968} + \frac{3962f_{n+3}}{4019} \\ + \frac{1427f_{n+4}}{2022} + \frac{2543f_{n+5}}{1822} + \frac{890f_{n+6}}{1483} \\ + \frac{761f_{n+7}}{565} + \frac{1025f_{n+8}}{2949} - \frac{77f_{n+9}}{11425} \end{array} \right)$$

$$h^3 y_{n+9}''' = -y_{n+1} + 3y_{n+2} - 3y_{n+3} + y_{n+4}$$

$$+h^4 \left(\begin{array}{l} \frac{184f_n}{26447} - \frac{155f_{n+1}}{2191} + \frac{1549f_{n+2}}{3184} - \frac{103f_{n+3}}{3123} \\ + \frac{2367f_{n+4}}{922} - \frac{1313f_{n+5}}{1332} + \frac{2115f_{n+6}}{754} \\ - \frac{558f_{n+7}}{2707} + \frac{871f_{n+8}}{528} + \frac{95f_{n+9}}{339} \end{array} \right)$$

(11)

From eq. (8) and from first eq. (9), eq. (10) and eq. (11), the coefficient matrices of

$$\begin{aligned}
 Y_N &= (y_{n+1}, y_{n+2}, \dots, y_{n+9}), Y_{N-1} = (y_{n-8}, y_{n-7}, \dots, y_n), \\
 Y'_{N-1} &= (y'_{n-8}, y'_{n-7}, \dots, y'_n), Y''_{N-1} = (y''_{n-8}, y''_{n-7}, \dots, y''_n), \\
 Y'''_{N-1} &= (y'''_{n-8}, y'''_{n-7}, \dots, y'''_n), F_N = (f_{n-8}, f_{n-7}, \dots, f_n), \\
 F_N &= (f_{n+1}, f_{n+2}, \dots, f_{n+9})
 \end{aligned}$$

Next multiplying each coefficient matrix by the inverse coefficient matrix of $(y_{n+1}, y_{n+2}, \dots, y_{n+9})$, we get the following equations.

$$\begin{aligned}
 B_0 Y_N &= C_1 Y_{N-1} + h C_2 Y'_{N-1} + h^2 C_3 Y''_{N-1} \\
 &+ h^3 C_4 Y'''_{N-1} + h^4 C_5 F_{N-1} + h^4 C_6 F_N
 \end{aligned} \tag{12}$$

$$\begin{aligned}
 Y_N &= [y_{n+i}]^T \quad i=1(1)9 \\
 Y_{N-1} &= [y_{n-i}]^T \quad i=8(1)0 \\
 Y'_{N-1} &= [y'_{n-i}]^T \quad i=8(1)0 \\
 Y''_{N-1} &= [y''_{n-i}]^T \quad i=8(1)0 \\
 Y'''_{N-1} &= [y'''_{n-i}]^T \quad i=8(1)0 \\
 F_{N-1} &= [f_{n-i}]^T \quad i=8(1)0 \\
 F_N &= [f_{n+i}]^T \quad i=1(1)9
 \end{aligned}$$

$$B_0 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$C_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$C_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 9 \end{pmatrix}$$

$$C_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{9}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{25}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 18 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{49}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 32 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{81}{2} \end{pmatrix},$$

$$C_4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{6} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{4}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{9}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{32}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{125}{6} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 36 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{343}{6} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{256}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{243}{2} \end{pmatrix},$$

$$C_5 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{15}{598} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{4243}{15646} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1207}{1185} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{2211}{868} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{3925}{764} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{2485}{274} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{3919}{268} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{5586}{253} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{4821}{152} \end{pmatrix}$$

$$C_6 = \begin{pmatrix} \frac{211}{5079} & \frac{-1757}{25476} & \frac{1087}{10637} & \frac{-335}{2981} & \frac{225}{2536} & \frac{-153}{3131} & \frac{230}{12891} & \frac{-92}{23679} & \frac{10}{26151} \\ \frac{317}{408} & \frac{-1715}{1649} & \frac{871}{573} & \frac{-931}{561} & \frac{1473}{1129} & \frac{-655}{914} & \frac{281}{1076} & \frac{-87}{1532} & \frac{80}{14329} \\ \frac{1112}{301} & \frac{-1897}{477} & \frac{2572}{421} & \frac{-3044}{455} & \frac{3428}{651} & \frac{-1282}{443} & \frac{903}{856} & \frac{-728}{3173} & \frac{171}{7580} \\ \frac{2016}{193} & \frac{-7795}{842} & \frac{3391}{214} & \frac{-4951}{288} & \frac{4454}{329} & \frac{-3312}{445} & \frac{2057}{758} & \frac{-683}{1157} & \frac{137}{2360} \\ \frac{6736}{297} & \frac{-5314}{311} & \frac{7307}{219} & \frac{-6122}{175} & \frac{997}{36} & \frac{-5011}{329} & \frac{7160}{1289} & \frac{-1536}{1271} & \frac{394}{3315} \\ \frac{3490}{83} & \frac{-3985}{144} & \frac{17031}{277} & \frac{-4103}{67} & \frac{8611}{174} & \frac{-3500}{129} & \frac{1613}{163} & \frac{-1083}{503} & \frac{3035}{14332} \\ \frac{14686}{351} & \frac{-18658}{3999} & \frac{17341}{8447} & \frac{-11643}{-13961} & \frac{15391}{20249} & \frac{-6509}{-6751} & \frac{19052}{2584} & \frac{-5419}{-1610} & \frac{963}{281} \\ \frac{135}{5} & \frac{21}{97} & \frac{131}{82} & \frac{47}{144} & \frac{25}{249} & \frac{76}{154} & \frac{159}{161} & \frac{323}{461} & \frac{1921}{818} \\ \frac{14351}{90} & \frac{-18658}{239} & \frac{17341}{73} & \frac{-11643}{58} & \frac{15391}{83} & \frac{-6509}{71} & \frac{19052}{523} & \frac{-5419}{725} & \frac{963}{1282} \end{pmatrix}$$

3. BASIC PROPERTIES OF THE BLOCK METHOD

This section contains some of the assumptions like order, convergence, and stability of present scheme.

3.1 Order of the method

The linear operator associated with eq. (12) can be defined as

$$L(y(z) : h) = D_0 y(z) + D_1 h y'(z) + D_2 h^2 y''(z) + \dots + D_p h^p y^{(p)}(z) + D_{p+1} h^{p+1} y^{(p+1)}(z) + D_{p+2} h^{p+2} y^{(p+2)}(z) + \dots \quad (13)$$

$y(z)$ is an arbitrary continuously differentiable function on $[a, b]$.

From eq. (12) we get the following Taylor series representation as

$$\begin{aligned} & \sum_{m=0}^{\infty} \frac{h^m y_n^{(m)}}{m!} - \sum_{m=0}^3 \frac{h^m y_n^{(m)}}{m!} - \frac{613h^4 y^{(4)}}{24769} - \\ & \left[\begin{aligned} & \sum_{m=0}^{\infty} \frac{h^{4+m} y_n^{(4+m)}}{m!} - \frac{577}{3158} (4)^m + \frac{76}{439} (5)^m - \frac{809}{6787} (6)^m + \frac{539}{9288} (7)^m \\ & - \frac{101}{5328} (8)^m + \frac{66}{17687} (9)^m - \frac{29}{86589} (10)^m \end{aligned} \right] \\ & \sum_{m=0}^{\infty} \frac{(2h)^m y_n^{(m)}}{m!} - \sum_{m=0}^3 \frac{(2h)^m y_n^{(m)}}{m!} - \frac{595h^4 y^{(4)}}{2234} - \\ & \left[\begin{aligned} & \sum_{m=0}^{\infty} \frac{h^{4+m} y_n^{(4+m)}}{m!} + \frac{1089}{431} (5)^m - \frac{1067}{615} (6)^m + \frac{795}{943} (7)^m - \frac{4595}{16709} (8)^m \\ & + \frac{140}{2589} (9)^m - \frac{59}{12167} (10)^m \end{aligned} \right] \\ & \sum_{m=0}^{\infty} \frac{(3h)^m y_n^{(m)}}{m!} - \sum_{m=0}^3 \frac{(3h)^m y_n^{(m)}}{m!} - \frac{964h^4 y^{(4)}}{965} - \\ & \left[\begin{aligned} & - \sum_{m=0}^{\infty} \frac{h^{4+m} y_n^{(4+m)}}{m!} + \frac{3327}{326} (5)^m - \frac{3372}{326} (6)^m + \frac{569}{167} (7)^m - \frac{3458}{3111} (8)^m \\ & + \frac{327}{1496} (9)^m - \frac{209}{10662} (10)^m \end{aligned} \right] \\ & \sum_{m=0}^{\infty} \frac{(4h)^m y_n^{(m)}}{m!} - \sum_{m=0}^3 \frac{(4h)^m y_n^{(m)}}{m!} - \frac{769h^4 y^{(4)}}{308} - \\ & \left[\begin{aligned} & \sum_{m=0}^{\infty} \frac{h^{4+m} y_n^{(4+m)}}{m!} + \frac{3440}{131} (5)^m - \frac{4511}{250} (6)^m + \frac{1421}{162} (7)^m - \frac{2344}{819} (8)^m \\ & + \frac{3187}{5662} (9)^m - \frac{93}{5804} (10)^m \end{aligned} \right] \\ & \sum_{m=0}^{\infty} \frac{(5h)^m y_n^{(m)}}{m!} - \sum_{m=0}^3 \frac{(5h)^m y_n^{(m)}}{m!} - \frac{740h^4 y^{(4)}}{147} - \\ & \left[\begin{aligned} & \sum_{m=0}^{\infty} \frac{h^{4+m} y_n^{(4+m)}}{m!} + \frac{48567}{2048} (1)^m - \frac{5370}{247} (2)^m + \frac{3067}{67} (3)^m - \frac{9129}{161} (4)^m \\ & + \frac{3763}{70} (5)^m - \frac{739}{20} (6)^m + \frac{20858}{1161} (7)^m - \frac{1237}{211} (8)^m \\ & + \frac{1529}{1326} (9)^m - \frac{145}{1402} (10)^m \end{aligned} \right] \end{aligned} \quad (14)$$

$$\begin{aligned} & \sum_{m=0}^{\infty} \frac{(6h)^m y_n^{(m)}}{m!} - \sum_{m=0}^3 \frac{(6h)^m y_n^{(m)}}{m!} - \frac{2781h^4 y^{(4)}}{313} - \\ & \left[\begin{aligned} & \sum_{m=0}^{\infty} \frac{h^{4+m} y_n^{(4+m)}}{m!} + \frac{6891}{157} (1)^m - \frac{3525}{98} (2)^m + \frac{8695}{104} (3)^m - \frac{2099}{21} (4)^m \\ & + \frac{20820}{217} (5)^m - \frac{6387}{97} (6)^m + \frac{5411}{169} (7)^m + \frac{3166}{303} (8)^m \\ & + \frac{1190}{579} (9)^m + \frac{1006}{5457} (10)^m \end{aligned} \right] \\ & \sum_{m=0}^{\infty} \frac{(7h)^m y_n^{(m)}}{m!} - \sum_{m=0}^3 \frac{(7h)^m y_n^{(m)}}{m!} - \frac{1017h^4 y^{(4)}}{71} - \\ & \left[\begin{aligned} & \sum_{m=0}^{\infty} \frac{h^{4+m} y_n^{(4+m)}}{m!} + \frac{5343}{73} (1)^m - \frac{1586}{29} (2)^m + \frac{24032}{173} (3)^m - \frac{26683}{167} (4)^m \\ & + \frac{28992}{185} (5)^m - \frac{17813}{167} (6)^m + \frac{3169}{61} (7)^m - \frac{4934}{291} (8)^m \\ & + \frac{2278}{683} (9)^m + \frac{581}{1942} (10)^m \end{aligned} \right] \\ & \sum_{m=0}^{\infty} \frac{(8h)^m y_n^{(m)}}{m!} - \sum_{m=0}^3 \frac{(8h)^m y_n^{(m)}}{m!} - \frac{14424h^4 y^{(4)}}{667} - \\ & \left[\begin{aligned} & \sum_{m=0}^{\infty} \frac{h^{4+m} y_n^{(4+m)}}{m!} + \frac{15752}{139} (1)^m - \frac{8543}{109} (2)^m + \frac{29485}{137} (3)^m - \frac{30279}{127} (4)^m \\ & + \frac{30737}{128} (5)^m - \frac{2089}{13} (6)^m + \frac{14135}{179} (7)^m - \frac{1106}{43} (8)^m \\ & + \frac{13307}{2630} (9)^m - \frac{241}{531} (10)^m \end{aligned} \right] \\ & \sum_{m=0}^{\infty} \frac{(9h)^m y_n^{(m)}}{m!} - \sum_{m=0}^3 \frac{(9h)^m y_n^{(m)}}{m!} - \frac{4939h^4 y^{(4)}}{159} - \\ & \left[\begin{aligned} & \sum_{m=0}^{\infty} \frac{h^{4+m} y_n^{(4+m)}}{m!} + \frac{64905}{391} (1)^m - \frac{9568}{98} (2)^m + \frac{12326}{39} (3)^m - \frac{13863}{41} (4)^m \\ & + \frac{10859}{31} (5)^m - \frac{20157}{88} (6)^m + \frac{8390}{73} (7)^m - \frac{11480}{311} (8)^m \\ & + \frac{16198}{2221} (9)^m - \frac{507}{775} (10)^m \end{aligned} \right] \\ & \sum_{m=0}^{\infty} \frac{(10h)^m y_n^{(m)}}{m!} - \sum_{m=0}^3 \frac{(10h)^m y_n^{(m)}}{m!} - \frac{4849h^4 y^{(4)}}{113} - \\ & \left[\begin{aligned} & \sum_{m=0}^{\infty} \frac{h^{4+m} y_n^{(4+m)}}{m!} + \frac{14674}{63} (1)^m - \frac{1568}{11} (2)^m + \frac{19131}{43} (3)^m - \frac{17060}{37} (4)^m \\ & + \frac{13753}{28} (5)^m - \frac{16265}{52} (6)^m + \frac{35437}{219} (7)^m - \frac{8351}{167} (8)^m \\ & + \frac{1683}{164} (9)^m - \frac{887}{979} (10)^m \end{aligned} \right] \end{aligned} \quad (14)$$

Now expanding each term in eq. (16) and comparing the coefficients of h^m and $y^{(m)}$ in eq. (14) gives $D_0 = D_1 = D_2 = \dots = D_{10} = D_{11} = D_{12} = 0$ and the error constant

$$D_{13} = \begin{pmatrix} -3 & -13 & -6 & -16 & -19 & -9 & 11 & -93 & -107 \\ 749116 & 194104 & 278119 & 127965 & 33950 & 7805 & 246274 & 31835 & 24366 \end{pmatrix}^T$$

Therefore according to the definition [19-20], the order is 11.

3.2 Zero stability of the method

The block eq. (12) is said to be zero stable [18] if the roots $r_i, i=1(1)9$ of the characteristic polynomial equation $P(t) = \det(tB_0 - C_1)$ satisfies $|t| \leq 1$ and $|t| = 1$ has multiplicity not greater than the fourth order differential equation. Now, $P(t) = \det(tB_0 - C_1) = 0$, where B_0 and C_1 coefficients of $y_{n+m}, m=1(1)9$ and y_n , respectively in (12)

$$P(t) = t \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} = 0$$

$$\Rightarrow P(t) = t^8(t-1) = 0$$

$$\Rightarrow t = 0, 0, 0, 0, 0, 0, 0, 0, 1$$

Hence the block is zero stable.

3.3 Convergence

Therefore by the new method eq (12) is convergent because it is zero stable and has order greater than one [13].

3.4. Absolute stability

The boundary locus method that is used to find the region of absolute stability of the block method was proposed by [8] and the adopted boundary locus method [17] has given $\bar{p}(t) = \frac{\alpha(t)}{t(t)}$ where the first characteristics polynomial $\alpha(t) = B_0 Y_N - C_1 Y_{N-1}$ and the second characteristics polynomial $t(t) = h^p \sum_{i=0}^1 C_{6-i} F_{N-i}$. The set problem of the form $y^{(d)} = \lambda^d y$ is substituted in eq(12) which gives

$$\text{and } B_0 Y_N = \sum_{i=0}^{d-1} h^d C_{i+1} Y_{N-1}^{(i)} + h^d \sum_{i=0}^1 \lambda^d C_{6-i} y_{N-i}$$

$$\bar{p}(t, h) = \frac{B_0 Y_N(t) - C_1 Y_{N-1}(t)}{C_6 y_N(t) + C_5 y_{N-1}(t)} \quad (15)$$

where $\bar{p} = \lambda^d h^d$.

Eq. (15) can be written in Euler's form by taking $t = e^{i\omega}$ and by using eq(15) in eq(12) of nine step block method we have

$$\bar{p}(\omega, h) = \frac{B_0 Y_N(\omega) - C_1 Y_{N-1}(\omega)}{C_6 y_N(\omega) + C_5 y_{N-1}(\omega)} \quad (16)$$

Eq. (16) is called the characteristics matrix [17].

Now by using eq. (16) to the new nine step block, stated in eq. (12), gives

$$\bar{p}(\omega, h) = \frac{W_1 - C_1}{W + C_5} W_1 = \begin{pmatrix} e^{i\omega} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & e^{2i\omega} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & e^{3i\omega} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{4i\omega} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{5i\omega} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & e^{6i\omega} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & e^{7i\omega} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{8i\omega} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{9i\omega} \end{pmatrix}$$

The determinant of the above matrix and its simplification yields

$$\bar{p}(\omega, h) = \frac{2155966188894 (e^{9i\omega} - 1)}{(24911e^{9i\omega} + 57697836)} \quad (17)$$

Eq. (17) is expanded trigonometrically and the imaginary parts are equated to zero. Hence the result

$$\bar{p}(\omega, h) = \frac{2155966188894 (\cos(9\omega) - 1)}{(24911\cos(9\omega) + 57697836)} \quad (18)$$

Evaluating eq.(18) in intervals of 30° gives the results as in the Table-1 below.

Table 1. Interval of absolute stability

ω	0	30	60	90	120	150	180
$\bar{p}(\omega, h)$	0	-37366.5	-74765.3	-37366.5	0	-37366.5	-74765.3

Therefore, the interval of absolute stability is (-74765.3,0).

4. NUMERICAL EXAMPLES

To test the method let us consider two fourth order initial value problems;

Table 2a. Result analysis of Example-1

z	Exact	Approximate solution
0.1	0.008951884436413	0.008951884436411
0.2	0.031267910608900	0.031267910608818
0.3	0.059528773414102	0.059528773413735
0.4	0.085929102584137	0.085929102583503
0.5	0.103045079418758	0.103045079418088
0.6	0.104954042902493	0.104954042901967
0.7	0.088806494399448	0.088806494399109
0.8	0.056973847769407	0.056973847769217
0.9	0.019922785200371	0.019922785200275
1	0.0	4.498300602208153e ⁻¹⁴

Table 2b. Result analysis of Example-1

z	Absolute error in ODE45	Absolute error in LMM in the same vein [16]	Absolute error in current method
0.1	1.0326e-7	2.8231e-12	1.5370e ⁻¹⁴
0.2	1.6792e-7	2.1030e-12	8.2021e ⁻¹⁴
0.3	2.5340e-7	1.8220e-10	3.6666e ⁻¹³
0.4	3.6477e-7	1.7711e-11	6.3424e ⁻¹³
0.5	5.0816e-7	3.2272e-10	6.7024e ⁻¹³
0.6	6.9093e-7	2.3688e-10	5.26084e ⁻¹³
0.7	9.2191e-7	2.3965e-10	3.3906e ⁻¹³
0.8	1.2116e-6	1.67216e-11	1.9011e ⁻¹³
0.9	1.5727e-6	1.2409e-10	9.6152e ⁻¹⁴
1	4.1649e-6	8.7041e-12	4.4983e ⁻¹⁴

$$1. y^{(4)} = e^z (z^4 + 14z^3 + 49z^2 + 32z - 12), 0 \leq z \leq 1,$$

$$h = 0.1, y(0) = y'(0) = 0, y''(0) = 2, y'''(0) = -6$$

Exact solution: $y(z) = z^2(1-z)^2 e^z$.

This result is shown in Table -2a and Table-2b.

2. $y^{(4)} = 4y''$, $0 \leq z \leq 1$,
 $h = 0.1$, $y(0) = 1$, $y'(0) = 3$, $y''(0) = 0$, $y'''(0) = 16$
Exact solution: $y(z) = 1 - z + e^{2z} - e^{-2z}$.
This result is shown in Table -3a and Table-3b.

Table 3a. Result analysis of Example 2

z	Exact Solution	Approximate solution
0.1	1.302672005082188	1.302672005082187
0.2	1.621504651605631	1.621504651605629
0.3	1.973307164296482	1.973307164296063
0.4	2.376211964375246	2.376211964375192
0.5	2.850402387287603	2.850402387286068
0.6	3.418922710824345	3.418922710824220
0.7	4.108603002903068	4.108603002901344
0.8	4.951135906400459	4.951135906400334
0.9	5.984348576191359	5.984348576190186
1	7.253720815694038	7.253720815693955

Table 3b. Result analysis of Example 2

z	Absolute error in ODE45	Absolute error in LMM [16],for h=0.1	Absolute error in current method
0.1	3.2660e-7	5.8570e-12	1.33228e ⁻¹⁵
0.2	3.2903e-7	1.0731e-10	1.9984e ⁻¹⁵
0.3	3.5880e-7	2.7926 e-10	4.1989e ⁻¹³
0.4	4.1756e-7	3.4386 e-10	5.4325e ⁻¹⁴
0.5	5.0869e-7	2.8673 e-10	1.5352e ⁻¹²
0.6	6.3749e-7	1.8874 e-10	1.2479e ⁻¹³
0.7	8.1149e-7	1.0612 e-10	1.7240e ⁻¹²
0.8	1.0409e-7	5.3352 e-11	1.2523e ⁻¹³
0.9	1.3387e-6	7.8518e-7	1.1733e ⁻¹²
1	7.9828e-6	1.0706e-11	8.3489e ⁻¹⁴

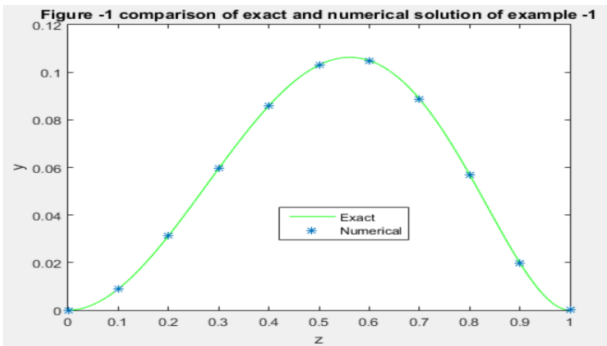


Figure 1. Comparison of exact and numerical solution of example-1

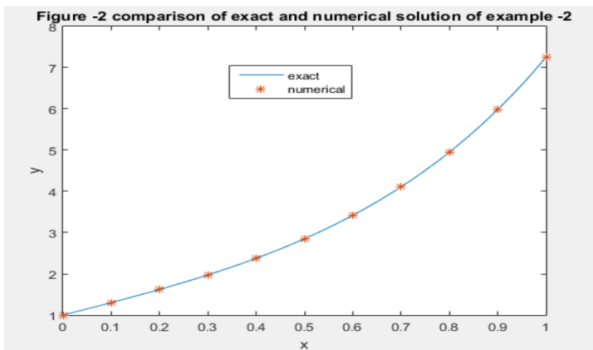


Figure 2. Comparison of exact and numerical solution of example-2

From the above Figure 1 the exact and numerical solutions coincide in the same plane.

In Figure 2 the exact and numerical solutions coincide in the same plane.

5. CONCLUSION

The numerical approximation of the current method is in good agreement with exact numerical solution. Figure 1, Figure 2, Table 2a, Table 2b and Table 3a, Table 3b it has been observed that the proposed method contributes a better approximation than the other two methods (ODE45 and LMM) to particular problems. This method helps to find accurate approximate solution of any linear or nonlinear fourth-order ordinary differential equations with initial conditions, which are complicated to find their exact solutions in the analytic manner.

REFERENCES

- [1] Alomari AK, Ratib Anakira N, Bataineh AS, Hashim I. (2013). Approximate solution of nonlinear system of BVP arising in fluid flow problem. *Mathematical Problems in Engineering* 201(3): 1-7.
- [2] Jator SN. (2008). Numerical integrators for fourth order initial and boundary value problems. *International Journal of Pure and Applied Mathematics* 47(4): 563-576.
- [3] Kelesoglu O. (2014). The solution of fourth order boundary value problem arising out of the beam-column theory using Adomian decomposition method. *Mathematical Problems in Engineering* 1-6.
- [4] Boutayeb A, Chetouani A. (2007). A mini-review of numerical methods for high-order problems. *International Journal of Computer Mathematics* 84(4): 563-579.
- [5] Kayode SJ. (2008). An efficient zero-stable numerical method for fourth-order differential equations. *International Journal of Mathematics and Mathematical Sciences* 2008: 1-10.
- [6] Awoyemi DO. (1992). On some continuous linear multistep methods for initial value problems. Ph.D. Dissertation, Dept. of Mathematics, University of Ilorin, Ilorin, Nigeria.
- [7] Senu N, Suleiman M, Ismail F, Othman M. (2011). A singly diagonally implicit Runge-Kutta-Nystrom method for solving oscillatory problems. *IAENG International Journal of Applied Mathematics* 41(2): 155-161.
- [8] Lambert JD. (1973). *Computational Methods in Ordinary Differential Equations*. John Wiley & Sons, London, UK.
- [9] Twizel EH, Khaliq AQM. (1984). Multi derivative methods for periodic IVPs. *SIAM Journal of Numerical Analysis* 21: 111-121.
- [10] Yusuf Y, Onumanyi P. (2005). New multiple FDMs through multistep collocation for $y'' = f(x, y)$ Proceeding of Conference Organised by the National Mathematics Center, Abuja, Nigeria.
- [11] Oluwaseun A, Zurni O. (2016). A new algorithm for developing block methods for solving fourth order ordinary differential equations. *Global Journal of Pure and Applied Mathematic* 12(2): 1465-1471.
- [12] Fatunla SO. (1988). Numerical methods for initial value problems in ordinary differential equations. *Computer*

- Science and Scientific Computing, Academic Press, Boston, Mass, USA.
- [13] Duromola MK. (2016). An accurate five off step points implicit block method for direct solution of fourth order differential equations. Scientific Research, An academic Publisher 3: 1-14.
- [14] Awari YS, Abada AA. (2014). A class of seven point zero stable continuous block method for solution of second order ordinary differential equation. International Journal of Mathematics and Statistics Invention 2: 47-54.
- [15] Zurni O, Oluwaseun A. (2016). Solving two-point second order boundary value problems using two-step block method with starting and non-Starting Values. International Journal of Applied Engineering Research 11(4): 2407-2410.
- [16] Awoyemi DO, Kayode SJ, Adoghe LO. (2015). A six step continuous multistep method for the solution of general fourth order initial value problems of ordinary differential equations. Journal of Natural Sciences Research 5(5): 131-138.
- [17] Kuboye JO. (2015). Block methods for direct solution of higher order ordinary differential equations using interpolation and collocation approach. Doctor of Philosophy University Utara Malaysia.
- [18] Zurni O, Kuboye JO. (2016). New seven-step numerical method for direct solution of fourth order ordinary differential equations. Journal of Mathematics and Fundamental Sciences 48(2): 94-105.
- [19] Adesanya A, Olaide AA, Momoh A, Adamu M, Tahir A. (2012). Five steps block method for the solution of fourth order ordinary differential equations. International Journal of Engineering Research and Applications 2(5): 991-998.
- [20] Oluwaseun A, Zurni O. (2017). Solving non linear fourth-order boundary value problems using a numerical approach: (m+1) th-Step block method. International Journal of Differential Equations 1-9.